THE TUTORIAL ALGEBRA

BY

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REVISED AND REWRITTEN

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PREFACE

IN the fifth edition of *The Tutorial Algebra* the text was completely revised, rewritten, and extended, to conform with current conceptions and teaching methods. Because of the additional matter incorporated it was issued in two volumes.

In the sixth edition of Volume I, the text has again been revised throughout, further material added, and many new examples inserted. Two new chapters have been included, the first dealing with important series and the second with the elementary theory of probability. The chapter in the fifth edition on interest and annuities has been omitted, as it is no longer required for certain examinations.

In its present form Volume I is designed to cover the algebra prescribed in the syllabuses of the various examining bodies for the General Certificate of Education at the Advanced Level of the University of London, and of other examinations of similar standard. Elementary ideas on complex numbers are discussed after considering the properties of surds. Particular stress is laid on the functional notation and, wherever the presentation is simplified, use has been made of those symbols which have become more familiar in recent times. The opportunity has been taken to draw attention at an early stage (Chapter VII) to properties of simple partial fractions. The notion of a limit, including that of infinity and fundamental ideas on convergence, are introduced at a comparatively early point in the book, emphasis being laid on the graphical interpretation. Rational functions (Chapter XX) are considered immediately after quadratic expressions, and the graphical interpretation is again emphasised. Special care has been taken in the choice of examples illustrating the method of mathematical induction, a branch of the subject which offers difficulty to many students. Important expansions, including the binomial, exponential, and logarithmic series, are discussed in Chapters XXV and XXVI, the emphasis being on methods of application. Volume I concludes with a treatment of the elementary mathematical theory of probability.

Volume II is intended for candidates for the B.A. and B.Sc. General examinations of the University of London, and provides an

introduction to the more advanced work of the Honours and Special examinations. It is hoped that this volume will be of material assistance to students reading for University Scholarship examinations as well as to those studying mathematics for the Natural Sciences and Mathematical Triposes examinations of the University of Cambridge.

In Volume II, Chapters I and II deal with limits, convergence, and continuous functions, while Chapter III considers the binomial theorem for a rational index. After a discussion of uniform convergence in Chapter IV, the properties of the exponential and logarithmic series are developed in Chapter V. This is followed by a chapter on the determination of functions from empirical data. Properties of complex numbers, sequences, and series are treated in Chapter VIII. Chapter IX contains a general discussion on partial fractions, recurring series, and introduces finite difference equations. Chapter X deals with finite differences, with applications to polynomials, interpolation, and linear finite difference equations. Methods of summation of series occupy Chapter XI. The two succeeding Chapters XII and XIII deal with the fundamental properties of determinants and matrices, while XIV covers the application of determinants to elimination. The last two Chapters XV and XVI, are devoted to general properties in the theory of equations, and include methods of numerical approximations to the roots of an equation.

For the seventh edition, the method of least squares has been introduced in Chapter VI, the part of Chapter IX dealing with partial fractions has been rewritten, and minor changes have been made elsewhere.

Throughout each volume emphasis has been laid on the diversity of types of examples. To assist appreciation of the theoretical development, a large number of examples (including many from past examination papers) have been worked in the text. Examples have been taken from trigonometry where these afford good illustrations of algebraic principles. The sets of examples for exercise include many selected from examination papers set at various General Certificate of Education examinations, at Intermediate, B.A., and B.Sc. examinations of the University of London, and at College Scholarship and Tripos examinations of the University of Cambridge. Sources of questions are indicated as follows:

General Certificate of Education as G.C.E. followed by a letter to indicate the particular examining body, University of Bristol (*B.*), University of Cambridge (*C.*), University of Durham (*D.*), University of London (*L.*), Oxford and Cambridge Schools Examination Board (*O.C.*), Joint Matriculation Board of the Northern Universities (*N.*), Oxford Local Examinations (*O.*), and Welsh Education Committee (*W.*); *Lond. Inter.*, *Lond. B.A.*, and *Lond. B.Sc.* refer to papers set by the University of London; *M.T.*, *N.Sc.*, and *Camb. Sch.* to Mathematical Tripos, Natural Sciences Tripos, and College Scholarship papers, respectively, of the University of Cambridge.

In conclusion, the author wishes to express his thanks to the various examining bodies who have given permission for the insertion of questions taken from past examination papers, to Mr F. E. Blamey for valuable criticism and help in checking the proofs of Volume I, to Mr J. W. Watts for his helpful advice and suggestions in the revision of Volume II, and to Mrs A. Mary Tropper, Ph.D., M.Sc., Lecturer in Mathematics, Queen Mary College, University of London, for the chapter on matrices specially written for the sixth edition of Volume II. The author will be very grateful to readers who point out any errors which the book may still contain.

G. WALKER.

CONTENTS

THE TUTORIAL ALGEBRA

VOLUME II

CHAPTER I

INFINITE SEQUENCES AND SERIES

ELEMENTARY ideas on limits and on convergence have already been considered in Vol. I., Chapters XII. and XIII. Here appeal was made to graphical representation in order to illustrate the principles involved. In the present chapter we pass to the precise arithmetical treatment of limits and convergence, and, in order to do this, use is made of the **ϵ-notation.** *The letter ϵ is taken to represent a fixed positive quantity which may be as small as we please.*

1·11. Formation of Sequences

Suppose we write down at random a collection of numbers, say 1, 3, $\frac{1}{2}$, 7, 2, 6, $\frac{1}{5}$, 9. Such a collection we can refer to as a set of numbers. Now examine the individual members of the set and see whether there is any general relation which is satisfied by every member of the set and relates the members to each other. There does not appear to be any obvious relation—thus some of the numbers are integers, others reciprocals of integers, some are less than adjoining members, others greater. If, however, the set had been 1, 2, 3, 6, 7, 9 there would be relationships. Thus each member is a positive integer and members are related to each other by the property

$$1 < 2 < 3 < 6 < 7 < 9,$$

i.e. each is less than the succeeding member.

If the set had been 1, 3, 5, 7, 9 there would have been two further relations for now all the numbers are odd and each member differs from its neighbour by a constant number 2. Thus given any set of numbers we can classify it according as there is or is not some known inter-relation between the individual members.

Now suppose we are told that the set is to consist of even numbers. We could write down any even number, say 12, and by adding or subtracting arbitrarily, 2 or multiples of 2 we can write the others down, say 12, 10, 14, 6, 24, ... This would not give us a sequence as there is no inter-relation between the individual members in the order in which they have been written down. If we say that the numbers are to be in ascending order, then if we start from 12 as before we obtain

$$12, 14, 16, 18, 20, 22, 24, \ldots.$$

Each member is obtained from the preceeding one by adding 2. We have obtained the sequence of even integers beginning with 12. The number 12 was chosen arbitrarily and so the starting member would need to be specified if the sequence is to be unique. We could represent the general term of our sequence by $2n$ where n takes the values 6, 7, 8, ... in succession. If, however, it is more convenient, we can represent the general term by $2(n + 5)$ where n now takes the values 1, 2, 3, ... in succession.

Consider a fraction of the form $(2n + 1)/2n$ and write down the numbers obtained by giving n the values 1, 2, 3, ... in succession. We obtain

$$\tfrac{3}{2}, \tfrac{5}{4}, \tfrac{7}{6}, \tfrac{9}{8}, \tfrac{11}{10},$$

This is a sequence in which the inter-relation between the individual members is given by the form of the general term $(2n + 1)/2n$. The first term of the sequence is determined by the initial value of n, i.e. $n = 1$.

Thus before considering properties of sequences it is first essential to understand the meaning of the term sequence and how sequences may arise. We can regard a sequence as a set of numbers arranged in a definite order or what is equivalent, a collection of numbers each successive member of which being formed in accordance with a definite rule. If there are only a finite number of members of the sequence we describe the sequence as finite; if there is more than a finite number of members, the sequence is said to be infinite. The words finite and infinite thus refer only to numbers of members and not to their magnitude.

We now show how sequences arise in elementary arithmetical and algebraic operations. Consider first the method of expressing a number as a continued fraction. Let a and b be positive integers which are prime to each other, so that a/b is a rational number

expressed as a fraction in its lowest terms. We can write

$$\frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{\frac{b}{r_1}}$$

where q_1 is the quotient, r_1 the remainder when a is divided by b, so that $b > r_1 > 0$.

If a/b is a proper fraction $q_1 = 0$ and $a = r_1$; if a/b is an improper fraction, q_1 is a positive integer. Similarly, if q_2 and r_2 are the quotient and remainder respectively when b is divided by r_1 we have

$$\frac{b}{r_1} = q_2 + \frac{r_2}{r_1} = q_2 + \frac{1}{\frac{r_1}{r_2}}$$

Hence q_2 is a positive integer, r_2 is a positive integer or zero and $r_1 > r_2$.

If we continue the process we have in general

$$\frac{r_{n-2}}{r_{n-1}} = q_n + \frac{r_n}{r_{n-1}}$$

where q_n is a positive integer and r_n is a positive integer or zero.

This process can be continued so long as $r_n > 0$. Since $r_1, r_2, \ldots, r_n, \ldots$ are all positive integers or zero, and $r_1 > r_2 > r_3 > \ldots$, *i.e.* follows that after a finite number of operations r_n must be zero. Suppose $r_n = 0$ for $n = s$. Then the last equation in the set of operations is

$$\frac{r_{s-2}}{r_{s-1}} = q_s.$$

Observe that we have constructed two finite sequences from our original number a/b, namely,

$$q_1, q_2, q_3, \ldots, q_s$$

our $$r_1, r_2, r_3, \ldots, r_{s-1}.$$

The first sequence contains s members and the second $(s - 1)$. Each member is determined uniquely in magnitude and position. For the first sequence there is no general relationship between the magnitudes of the terms but in the second we have the general property that each term exceeds all the terms which follow it.

To return to the original operation we can now write a/b as a continued fraction.

$$\frac{a}{b} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \dots}}$$

or in the more concise form

$$\frac{a}{b} = q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +}\,\dots\frac{1}{q_s}.$$

Observe that in the above operation we have defined another set of s numbers in a unique order, which we can represent as follows:

$$u_1 = q_1,\ u_2 = q_1 + \frac{1}{q_2},\ u_3 = q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3},\ \dots,$$

$$u_n = q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +}\dots\frac{1}{q_n},\ \dots u_s = q_1 + \frac{1}{q_2 +}\frac{1}{q_3 +}\dots\frac{1}{q_s} = \frac{a}{b}.$$

Readers familiar with the theory of continued fractions will know that u_n can be expressed as an ordinary fraction x_n/y_n where

$$x_1 = q_1,\ x_2 = q_1q_2 + 1,\ x_n = q_nx_{n-1} + x_{n-2} \text{ for } n \geqslant 3;$$

$$y_1 = 1,\ y_2 = q_2,\ y_n = q_ny_{n-1} + y_{n-2} \text{ for } n \geqslant 3.$$

These results are easily proved by induction.

It is clear that u_1, u_2, u_3, ... are rational numbers which can be regarded as approximations to the rational number a/b. By considering the set of numbers as a sequence we can assert something more than this. The property is that as we pass from one member of the sequence to the succeeding member the approximation is closer, *i.e.* u_2 is a closer approximation than u_1, u_3 is a closer approximation than u_2, and so on. In Ex. 3 below it will be seen that a surd, which is an irrational number, can be expressed as a continued fraction, the number of quotients q_1, q_2, ... no longer being finite. Thus the sequence $\{u_n\}$ will have more than a finite number of terms, *i.e.* we have an infinite sequence. All members of the sequence will be rational numbers and by taking n sufficiently great we can find a rational number u_n which differs from the irrational number by a quantity as small as we please.

Now take the original sequence $\{u_n\}$ and from its terms form two new sequences.

$$u_1,\ u_3,\ u_5,\ \dots,\ u_{2n-1},\ \dots$$

$$u_2,\ u_4,\ u_6\ \dots\ u_{2n},$$

The first which we can denote by $\{u_{2n-1}\}$, consists of the terms of $\{u_n\}$ with odd suffixes, the second of the terms with even suffixes and can be represented by $\{u_{2n}\}$. The sequences $\{u_{2n-1}\}$, $\{u_{2n}\}$ have the following property not possessed by the original sequence $\{u_n\}$.

$$u_1 < u_3 < u_5 < \dots < u_{2n-1} < \dots$$

$$u_2 > u_4 > u_6 > \dots > u_{2n} > \dots$$

In words, $\{u_{2n-1}\}$ is a sequence of rational numbers such that each term is less than the succeeding term; $\{u_{2n}\}$ is such that each term is greater than the succeeding term.

The above properties are illustrated in the examples given below; the general results can be proved from the theory of continued fractions.

Examples.—(1) *Express* 2·6 *in the form of a continued fraction.*

Observe that in the calculation of the quotients $q_1, q_2, q_3, \dots$ we use the elementary arithmetical method of finding the highest common factor of two integers. Thus we can set the working out as follows. Write $2{\cdot}6 = \frac{13}{5}$.

$$\begin{array}{l} 5)\ 13\ (2 = q_1 \\ r_1 = 3)\ 5\ (1 = q_2 \\ \quad r_2 = 2)\ 3\ (1 = q_3 \\ \qquad r_3 = 1)\ 2\ (2 = q_4 \end{array}$$

Thus the $\{q\}$ sequence is 2, 1, 1, 2 and we have

$$\frac{13}{5} = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}.$$

For the $\{u_n\}$ sequence we have

$$u_1 = 2,\ u_2 = 2 + \frac{1}{1},\ u_3 = 2 + \frac{1}{1+}\,\frac{1}{1},\ u_4 = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2},$$

$$\text{or } u_1 = 2,\ u_2 = 3,\ u_3 = 2{\cdot}5,\ u_4 = 2{\cdot}6.$$

(2) *Express* 55/34 *as a continued fraction and deduce a sequence of simpler fractions which approximates to it. Consider the closeness of the approximations.*

$$\begin{array}{l} 34)\ 55\ (1 = q_1 \\ \quad 21)\ 34\ (1 = q_2 \\ \qquad 13)\ 21\ (1 = q_3 \\ \qquad\quad 8)\ 13\ (1 = q_4 \\ \qquad\qquad 5)\ 8\ (1 = q_5 \\ \qquad\qquad\quad 3)\ 5\ (1 = q_6 \\ \qquad\qquad\qquad 2)\ 3\ (1 = q_7 \\ \qquad\qquad\qquad\quad 1)\ 2\ (2 = q_8. \end{array}$$

The $\{q\}$ sequence is 1, 1, 1, 1, 1, 1, 1, 2, and

$$\frac{55}{34} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$$

Write $u_1 = 1$, $u_2 = 1 + \frac{1}{1}$, $u_3 = 1 + \frac{1}{1+}\,\frac{1}{1}$, $u_4 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1}$,

$$u_5 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1}, \quad u_6 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1},$$

$$u_7 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1}, \quad u_8 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}.$$

Expressing $u_1, u_2, \ldots u_8$ as ordinary fractions we find that the $\{u\}$ sequence is 1, 2, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, $\frac{21}{13}$, $\frac{55}{34}$. The first seven members of the sequence are rational numbers which approximate more and more closely to 55/34 as we move along the sequence. The closeness of the approximation can be seen by expressing $u_1, u_2, \ldots, u_7, u_8$ correct to three decimal places.

Write $u = u_8 = 55/34 = 1{\cdot}617$ correct to three decimal places;

$$v_n = |\,u - u_n\,|, \; n = 1, 2, \ldots, 7.$$

Then the values of v_n give the measure of the numerical differences between the given fraction and the member of the sequence.

n	u_n	v_n
1	1·000	0·617
2	2·000	0·383
3	1·500	0·117
4	1·667	0·050
5	1·600	0·017
6	1·625	0·008
7	1·615	0·002

The value of v_n decreases steadily as n increases. Further, we observe that the members of $\{u_n\}$ with odd suffixes, i.e. u_1, u_3, u_5, u_7 are all less than u, while the members of u with even suffixes are all greater than u.

(3) *Express the irrational number $\sqrt{3}$ in the form of a continued fraction and hence show how a sequence of rational numbers can be obtained whose succeeding terms approximates more and more closely to $\sqrt{3}$. Find the first member of the sequence which differs from $\sqrt{3}$ by less than* 0·0001.

The greatest integer which is less than $\sqrt{3}$ is 1. Hence we write

$$\sqrt{3} = 1 + (\sqrt{3} - 1) = 1 + \frac{2}{\sqrt{3} + 1} \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

giving $q_1 = 1$.

We can regard $(\sqrt{3} - 1)$ or $2/(\sqrt{3} + 1)$ as a remainder whose value is less than unity, so that its reciprocal is greater than unity. Proceeding as before, the greatest integer which is less than $(\sqrt{3} + 1)/2$ is 1. Write—

$$\frac{\sqrt{3}+1}{2} = 1 + \left(\frac{\sqrt{3}+1}{2} - 1\right) = 1 + \frac{\sqrt{3}-1}{2} \quad \ldots\ldots\ldots \quad \text{(ii)}$$

giving $q_2 = 1$.

The reciprocal of the remainder $\frac{\sqrt{3}-1}{2}$ is $\frac{2}{\sqrt{3}-1} = \sqrt{3}+1$.

The greatest integer less than $\sqrt{3}+1$ is 2. Hence we write

$$\sqrt{3}+1 = 2 + (\sqrt{3}+1-2) = 2 + (\sqrt{3}-1) \quad \text{........ (iii)}$$

giving $q_3 = 2$.

Observe that the remainder is now $(\sqrt{3}-1)$ which is the same as its value in (i).

Hence the next step will be the same as (ii), the following step is the same as (iii), and so on. Thus $q_4 = 1$, $q_5 = 2$. We see that the process continues indefinitely, giving $q_{2n} = 1$, $q_{2n+1} = 2$, $n \geqslant 1$.

Accordingly the $\{q\}$ sequence is

$$1, 1, 2, 1, 2, 1, 2, \ldots.$$

and the continued fraction is

$$\sqrt{3} = 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots.$$

We can regard $\frac{1}{1+}\frac{1}{2+}$ as a repeating part similar to what occurs in an ordinary repeating decimal. As the $\{q\}$ sequence has more than a finite number of terms, the continued fraction is spoken of as an infinite continued fraction. Observe that as in the case of sequences the word infinite does not refer to the magnitude of the fraction but to the number of operations used in defining it.

Write $u_1 = 1$, $u_2 = 1 + \frac{1}{1}$, $u_3 = 1 + \frac{1}{1+}\frac{1}{2}$, $u_4 = 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1}, \ldots.$
so that $u_1 = 1$, $u_2 = 2$, $u_3 = \frac{5}{3}$.

For $n > 3$ we can calculate u_n from the formula $u_n = x_n/y_n$ where $x_n = q_n x_{n-1} + x_{n-2}$, $y_n = q_n y_{n-1} + y_{n-2}$, referred to earlier in this section. We find $u_4 = \frac{7}{4}$, $u_5 = \frac{19}{11}$, $u_6 = \frac{26}{15}$, $u_7 = \frac{71}{41}$, $u_8 = \frac{97}{56}$, ...

Hence the first 8 members of the required sequence are

$$1, 2, \tfrac{5}{3}, \tfrac{7}{4}, \tfrac{19}{11}, \tfrac{26}{15}, \tfrac{71}{41}, \tfrac{97}{56}.$$

Write $u = \sqrt{3} = 1{\cdot}7321\ldots$, and express the members of the sequence as decimals correct to four decimal places. The results are shown in the form of a table where $v_n = |u - u_n|$. Observe that $u - u_n$ being the difference of an irrational and a rational number must be irrational.

r	u_r	$u - u_r$ +	$u - u_r$ −	v_r
1	1·0000	0·7321		0·7321
2	2·0000		0·2679	0·2679
3	1·6667	0·0654		0·0654
4	1·7500		0·0175	0·0175
5	1·7272	0·0049		0·0049
6	1·7333		0·0012	0·0012
7	1·7315	0·0006		0·0006
8	1·7321		0·0000	0·0000

The rational number u_8 agrees with $\sqrt{3}$ correct to four decimal places. The sequence v_r is a set of positive irrational numbers whose terms decrease steadily towards zero. Observe that we have defined *three* sequences of

rational numbers whose successive members approximate more closely to the given irrational number.

(1) $u_1, u_2, u_3, u_4, \ldots.$
(2) $u_1, u_3, u_5, u_7, \ldots.$
(3) $u_2, u_4, u_6, u_8, \ldots.$

Sequence (1) has the property that a given term is a closer approximation to $\sqrt{3}$ than early terms in the sequence. Also the terms are alternately less than and greater than $\sqrt{3}$.

In addition to the approximation property, sequence (2) has the following properties: (*a*) the terms of the sequence steadily increase, *i.e.* each term is less than the preceding term; (*b*) every term of the sequence is less than $\sqrt{3}$.

Sequence (3) has similar properties except that the terms steadily decrease and every term of the sequence is greater than $\sqrt{3}$.

Further examples of infinite sequences are provided by the sets of decimals which arise in elementary arithmetic as successive approximations to rational numbers which cannot be expressed as terminating decimals. In this way we obtain an infinite sequence of rational numbers whose successive terms approximate more and more closely to another rational number.

Consider the fraction $\frac{1}{3}$ which can be expressed as the repeating decimal $0{\cdot}\dot{3}$.

Write $u_1 = 0{\cdot}3$, $u_2 = 0{\cdot}33$, $u_3 = 0{\cdot}333, \ldots, u_n = 0{\cdot}33\ldots3$ (n decimal places). We can express u_n more conveniently in another form.

$$u_n = \frac{3}{10} + \frac{3}{10^2} + \ldots + \frac{3}{10^n} = \frac{3}{10}\left(1 + \frac{1}{10} + \ldots + \frac{1}{10^{n-1}}\right)$$
$$= \tfrac{1}{3}(1 - 10^{-n}).$$

$$u_{n+1} - u_n = \tfrac{1}{3}(1 - 10^{-n-1}) - \tfrac{1}{3}(1 - 10^{-n}) = \frac{3}{10^{n+1}} > 0.$$

Hence $u_{n+1} > u_n$.

Also, if we write $u = \frac{1}{3}$, $u - u_n = \frac{1}{3} - \frac{1}{3}(1 - 10^{-n}) = \dfrac{10^{-n}}{3} > 0$.

We have thus defined an infinite sequence $\{u_n\}$ with the following properties:

(1) each term of the sequence is a positive rational number;

(2) each term is less than any term which follows it in the sequence, *i.e.* the terms steadily increase, or in symbols $u_n < u_{n+1}$;

(3) no matter how large n may be, u_n is always less than a fixed positive number k. This number can be $\frac{1}{3}$ or any number greater than $\frac{1}{3}$. In symbols $u_n < k$.

We now consider another property of the sequence. Choose an *arbitrary positive* number as small as we please and denote it by ϵ. This number is at our disposal, *e.g.* $\epsilon = 10^{-10}$, or 10^{-20}, but once it is chosen it *must not be altered.* Consider the question of finding how many terms of the sequence it would be necessary to take before we reach a number which differs from $\frac{1}{3}$ by less than ϵ. In symbols, for what value of n is $\frac{1}{3} - u_n < \epsilon$ or $\frac{10^{-n}}{3} < \epsilon$, or $10^n > 1/3\epsilon$. Since ϵ is a *fixed* positive number we can always find a positive integer to satisfy the inequality. Suppose N is such a value of n. Then clearly $N + 1$, $N + 2$, $N + 3, \ldots$, *i.e.* any positive integer greater than N will also satisfy the inequality. Thus, for example, if we choose $\epsilon = 10^{-20}$, we require $10^{-n} < 3.10^{-20}$ or $10^n > \frac{10^{20}}{3}$. This is clearly satisfied for $n = 20$ or any positive integer n greater than 20.

Thus the sequence has following property. If ϵ is any fixed positive number chosen at will we can always find a positive integer N such that for every $n \geqslant N$

$$\left| \tfrac{1}{3} - u_n \right| < \epsilon.$$

1·12. Variable which Assumes Integral Values only

In considering the construction of sequences we were able to represent the general term by the symbol u_n, *i.e.* by giving n a particular positive integral value we obtained a particular member of the sequence. When n is used in this way we speak of it as an integral variable. It may range through a finite or infinite set of values. Thus in the sequence of even numbers

$$12, 14, 16, 18, 20, 22, 24, \ldots.$$

we can write $u_n = 2n$, where $n = 6, 7, 8, \ldots$

$$\text{or } u_n = 2(n + 5), \text{ where } n = 1, 2, 3, \ldots.$$

In each representation the sequence is the same. Hence we have a choice in the range of the variable and in practice we choose the most convenient. Clearly, to make the correspondence obvious we would choose u_1 to correspond to $n = 1$, u_2 to $n = 2$, u_3 to $n = 3$, and so on, *i.e.* the integral variable takes the values $1, 2, 3, \ldots.$ If the general term is not given in a suitable form it may be modified as in the example just considered. Thus in discussing sequences in general we may assume without loss of

generality that n ranges through integral values beginning with unity.

If n takes more than a finite number of values we are faced immediately with the necessity of attaching a precise arithmetical meaning to the expressions n *very large*, or n *tends to infinity*, which we write in symbols as $n \to \infty$. We could fix a positive integer n_0, as large as we please, and then say that "n very large" is to mean a value of n which is greater than or equal to n_0, i.e. $n \geqslant n_0$. Then consider the values of the terms of the sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$ for $n \geqslant n_0$. There are two possibilities. In the first place there may exist a number u with the property that for all values of $n \geqslant n_0$, u_n is always "very near" u; in the contrary case there is no such number. It will be necessary to give the phrase "very near" a precise arithmetical meaning and this is done in the following way. Choose a fixed positive number and denote it by ϵ. The choice of ϵ is arbitrary and so completely at our disposal, and can be as small as we please. We might say that u_n is very near u if $|u - u_n| < \epsilon$. With the help of these ideas we can give a precise arithmetical definition to the limit of a sequence.

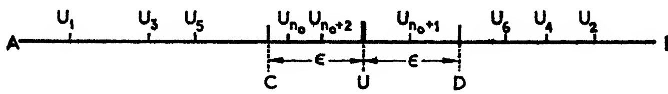
Take an arbitrary positive number ϵ. Then if there exists a number u and an integer n_0 such that

$$|u - u_n| < \epsilon$$

for all values of $n \geqslant n_0$, then u is defined to be the limit of u_n as n tends to infinity; in symbols $u = \lim_{n \to \infty} u_n$. Observe that the only number which is arbitrary here is ϵ and that when it has been chosen it will, in general, determine a value of n_0; in other words, n_0 is a function of ϵ and to emphasise this fact it is sometimes written as $n_0(\epsilon)$. An example of the dependence of n_0 on ϵ is given below.

We can illustrate diagrammatically the definition of the limit. Suppose real numbers are represented as points along a straight line AB produced in both directions as necessary and adopt the usual convention that the real numbers will increase as we move from left to right along the line. As the numbers of our sequences are real numbers they can be represented as points on AB. To denote the correspondence between points and the members of sequence capital letters will denote corresponding points so that the point U_n will represent u_n.

First mark in U corresponding to u. The inequality $|u - u_n| < \epsilon$ asserts that $u - \epsilon < u_n < u + \epsilon$. Mark off intervals on each side of U of length ϵ so that $CU = UD = \epsilon$. Now put in the points $U_1, U_2, U_3, \quad U_{n_0}, U_{n_0+1}, U_{n_0+2}, \quad$ Then U_{n_0} will lie in the interval CD, as also will all the points $U_{n_0+1}, U_{n_0+2}, U_{n_0+3}, \ldots$



which follow U_{n_0}. It is immaterial whether these points are to the left or right of U so long as they lie in CD. In other words, if when we choose an arbitrarily small interval on each side of U we can find n_0 such that U_n always lies in one of these intervals when $n \geqslant n_0$ then the limit of U_n is U.

In §1·11 we defined a sequence $\{u_n\}$ whose terms formed, with increasing n, closer and closer approximations to $\frac{1}{3}$. Here, $u_n = \frac{1}{3}(1 - 10^{-n})$. If we take $u = \frac{1}{3}$, then $\frac{1}{3} - u_n = \frac{1}{3}10^{-n}$. Now choose an arbitrary positive number ϵ. Then $|\frac{1}{3} - u_n| < \epsilon$ provided $\frac{1}{3}10^{-n} < \epsilon$, or $10^{-n} > 1/3\epsilon$. Since ϵ is a fixed number, and 10^n increases as n increases we can always find a positive integer n_0 such that $10^{n_0} > 1/3\epsilon$. Thus if $k = \log_{10}(1/3\epsilon)$ we can take n_0 to be the first positive integer greater than k. Also

$$10^N > 10^{n_0} > 1/3\epsilon, N > n_0.$$

Hence, in accordance with the definition, $\lim\limits_{n \to \infty} u_n = \frac{1}{3}$.

The dependence of n_0 on ϵ is easily seen by giving ϵ particular values. Thus if $\epsilon = 10^{-10}$ we require $10^{n_0} > \frac{1}{3}.10^{10}$. Thus we can take $n_0 = 10$. If on the other hand we write $\epsilon = 10^{-20}$ the inequality to be satisfied is $10^{n_0} > \frac{1}{3}10^{20}$ and we take $n_0 = 20$. The smaller value $n_0 = 10$ will not satisfy the second inequality but the greater value $n_0 = 20$ will satisfy *both* inequalities

$$10^{n_0} > \tfrac{1}{3}10^{10} \quad \text{and} \quad 10^{n_0} > \tfrac{1}{3}10^{20}.$$

In general, the effect of choosing a smaller ϵ will be to give an increased n_0.

Consider now the sequence $\{u_n\}$ where $u_n = (-1)^n/n$,

i.e. $$-1, +\tfrac{1}{2}, -\tfrac{1}{3}, +\tfrac{1}{4}, -\tfrac{1}{5}, \ldots, \frac{(-1)^n}{n}, \ldots.$$

Here the limit is zero. For

$$|0 - u_n| = |u_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

$$|0 - u_n| < \epsilon, \text{ provided } \frac{1}{n} < \epsilon, \text{ or } n > \frac{1}{\epsilon}.$$

Choose n_0 to be the first integer greater than $1/\epsilon$. Then for all values of $n > n_0$,

$$\frac{1}{n} < \epsilon.$$

Hence $$\lim_{n \to \infty} u_n = 0.$$

This example shows that if the terms of a sequence are alternately positive and negative the sequence cannot have a limit unless the numerical value of u_n tends to zero with increasing n.

1·13. Continuous Variable

Now consider a *continuous variable* as distinct from an *integral variable* such as n in the previous section.

Let a denote a finite number. Then *by x being a continuous variable which tends to a we mean one which takes all values in the "neighbourhood" of a.*

A **neighbourhood** of a number a may be defined precisely as follows. Let η denote a positive number which may be as small as we please. Then by a neighbourhood of a is meant the interval from $a - \eta$ to $a + \eta$. *The value a itself is excluded from the interval.*

This statement may seem rather indefinite at first sight, but it must be remembered that a neighbourhood of a point is arbitrary and we can choose as many neighbourhoods as we please.

Now suppose that $f(x)$ is a function which depends on x, and that l is a number with the following property. There exists a neighbourhood of a with the property that the numerical difference between $f(x)$ and l is less than an arbitrary positive number ϵ *at all points of the neighbourhood.*

It is important to observe that the value a itself is excluded and that we are considering a property which is defined at all points *near* a. If the number l possesses the property indicated above then l is said to be the limit of $f(x)$ as x tends to a. In symbols

$$\lim_{x \to a} f(x) = l$$

The definition, stated precisely in symbols, is as follows: ***Let ϵ denote an arbitrary positive number. Then l is the limit of $f(x)$ as x tends to a if we can find a positive number η with the property that***

$$|f(x) - l| < \epsilon$$

for all values of x satisfying the inequality $0 < |x - a| \leqslant \eta$.

The first part of the inequality $0 < |x - a| \leqslant \eta$ is expressly inserted so as to *exclude* the value $x = a$. The inequality is equivalent to the interval

$$a - \eta \leqslant x \leqslant a + \eta$$

where $x = a$ is excluded or to

$$a - \eta \leqslant x < a, \quad a < x \leqslant a + \eta.$$

The fact that η *depends on* ϵ is frequently indicated by writing it in the form $\eta(\epsilon)$, the symbol denoting in the usual way that η is a function of ϵ. The dependence of η on ϵ is seen in the following example:

Let $f(x) = x^2 + 1$ and consider $\lim\limits_{x \to 1} (x^2 + 1)$. Clearly, as x becomes nearer and nearer to unity $x^2 + 1$ becomes nearer and nearer to 2.

Take, *e.g.*, $\epsilon = 10^{-6}$. Then the inequality $|f(x) - l| < \epsilon$ becomes

$$|x^2 + 1 - 2| < 10^{-6}, \textit{ i.e. } |x^2 - \quad < 10^{-}$$

This inequality is equivalent to

$$-10^{-6} < x^2 - 1 < 10^{-}$$

$$\textit{i.e. } 1 \quad 10^{-6} < x^2 < 10^{-6} + 1.$$

Taking the positive square root, this inequality is equivalent to

$$\{1 - 10^{-6}\}^{\frac{1}{2}} < x < \{1 + 10^{-6}\}^{\frac{1}{2}},$$

$$\textit{i.e. } \{1 - 10^{-6}\}^{\frac{1}{2}} - 1 < x - 1 < \{1 + 10^{-6}\}^{\frac{1}{2}} - 1,$$

$$\textit{i.e. } -\lambda_1 < x - 1 < \lambda_2, \text{ say.}$$

Let η be the greater of positive numbers λ_1, λ_2. Then

$$|x - 1| < \eta.$$

Thus we have found a number η with the required property.

1·2. Bounded Sequences

Let $u_1, u_2, u_3, \ldots u_n, \ldots$ denote a sequence of numbers. Then the sequence is said to be *bounded above* if there exists a finite number A with the property that $u_n \leqslant A$ for all values of n. Similarly if there exists a finite number B with the property that $u_n \geqslant B$ for all values of n, the sequence is said to be *bounded below*. The sequence is said to be *bounded* if it is *bounded above and below*.

Such a sequence has the property that $|u_n| \leqslant C$ for all n, where C is a fixed positive number. For we can choose C equal to the greater of $|A|, |B|$.

Observe that although the numbers A, B, C bound the sequence they are *not unique* and so must not be described as the bounds of the sequence. For if $A_1 > A$, then $u_n \leqslant A$ implies $u_n \leqslant A_1$. In a subsequent section it will be shown how to define *unique* numbers which can be described as upper and lower bounds. It will be seen there that these bounds represent the best possible values of A and B.

Examples of bounded sequences have occurred in earlier sections. Thus the sequence $\{u_n\}$ where $u_n = \frac{1}{3}(1 - 10^{-n})$ has the property $\frac{3}{10} \leqslant u_n \leqslant \frac{1}{3}$ for n a positive integer and so $\{u_n\}$ is bounded above and below. The same property could have been asserted if the inequality had been written in the cruder form $|u_n| \leqslant 1$.

Sequences which do not possess the property $|u_n| \leqslant C$ where C is a fixed positive number, can be described as unbounded. They may, however, be bounded above or below. Thus consider the sequence $\{u_n\}$ where $u_n = n^2$, *i.e.* the sequence consisting of the squares of positive integers. It is bounded below for $u_n \geqslant 1$ for all values of n. It is not bounded above because, however large A is, we can always find a value of n such that $n^2 > A$. The sequence

$$-1, +2, -3, +4, -5, \ldots, (-1)^n n, \ldots$$

is not bounded above or below. For $|u_n| = n$ which can always be made greater than the fixed number C, however large C is.

1·21. Monotonic Sequences

A sequence may be monotonic increasing or decreasing. The sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$ is said to be **monotonic increasing** if

$$u_1 \leqslant u_2 \leqslant u_3 \leqslant \ldots \leqslant u_n \leqslant \ldots,$$

that is if u_n does *not decrease* as n increases. An example of a

monotonic increasing sequence is

$$\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \ldots, \frac{n}{n+1}, \ldots.$$

For $u_{n+1} - u_n = \dfrac{n+1}{n+2} - \dfrac{n}{n+1} = \dfrac{1}{(n+1)(n+2)} > 0$, giving $u_n < u_{n+1}$. This sequence is also bounded for $0 \leqslant u_n \leqslant 1$ for all positive integral values of n.

The sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$ is said to be **monotonic decreasing** if

$$u_1 \geqslant u_2 \geqslant u_3 \geqslant u_4 \geqslant \ldots \geqslant u_n \geqslant \ldots,$$

that is, if the members of the sequence do *not increase* as n increases. An example is the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \frac{1}{n}, \ldots.$$

It will be observed that this sequence is also bounded, for none of its members lie outside the interval $(0, 1)$.

A sequence which is either monotonic increasing or monotonic decreasing is said to be **monotonic**.

It will be observed in the above definitions that the equality sign is included. If we wish to exclude this possibility so that none of the members of the sequence are equal to one another we say that the sequence is *strictly monotonic increasing* or *strictly monotonic decreasing* as the case may be. Both examples given above are *strictly* monotonic. The sequence $\{u_n\}$ where $u_n = \frac{1}{3}(1 - 10^{-n})$ considered in § 1·2 is strictly monotonic increasing.

1·22. Upper and Lower Bounds

We have stated above, in § 1·2, the conditions under which a sequence is bounded. We now consider what is meant by the *bounds* of the sequence.

Let ϵ be an arbitrary positive number which may be as small as we please. Then if the sequence is bounded above we can find a number M with the following properties.

(*a*) $u_n \leqslant M$ for *all* n.

(*b*) *There exists some value of* n such that $u_n > M - \epsilon$.

This number M is defined to be the **upper bound*** of the sequence.

* For a rigid proof of the existence of M in the case of bounded sequences the student is referred to books on mathematical analysis.

Again, if the sequence is bounded below there exists a number m with the properties,

(a) $u_n \geqslant m$, *for all* n.

(b) *There exists a value of* n such that $u_n < m + \epsilon$.

This number m is defined to be the **lower bound** of the sequence.

Consider the first sequence $\{u_n\}$ defined in § 1·21. Here $u_n = n/(n + 1)$. We prove that the lower bound is $\frac{1}{2}$ and the upper bound 1. First observe that the sequence is monotonic increasing so that the least term is $u_1 = \frac{1}{2}$ and $u_n \geqslant \frac{1}{2}$ for all positive integral values of n. Further, however small ϵ may be there is a member of the sequence greater than $\frac{1}{2} + \epsilon$, viz. u_1. Hence the lower bound is $\frac{1}{2}$ and there is a member of the sequence which is equal to the lower bound. For the upper bound we note that $u_n < 1$ for all finite values of n. Next consider whether there is a member of the sequence greater than $1 - \epsilon$. We require a value of n such that

$$\frac{n}{n+1} > 1 - \epsilon, \quad \text{or} \quad n > \frac{1-\epsilon}{\epsilon}.$$

Since ϵ is fixed we can always find a value of n satisfying this inequality. To make the value definite take n_0 to be the first positive integer greater than $(1 - \epsilon)/\epsilon$. Then $u_{n_0} > 1 - \epsilon$ and so 1 is the upper bound. We might, for example, take $\epsilon = 10^{-6}$ which would require $n > 10^6 - 1$. In this case we can take $n_0 = 10^6$. Since the sequence is strictly monotonic increasing we have

$$1 - \epsilon < u_{n_0} < u_{n_0+1} < u_{n_0+2} < \quad\quad 1.$$

and so there are infinite numbers of members greater than $1 - \epsilon$. There is no finite value of n which makes $u_n = 1$, *i.e.* no member of the sequence is equal to the upper bound. In such a case we say that the bound is not *attained.*

For the sequence $\{u_n\}$ where $u_n = 1/n$ it can be shown in a similar way that the upper bound is 1, and the lower bound 0, and the latter is not attained. Similarly, if $u_n = \frac{1}{3}(1 - 10^{-n})$, it is easily seen that the lower bound is $\frac{3}{10}$ and the upper bound $\frac{1}{3}$, the latter not being attained.

As an example of a sequence in which both bounds are attained consider $\{u_n\}$ where $u_n = (-1)^n$. The sequence is

$$-1, +1, -1, +1, -1, +1, \ldots.$$

Here $| u_n | = 1$; the lower bound is -1, the upper bound is $+1$ and there are an infinite number of members equal to both bounds.

1·23. Limits of a Monotonic Sequence

It will first be observed that a monotonic increasing sequence is always bounded below, for its lower bound is its first term. Also a monotonic decreasing sequence is bounded above by its first term.

We have the following general theorem. *A bounded monotonic sequence always converges to a limit as n tends to infinity.*

Suppose that $u_n \leqslant u_{n+1}$ so that sequence is monotonic increaing. We assume the result that a sequence which is bounded has an upper bound. Let this upper bound be M. Then M has the following properties:

(*a*) $u_n \leqslant M$ for all n.

(*b*) There exists a value n_0 of n such that $u_{n_0} > M - \epsilon$ where ϵ is an arbitrary positive number.

But if $n > n_0$, $u_n \geqslant u_{n_0}$, i.e. $M - \epsilon < u_n \leqslant M$. Thus we have proved that

$$| u_n - M | < \epsilon \text{ for } n > n_0.$$

It follows that M is the limit of u_n as $n \to \infty$.

A similar argument applies when the sequence is monotonic decreasing.

Hence a bounded monotonic increasing sequence has a limit which is equal to its upper bound, while a bounded monotonic decreasing sequence has a limit which is equal to its lower bound.

Example 1.—If $u_n = 1/n$ (monotonic decreasing) $\lim\limits_{n \to \infty} u_n = 0$.

Example 2.—If $u_n = n/(n+1)$, (monotonic increasing) $\lim\limits_{n \to \infty} u_n = 1$.

Example 3.—If $u_n = \frac{1}{3}(1 - 10^{-n})$, (monotonic increasing $\lim\limits_{n \to \infty} u_n = \frac{1}{3}$.

Note.—A monotonic sequence must either tend to a limit or to infinity. Thus if the sequence is increasing and does not tend to a limit it must tend to $+\infty$. If the sequence is decreasing it will either tend to a limit or to $-\infty$.

1·3. General Principle of Convergence for a Bounded Sequence.

Let $\{u_n\}$ denote the sequence whose nth term is u_n. Then *the necessary and sufficient condition that the sequence $\{u_n\}$ converge to a limit (or simply converge) is that corresponding to an arbitrary positive number ϵ there exists a number n_0 such that for all $n \geqslant n_0$*

$$| u_{n+p} - u_n | < \epsilon$$

for all positive integral values of p.

(*a*) *The condition is necessary.* For suppose u_n converges to the limit l. Then corresponding to $\frac{1}{2}\epsilon$ we can find a number n_0 such that

$$| u_n - l | < \tfrac{1}{2}\epsilon, \text{ for all } n \geqslant n_0.$$

Since $n + p > n_0$,

$$| u_{n+p} - l | < \tfrac{1}{2}\epsilon, \text{ for all positive integral } p;$$

$$\therefore\ | u_{n+p} - u_n | = | u_{n+p} - l + l - u_n |$$
$$\leqslant | u_{n+p} - l | + | l - u_n | < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon$$

for $n \geqslant n_0$ and for all positive integral values of p.

Here we have made use of the property that if A and B are real numbers, $| A + B | \leqslant | A | + | B |$. This is easily seen to be true.

(*b*) *The condition is sufficient.* For suppose $| u_{n+p} - u_n | < \epsilon$ under the stated conditions. This inequality implies that

$$u_n - \epsilon < u_{n+p} < u_n + \epsilon, \text{ for all positive integral values of } p.$$

Thus the sequence $\{u_{n+p}\}$ is bounded as p tends to infinity.

Let m and M be the lower and upper bounds respectively. Then

$$m \geqslant u_n - \epsilon,\ M \leqslant u_n + \epsilon;$$

$$\therefore\ M - m \leqslant (u_n + \epsilon) - (u_n - \epsilon) = 2\epsilon.$$

Since ϵ is arbitrary this inequality can only be true if $M - m = 0$. Thus $M - \epsilon \leqslant u_n \leqslant M + \epsilon$.

It follows that $M - 2\epsilon < u_{n+p} < M + 2\epsilon$; that is

$$| u_{n+p} - M | < 2\epsilon.$$

Hence $\lim\limits_{p \to \infty} u_{n+p} = M$ and thus the sequence converges.

Example.—*Prove that if* λ *and* a_1 *are positive, the sequence* $\{a_n\}$ *defined by* $a_n = \sqrt{(\lambda + a_{n-1})}$ *is monotonic and converges to the positive root of* $x^2 - x - \lambda = 0$.

The roots of the equation $x^2 - x - \lambda = 0$ are $\frac{1}{2}\{1 \pm \sqrt{(1 + 4\lambda)}\}$. The positive root α is $\frac{1}{2}\{1 + \sqrt{(1 + 4\lambda)}\}$.

Suppose first that $a_1 > \frac{1}{2}\{1 + \sqrt{(1 + 4\lambda)}\}$. Then

$$a_2 - a_1 = \sqrt{(\lambda + a_1)} - a_1.$$

$$\text{Now } a_2 - a_1 < 0, \text{ provided } \sqrt{(\lambda + a_1)} - a_1 < 0,$$

$$\text{i.e. } \sqrt{(\lambda + a_1)} < a_1, \text{ i.e. } \lambda + a_1 < a_1^2,$$

$$\text{i.e. } a_1^2 - a_1 - \lambda > 0.$$

Since a_1 is greater than the positive root of the equation $x^2 - x - \lambda = 0$ this condition is satisfied, i.e. $a_2 < a_1$. Thus *if the sequence is monotonic it is monotonic decreasing.*

Next we show that $a_2 > \alpha$. The condition for this is that

$$a_2 - \alpha > 0, \text{ i.e. } \sqrt{(\lambda + a_1)} - \alpha > 0.$$

This reduces to $\lambda + a_1 > \alpha^2 = \alpha + \lambda$

since $\alpha^2 - \alpha - \lambda = 0$, i.e. $a_1 > \alpha$ which is known to be true. Thus we have shown that $\alpha < a_2 < a_1$.

To prove the result in general use the method of induction. Assume the result is true for $n = p$, *i.e.*

$$\alpha < a_p < a_{p-1};$$

then it is sufficient to show that $\alpha < a_{p+1} < a_p$.

$$\text{Now } a_{p+1} = \sqrt{(\lambda + a_p)};$$

$$\therefore a_p - a_{p+1} = a_p - \sqrt{(\lambda + a_p)} > 0,$$

provided $a_p^2 > a_p + \lambda$, i.e. $a_p^2 - a_p - \lambda > 0$. Since $a_p > \alpha$ this condition is satisfied. Thus $a_{p+1} < a_p$.

$$\text{Again } a_{p+1} - \alpha = \sqrt{(\lambda + a_p)} - \alpha > 0,$$

$$\text{provided } \lambda + a_p > \alpha^2 = \alpha + \lambda, \text{ i.e. } a_p > \alpha.$$

Since this is true $a_{p+1} > \alpha$. Thus $\alpha < a_{p+1} < a_p$.

Hence *the sequence* $\{a_n\}$ *is a monotonic decreasing sequence which is bounded below by* α.

It follows that the limit of the sequence exists. Let it be l. Then $l \geqslant \alpha$.

From $a_n = \sqrt{(\lambda + a_{n-1})}$ it follows that in the limit $l = \sqrt{(\lambda + l)}$, i.e. $l^2 - l - \lambda = 0$. Thus l is a root of the equation $x^2 - x - \lambda = 0$.

Since l is positive and the equation has only one positive root α, $l = \alpha$.

If we suppose initially that $a_1 < \alpha$ we would find in a similar way that $\{a_n\}$ is a monotonic increasing sequence which tends to α.

1·31. Relation between Sequences and Infinite Series.

Let $u_1, u_2, u_3, \ldots u_n, \ldots$ denote an infinite sequence. Consider a new sequence defined by the equations,

$$S_1 = u_1, \qquad S_2 = u_1 + u_2, \qquad S_3 = u_1 + u_2 + u_3,$$
$$\cdots\cdots\cdots$$
$$S_n = u_1 + u_2 + u_3 + \ldots + u_n,$$
$$\cdots\cdots\cdots$$

Clearly, the new sequence also has an infinite number of terms. If the sequence $S_1, S_2, \ldots S_n \ldots$ tends to a limit as $n \to \infty$ then the *series*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is *convergent.*

Thus the question of the convergence of a series reduces to the question of the limit of a sequence. If we suppose that u_n is positive for all values of n then S_n is monotonic increasing. Thus the series *either converges or tends to* $+\infty$. There are no other possibilities.

On the other hand, if the terms are not all positive the series may converge, diverge to $+\infty$, diverge to $-\infty$, or oscillate.

1·32. General Principle of Convergence for Series

The necessary and sufficient condition that the series

$$u_1 + u_2 + r_3 + \ldots + u_n + \ldots$$

converge is that corresponding to the arbitrary positive number ϵ ***there exists a number*** n_0 ***such that***

$$| S_{n+p} - S_n | < \epsilon,$$

for all $n \geqslant n_0$ ***and for all positive integral values of*** p***, where***

$$S_n = \sum_{r=1}^{n} u_r$$

Since $S_n - S_{n-1} = u_n$ it follows immediately that $\lim_{n \to \infty} u_n = 0$.

This is *a necessary condition for convergence,* but as will be seen later it is *not sufficient.*

The following notation is usually adopted. The sum S_n is called a *partial sum*; the sum

$$u_{n+1} + u_{n+2} + \ldots + u_{n+p} = S_{n+p} - S_n = {}_pR_n$$

is called a *partial remainder*; the sum

$$R_n = u_{n+1} + u_{n+2} + \ldots + u_{n+p} + \ldots \textit{ ad inf.}$$

is called *the remainder of the series after n terms.*

If S denote the sum of the series then $S = S_n + R_n$, and the conditions for convergence may be stated in the forms

$$| R_n | < \epsilon, \text{ for all } n \geqslant n_o,$$

$$_pR_n \quad < \epsilon, \text{ for all } n \geqslant n_o, \text{ and for all positive}$$

integral values of p.

1·33. Absolute and Conditional Convergence of Series

From the remarks in § 1·31 it is clear that the discussion for series whose terms are all positive (or what is substantially the same thing, all terms of the same sign) will be simpler than for series whose terms are positive and negative.

There are certain types of series which behave much in the same way as series whose terms are all of the same sign.

First of all there are series which contain only a *finite* number of negative terms. Then we can divide our series into two parts. The first part will consist of a finite number of negative terms, while the second part will consist of an infinite series all of whose terms are positive. The sum of the first part can be written down immediately as there are only a *finite* number of terms to be added together. Hence this part of the series will not affect convergence or divergence properties, so the problem reduces to that of an infinite series whose terms all have the same sign. Similarly for the case in which there are only a finite number of positive terms.

Secondly, the series contains an infinite number of both positive and negative terms. Convergent series of this type may be divided into two distinct groups:—

(*a*) Those which remain convergent when all the terms are made positive.

(*b*) Those which diverge to $+ \infty$ when all the terms are made positive.

If the condition (*a*) is satisfied the series is said to be **absolutely convergent,** while if (*b*) is satisfied the series is said to be **conditionally convergent.** It is the former class, *i.e.* absolutely convergent series, whose properties are in many respects similar to series whose terms are all of the same sign. Thus the series:

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} \cdots$$

is absolutely convergent; for the series obtained by making all the terms positive, *i.e.*

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

is convergent. The student will recognise the convergence property from the theory of geometrical progressions. Again the series:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots$$

is conditionally convergent; for the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \dots$$

diverges to $+\infty$, as will be proved later.

The reasons for the names *absolutely* convergent and *conditionally* convergent are as follows.

An absolutely convergent series possesses the property that *the sum remains unaltered no matter how we alter the given order of the terms.*

A conditionally convergent series has the property that by *suitably rearranging the given order of the terms we can make its sum equal to any given number, including $+\infty$ and $-\infty$.* These properties are not immediately obvious and will be considered below.

1·34. The sum of an absolutely convergent series remains unaltered no matter how the order of the terms is deranged

Let s denote the sum of the given series, σ' the sum of the series formed by the positive terms of the series in the order in which they occur, $-\sigma''$ the sum of the series formed by taking the negative terms of the given series, in the order in which they occur. Then from § 1·33 it is clear that it is only necessary to consider the case in which both the series σ' and σ'' have an infinite number of terms.

Let σ denote the sum of the series obtained by making all the terms positive.

Let s_ν be the sum of the first ν terms of the given series, and suppose that these first ν terms contain p positive and n negative terms.

If σ'_p denote the sum of these p positive terms, $-\sigma''_n$ the sum of the n negative terms, then

$$s_\nu = \sigma'_p - \sigma''_n.$$

Also it is clear that

$$\sigma'_p < \sigma, \quad \sigma''_n < \sigma.$$

As ν increases p and n will increase and σ'_p, σ''_n cannot decrease. Thus σ'_p, σ''_n are typical members of two monotonic increasing sequences which are bounded above. Hence as $\nu \to \infty$, σ'_p, σ''_n both tend to limits. These limits have been denoted by σ' and σ''.

$$\text{Since} \quad s\nu = \sigma'_p - \sigma''_n, \; s = \lim_{\nu \to \infty} s_\nu = \sigma' - \sigma''.$$

Now any alteration of the order of the terms in the original series will not affect the values of σ' and σ'' for each of these series consists of terms all of which are positive. Thus no matter how we rearrange the order of the terms of the original series the sum is always $\sigma' - \sigma''$, i.e. *is always equal to the sum of the sums of the two infinite series, the first formed with the positive terms in the order in which they occur, the second formed with the negative terms in the order in which they occur.*

1·35. Comparison Test for Absolutely Convergent Series

We now demonstrate the following important **comparison test** for absolute convergence.

Let $u_1, u_2, u_3, \ldots u_n, \ldots$ *denote the terms of an absolutely convergent series,* and $v_1, v_2, v_3, \ldots v_n, \ldots$ *the terms of another series such that*

$$| v_n | \leqslant k | u_n |$$

where k is a positive constant, for all values of n. Then the series Σv_n is absolutely convergent.

To prove this result it is necessary to show that the series of positive terms $| v_1 | + | v_2 | + \ldots + | v_n | + \ldots$ is convergent. Write:

$$\sigma_n = \sum_{r=1}^{n} | v_r |, \quad s_n = \sum_{r=1}^{n} | u_r |,$$

$$\sigma = \sum_{r=1}^{\infty} | v_r |, \quad s = \sum_{r=1}^{\infty} | u_r |.$$

Then since $| v_r | \leqslant k | u_r |$ for all values of r,

$$\sigma_n \leqslant ks_n.$$

Now s_n is a member of a monotonic increasing sequence which is bounded above by s. Hence

$$ks_n \leqslant ks, \qquad i.e. \; \sigma_n \leqslant ks.$$

Now σ_n is itself a member of a monotonic increasing sequence and is bounded above. Hence σ_n tends to a limit as $n \to \infty$.

Thus the series $\sum_{r=1}^{\infty} |v_r|$ is convergent, and the given series $\sum_{r=1}^{\infty} v_r$ is absolutely convergent.

1·36. Sum and Difference of Two Absolutely Convergent Series*

Let $\sum_{r=1}^{\infty} u_r$, $\sum_{r=1}^{\infty} v_r$ *be two absolutely convergent series whose sums are* s *and* σ *respectively. Then the series*

$$\sum_{r=1}^{\infty} (u_r + v_r), \qquad \sum_{r=1}^{\infty} (u_r - v_r)$$

are absolutely convergent and their sums are $s + \sigma$ *and* $s - \sigma$ *respectively.*

Consider the series $\sum_{r=1}^{\infty} (u_r + v_r)$ and write $\rho_n = \sum_{r=1}^{n} (u_r + v_r)$, $s_n = \sum_{r=1}^{n} u_r$, $\sigma_n = \sum_{r=1}^{n} v_r$. Then

$$\rho_n = s_n + \sigma_n. \quad \text{Also } |u_r + v_r| \leqslant |u_r| + |v_r|.$$

This property is obvious since it is assumed that u_r and v_r represent real numbers. Thus

$$\sum_{r=1}^{n} |u_r + v_r| \leqslant \sum_{r=1}^{n} |u_r| + \sum_{r=1}^{n} |v_r|.$$

Now if $s_n' = \sum_{r=1}^{n} |u_r|$, $\sigma_n' = \sum_{r=1}^{n} |v_r|$, s_n' and σ_n' are the nth terms of monotonic increasing sequences which are bounded above by $\sum_{r=1}^{\infty} |u_r|$ and $\sum_{r=1}^{\infty} |v_r|$. Hence $\sum_{r=1}^{n} |u_r + v_r| < k$, where k denotes some constant.

But $\sum_{r=1}^{n} |u_r + v_r|$ is the nth term of a monotonic increasing sequence. Since the sequence is bounded above its limit exists, *i.e.* the series $\sum_{r=1}^{\infty} (u_r + v_r)$ is absolutely convergent.

* Cf. §§ 1·7, 1·71, and the general result that if Σu_r, Σv_r, Σw_r are three series such that $w_r = u_r \pm v_r$ then the convergence of any two of the series implies the convergence of the third and $\Sigma w_r = \Sigma u_r \pm \Sigma v_r$.

Again, since $\rho_n = s_n + \sigma_n$ for all values of n and s_n, σ_n approach the limits s and σ as $n \to \infty$, it follows that $\lim\limits_{n \to \infty} \rho_n = s + \sigma$. Similarly for the series $\sum\limits_{r=1}^{\infty} (u_r - v_r)$.

1·37. Riemann's Theorem on Conditionally Convergent Series

If a series converges conditionally its sum can be made to have any arbitrary value by a suitable rearrangement of the order of the terms.

Let (s) be a conditionally convergent series, $\sum\limits_{r=1}^{\infty} u_n$. Then we know that there must be an infinite number of both positive and negative terms. The series formed by the positive terms in the order in which they occur and the series formed by the negative terms in the order in which they occur must *both* be divergent. For if both were convergent the original series would be absolutely convergent, while if one were convergent and the other divergent the original series could not be convergent.

Hence we can take sufficient terms from the positive series to make their sum exceed any arbitrary number. In the same way by taking a sufficient number of terms from the negative series we can ensure that the sum of the terms taken is less than any given negative number.

Suppose we wish the series to converge to the positive number λ.

$$\mathrm{A}_2 \quad \mathrm{A}_4 \quad \mathrm{A}_6 \quad \lambda \quad \mathrm{A}_5 \quad \mathrm{A}_3 \quad \mathrm{A}_1$$

To illustrate the argument we represent the numbers by points along a straight line adopting the usual convention for the positive direction, viz. from left to right. First take a set of positive terms from (s) in the order in which they occur and stop *as soon as the sum of the terms taken is greater than* λ. Denote the sum of the terms taken by A_1. Thus $\mathrm{A}_1 > \lambda$.

Now take a set of negative terms from (s) in the order in which they occur but *stop as soon as the combined sum A_2 is less than* λ. Thus $A_2 < \lambda$.

Now return to the positive series beginning where we left off previously and take a set of terms stopping when the total sum of the three sets A_3 first exceeds λ. Thus $A_3 > \lambda$.

Next returning to the negative series and beginning where we left off before, take a set of terms just sufficient to ensure that the new total sum A_4 less than λ. Thus $A_4 < \lambda$.

Continuing in this way we obtain a new series composed of terms of (s) whose sum is sometimes greater than λ and sometimes less than λ.

We have defined a sequence of numbers

$$A_1, A_2, A_3, \ldots A_{2n}, A_{2n+1} + \ldots$$

which represent partial sums of the new series. Also

$$A_{2n} < \lambda, \quad A_{2n+1} > \lambda$$

$$\text{and} \quad \lambda - A_{2n} < u_{n_1}, \quad A_{2n+1} - \lambda < u_{n_2}$$

for some values n_1, n_2. Again as $n \to \infty$, n_1 and n_2 both $\to \infty$.

Since the original series converges

$$| u_n | < \epsilon, \quad \text{all } n \geqslant v,$$

where ϵ is an arbitrary positive number. Hence there exists a number v such that

$$| u_{n_1} | < \epsilon, \ | u_{n_2} | < \epsilon,$$

where n_1 and n_2 are values of $n \geqslant v$

$$\text{Thus} \quad | \lambda - A_n | < \epsilon, \ \text{all } n \geqslant v.$$

Hence $A_n \to \lambda$, *i.e.* λ is the sum of the series (s').

If we wished to obtain a new series which converged to a negative value we would begin with the negative series instead of the positive series. It should be observed that by a suitable rearrangement of the terms of (s) we could obtain series which were divergent or oscillatory.

1·38. The Ratio Test for Absolute Convergence

Let $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ denote a given series. Then ***the series is absolutely convergent if, from and after some fixed term, the absolute value of ratio of each term to the preceding term is less than some quantity which is itself less than unity.***

Since the presence of a finite number of terms does not affect convergence we can omit the finite number of terms at the beginning of the series for which the property is not true and assume that we have a series in which the property is true for all of its terms. Let the given ratio be r where $r < 1$.

Then $$\left|\frac{u_2}{u_1}\right| < r, \text{ i.e. } |u_2| < r|u_1|.$$

$$\left|\frac{u_3}{u_2}\right| < r, \text{ i.e. } |u_3| < r|u_2| < r^2|u_1|.$$

$$\left|\frac{u_4}{u_3}\right| < r, \text{ i.e. } |u_4| < r|u_3| < r^3|u_1|$$

. . . .

In general $$\left|\frac{u_{n+1}}{u_n}\right| < r, \text{ i.e. } |u_{n+1}| < r|u_n| < r^n|u_1|.$$

Thus $$\sum_{n=1}^{\infty} |u_n| < |u_1| \sum_{n=1}^{\infty} r^{n-1} = |u_1|/(1-r),$$

since $0 < r < 1$, and the series $\sum_{n=1}^{\infty} r^n$ is a geometrical progression. Hence the series $\sum_{n=1}^{\infty} |u_n|$ is convergent, *i.e.* $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

1·381. Alternative Form for Ratio Test

The result of § 1·38 may be stated in an equivalent form which is more convenient to apply in practice. ***If the limit as n tends to infinity of the absolute value of the ratio of the (n + 1)th term to the nth term is less than unity the series is absolutely convergent.*** In symbols the condition is

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1.$$

Write $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = r$ where $0 < r < 1$. Choose any number ϵ such that $\epsilon < 1 - r$, i.e. $r + \epsilon < 1$. Thus, *e.g.*, we can take $\epsilon = \frac{1}{2}(1 - r)$ so that $r + \epsilon = \frac{1}{2} + \frac{1}{2}r = \lambda$ say, where $0 < \lambda < 1$. Then we can find a number n_0 such that

$$\frac{u_{n+1}}{u_n}\Big| - r < \epsilon \text{ for all } n \geqslant n_0$$

This inequality may be written in the form:

$$\epsilon < \left|\frac{u_{n+1}}{}\right| - r < \epsilon; \qquad < r + \epsilon = \lambda.$$

Then since $0 < \lambda < 1$ it follows from the argument of § 1·38 that the series is absolutely convergent.

Examples.—(1) *Discuss the convergence of the following infinite series:*

$$\frac{1}{4} + \frac{1.3}{4.7} + \frac{1.3.5}{4.7.10} + \frac{1.3.5.7}{4.7.10.13} + \ldots \qquad [\textit{Madras, B.Sc.}]$$

In this case u_n is always positive so that the modulus signs may be omitted.

$$u_n = \frac{1.3.5.7.\ldots.(2n-1)}{4.7.10.\ldots.(3n+1)}, \quad u_{n+1} = \frac{1.3.5.7.\ldots.(2n+1)}{4.7.10.\ldots.(3n+4)}$$

$$\text{Hence } \frac{u_{n+1}}{u_n} = \frac{2n+1}{3n+4} = \frac{2+\frac{1}{n}}{3+\frac{4}{n}}.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{2}{3} < 1. \quad \text{Thus series converges absolutely.}$$

(2) *Prove that if* $|r| < 1$ *the series*

$a + (a+d)r + \ldots + (a+nd)r^n + \ldots$ *is convergent.*

$$u_{n+1} = (a+nd)r^n, \; u_n = \{a + (n-1)d\}r^{n-1};$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{a+nd}{a+(n-1)d}.r = \frac{\frac{a}{n}+d}{\frac{a}{n}+\left(1-\frac{1}{n}\right)d}.r.$$

As $n \to \infty$, $a/n \to 0$. $1/n \to 0$ Thus

$$\lim. \qquad = \lim. \frac{+d}{+\left(1-\frac{1}{n}\right)} \quad r| = |r|.$$

Hence the series converges if $|r| < 1$.

(3) *Prove that the infinite series* $\sum\limits_{n=1} n^r x^n$ *is convergent if* r *is a positive integer and* $|x| < 1$.

$$u_{n+1} = (n+1)^r x^{n+1}, \; u_n = n^r x^n.$$

$$\text{Thus } \left|\frac{u_{n+1}}{u_n}\right. \qquad \left(\frac{n+1}{}\right)^r$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^r |x|\right\}$$

$$= |x|.$$

Thus the series converges if $|x| < 1$.

1·382. Limiting Ratio Greater than Unity

It is easily seen that if $\lim_{n \to \infty} \qquad > 1$ the series cannot converge.

Write $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = k$ and take $\epsilon = \frac{1}{2}(k - 1)$. Then corresponding to ϵ we can find a number n_0 such that for all $n \geqslant n_0$

$$\left| \frac{u_{n+1}}{u_n} \right| - k < \epsilon.$$

Thus for such values of n, $\frac{u_{n+1}}{u_n} > k - \epsilon = \frac{1}{2}(k + 1) > 1.$

$$\text{Thus } | u_{n+1} > \qquad (k - \epsilon) > | u_n |.$$

But a necessary condition for convergence is that $\lim_{n \to \infty} u_n = 0$. This contradicts the previous statement and hence the series cannot converge.

Example.—*Discuss the convergence of the series*

$$\sum_{=0}^{\infty} \frac{\alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 1)}{\beta(\beta + 1)(\beta + 2) \ldots (\beta + n - 1)} x^n$$

First we observe that β must not be a negative integer. For if β is a negative integer, it will be seen that all except a finite number of terms will be infinite as there will be a zero factor in the denominator. Also if α is a negative integer there will only be a finite number of terms in the series, for beyond a certain point there will always be a zero factor in the numerator. We assume then that α and β are not negative integers.

$$u_{n+1} = \frac{\alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 1)}{\beta(\beta + 1)(\beta + 2) \ldots (\beta + n - 1)} x^n.$$

$$\frac{\alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 2)}{\beta(\beta + 1)(\beta + 2) \ldots (\beta + n - 2)} x^{n-1}.$$

$$\frac{u_{n+1}}{u_n} = \frac{\alpha +}{\beta + n - 1} x = \frac{}{1 + \beta/(n - 1)} x.$$

$$\text{Hence } \lim_{n \to \infty} \left| \frac{u_{n+}}{u_n} \right.$$

Thus if $|x| < 1$ the series converges absolutely; if $|x| > 1$ the series will not converge. For the present we must omit discussion of the behaviour of the series at $|x| = 1$, *i.e.* when $x = \pm 1$.

1·4. Comparison Series

It has already been proved in § 1·35 that if $\sum_{r=1}^{\infty} u_r$ is absolutely convergent, so also is $\sum_{r=1}^{\infty} v_r$ where $|v_r| \leqslant k|u_r|$ and k is a constant, *i.e.* is independent of r. The ratio test of § 1·38 is essentially a particular case of this, $\sum_{r=1}^{\infty} u_r$ being a geometric progression whose common ratio is less than unity.

A known divergent series may be used to prove divergence of another series. Thus if $\sum_{r=1}^{\infty} u_r$ is a divergent series of positive terms and $\sum_{r=1}^{\infty} v_r$ possesses the property that $v_r \geqslant ku_r$, where k is positive and independent of r, then the series $\sum_{r=1}^{\infty} v_r$ also diverges.

For if s_n denote $\sum_{r=1}^{n} u_r$ and σ_n denote $\sum_{r=1}^{n} v_r$, it is clear that $\{s_n\}$, $\{\sigma_n\}$ are both monotonic increasing sequences. Also $\sigma_n \geqslant ks_n$.

Now as $n \to \infty$, $s_n \to \infty$ and hence the monotonic increasing sequence is not bounded above. Thus

$$\sigma_n \to \infty \text{ as } n \to \infty \text{ and } \Sigma v_r \text{ diverges.}$$

1·41. Discussion of the series $\sum_{n=1}^{\infty} n^{-p}$

This forms an important comparison series. *If* $p > 1$ *the series is absolutely convergent; if* $p \leqslant 1$ *the series diverges to* $+\infty$.

(*a*) Suppose $p = 1$ and the series becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots.$$

We may group the terms of the series as follows. The first group is to contain the first two terms, the second group 2 terms, the third group 2^2 terms, the fourth group 2^3 terms, . . . the nth group 2^{n-1} terms, and so on. Written in this way the series has the form

$$(1 + \tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \tfrac{1}{10} + \ldots + \tfrac{1}{16}) + \ldots$$

$$\text{The second group} > \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

$$\text{The third group} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{2}$$

$$\text{The } n\text{th group} = \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n}$$

$$> \frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n} = \frac{1}{2},$$

since there are 2^{n-1} members of the group. Thus the sum of n groups is greater than

$$1 + \tfrac{1}{2} + \tfrac{1}{2} + \dots + \tfrac{1}{2} = 1 + \tfrac{1}{2}n.$$

As n tends to infinity, $1 + \frac{1}{2}n \to \infty$. Hence $\sum\limits_{n=1}^{\infty} n^{-1}$ diverges to $+\infty$.

(*b*) $p < 1$. We may prove divergence in this case by comparison with the series $\Sigma\, n^{-1}$. For if $p < 1$, $\dfrac{1}{n^p} > \dfrac{1}{n}$.

(*c*) $p > 1$. In this case the terms are grouped as follows:

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots$$

The first group contains 1 term, the second group 2 terms, the third group 2^2 terms, ... the nth group 2^{n-1} terms, ... Consider the sum of the terms in the nth group.

$$\frac{1}{2^{(n-1)p}} + \frac{1}{(2^{n-1}+1)^p} + \dots + \frac{1}{(2^n-1)^p}.$$

The terms of the group steadily decrease so that the greatest member of the group is the first term. Also there are 2^{n-1} members of the group. Thus the sum of the group is less than

$$2^{n-1} \cdot \frac{1}{2^{(n-1)p}} = \frac{1}{2^{(n-1)(p-1)}} = u_n, \text{ say.}$$

The series Σu_n is a geometric progression whose terms are all positive and whose common ratio is $1/2^{p-1}$. Since $p > 1$, $1/2^{p-1} < 1$ and so the geometric progression converges. It follows that if $p > 1$, the series $\sum\limits_{n=1}^{\infty} 1/n^p$ is absolutely convergent.

Examples.—(1) *Prove that the series* $\sum\limits_{n=0}^{\infty} x^n/(1+n^2)$ *converges if* $|x| < 1$ *and diverges if* $|x| > 1$.

Write $u_{n+1} = x^n/(1+n^2)$. Then if $|x| < 1$, we have $|u_{n+1}| < 1/(1+n^2)$.

Now $\dfrac{1}{1+n^2} < \dfrac{1}{n^2}$ and since $\Sigma\dfrac{1}{n^2}$ converges, it follows that $\Sigma x^n/(1+n^2)$ converges absolutely for $|x| < 1$.

Next consider $|x| > 1$.

$$\frac{u_{n+1}}{u_n} = \frac{1 + (n-1)^2}{1+n^2} x = \frac{\frac{1}{n^2} + \left(1 - \frac{1}{n}\right)^2}{\frac{1}{n^2} + 1}$$

Hence $\lim\limits_{n \to \infty} = |x| > 1$. Thus the series diverges.

It will be observed that the ratio test will prove convergence for $|x| < 1$ but fails in the case $|x| = 1$.

(2) *Discuss the convergence of the series*

$$\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \dots .$$

(*a*) If $p > 1$ compare with the series $\Sigma \frac{1}{n^p}$. The nth term of the given series is $1/(2n-1)^p$. Since $1/(2n-1)^p < 1/n^p$, $n > 1$ it follows that the series converges.

(*b*) Next consider $p = 1$. The series becomes

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n-1} + \dots .$$

Compare with the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots .$

Now $\frac{1}{2n-1} > \frac{1}{2}\cdot\frac{1}{n}$ and since the series $\Sigma \frac{1}{n}$ diverges so also does $\Sigma \frac{1}{2n-1}$.

(*c*) Finally if $p < 1$, $\frac{1}{(2n-1)^p} > \frac{1}{2n-1}$. Hence by comparison with the case $p = 1$, the series diverges.

1·42. A necessary Condition for Absolute Convergence

The following useful result is due to Pringsheim. We consider first series whose terms are all positive.

Let $\{u_n\}$ denote a monotonic decreasing sequence whose terms are all positive. Then a necessary condition for convergence of the series $\sum\limits_{n=1}^{\infty} u_n$ *is that* $\lim\limits_{n \to \infty} nu_n = 0$.

Write $r = [\frac{1}{2}n]$ where $[x]$ denotes the integral part of x.

Then $r = \frac{1}{2}n$ or $\frac{1}{2}n - \frac{1}{2}$ according as n is even or odd. The corresponding values of $n - r$ are $\frac{1}{2}n$ and $\frac{1}{2}n + \frac{1}{2}$. If $s_n = \sum\limits^{n} u_r$, then

$$s_n - s_r = \sum_{s=r+1} u_r \geqslant (n - r)\, u_n \geqslant \tfrac{1}{2} n u_n.$$

If the series converges, then corresponding to the arbitrary positive number ϵ there exists a positive integer n_0 such that

$$| s_{n+p} - s_n | < \epsilon, \; n \geqslant n_0, \text{ all positive integral values of } p.$$

Hence if $\quad n > 2n_0, \; | s_n - s_r | < \epsilon$, or $\frac{1}{2}nu_n < \epsilon$.

Thus
$$\lim_{n \to \infty} nu_n = 0.$$

If the terms of the sequence $\{u_n\}$ are not all positive, but $\{| u_n |\}$ is a monotonic decreasing sequence then we can extend the result to absolute convergence. For if the series converges absolutely then $\Sigma | u_n |$ is convergent and the terms of this series are positive.

Hence $\quad \lim\limits_{n \to \infty} n | u_n | = 0$ and thus $\lim\limits_{n \to \infty} nu_n = 0$.

Notice that, in general, the existence of $\lim\limits_{n \to \infty} | v_n |$ does *not* imply the existence of $\lim\limits_{n \to \infty} v_n$, but it does so in the special case where the limit is zero.

1·5. Series whose Terms are alternately Positive and Negative

Consider the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}u_n$, where $u_n > 0$ and $u_{n+1} \leqslant u_n$ and $\lim\limits_{n \to \infty} u_n = 0$.

Without altering the given order of the terms the series may be written in the form

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \dots .$$
$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \dots .$$

Thus if s_n denote the sum to n terms,

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}) > 0.$$
$$s_{2n+1} = u_1 - (u_2 - u_3) - (u_4 - u_5) \dots - (u_{2n} - u_{2x+1}) < u_1$$
$$s_{2n} = s_{2n+1} - u_{2n+1} < u_1.$$

Thus $\{s_{2n}\}$ is a monotonic increasing sequence which is bounded above, and hence tends to a limit as n tends to infinity.

Again $\; s_{2n+1} = s_{2n} + u_{2n+1} > u_{2n+1} > 0$. Hence $\{s_{2n+1}\}$ is a

monotonic decreasing sequence which is bounded below. Hence it tends to a limit as n tends to infinity.

Since $s_{2n+1} - s_{2n} = u_{2n+1}$ and $\lim\limits_{n \to \infty} u_{2n+1} = 0$, it follows that the limit is the same in each case. Thus the given series converges.*

Example.—*Discuss the convergence of the series* $\Sigma(-1)^{n-1}n^{-p}$.

If $p > 1$, $|(-1)^{n-1} n^{-p}| = n^{-p}$ and since Σn^{-p} converges, the given series converges absolutely.

(*b*) If $p = 1$ the series becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

The terms are alternately positive and negative, and the absolute magnitude of the terms tends steadily to zero. Thus the series converges.

It is conditionally convergent; for if all the terms be made positive we obtain the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

which is known to be divergent.

(*c*) Suppose $0 < p < 1$. Write the series in the form $\Sigma(-1)^{n-1}u_n$, where $u_n = 1/n^p$. Now

$$\frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p} = 1\Big/\left(1 + \frac{1}{n}\right)^p,$$

and since $p > 0$, $\left(1 + \frac{1}{n}\right)^p > 1$. Thus $u_{n+1} < u_n$.

$$\text{Also} \quad \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n^p} = 0, \text{ since } p > 0.$$

Hence the series converges.

It will be observed that in this case also the convergence is conditional and not absolute for Σn^{-p} diverges for $0 < p < 1$.

(*d*) $p = 0$. In this case the series becomes $1 - 1 + 1 - 1 + 1 - \ldots.$ which oscillates between 0 and 1.

$p < 0$. In this case $1/n^p \to \infty$ as $n \to \infty$, so that the series oscillates *infinitely*.

Collecting results we have:—

(i) If $p > 1$, the series converges absolutely.

(ii) If $0 < p \leqslant 1$, the series converges conditionally.

(iii) If $p = 0$, the series oscillates finitely.

(iv) If $p < 0$, the series oscillates infinitely.

* The test for convergence for series whose terms are alternately positive and negative is stated in another form in § 7·97.

1·51. Comparison Test for two Series whose Terms are all Positive.

Let u_n, v_n be the nth terms of two series whose terms are all positive. Then if Σu_n converges and $\frac{v_{n+1}}{v_n} < \frac{u_{n+1}}{u_n}$ for all n greater than a fixed value then Σv_n converges. Also if Σu_n diverges and $\frac{v_{n+1}}{v_n} > \frac{u_{n+1}}{u_n}$ then Σv_n diverges.

(*a*) Σu_n converges and $\frac{v_{n+1}}{v_n} < \frac{u_{n+1}}{u_n}$.

Since the omission of a finite number of terms from a series does not affect convergence we can assume that the inequality holds for all positive integral values of n. Thus

$$\frac{v_2}{v_1} < \frac{u_2}{u_1}, \quad \frac{v_3}{v_2} < \frac{u_3}{u_2}, \ldots .$$

$$\begin{aligned}
v_1 + v_2 + v_3 + v_4 + \ldots & \\
&= v_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_1} + \frac{v_4}{v_1} + \ldots\right) \\
&= v_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2}\cdot\frac{v_2}{v_1} + \frac{v_4}{v_3}\cdot\frac{v_3}{v_2}\cdot\frac{v_2}{v_1} + \ldots\right) \\
&< v_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right) \\
&= \frac{v_1}{u_1}(u_1 + u_2 + u_3 + u_4 + \ldots)
\end{aligned}$$

Since v_1/u_1 is a constant it follows that Σv_n converges.

(*b*) Σu_n diverges and $\frac{v_{n+1}}{v_n} > \frac{u_{n+1}}{u_n}$.

$$\text{Thus} \quad \frac{v_2}{v_1} > \frac{u_2}{u_1}, \quad \frac{v_3}{v_2} > \frac{u_3}{u_2}, \ldots .$$

$$\begin{aligned}
v_1 + v_2 + v_3 + v_4 + \ldots & \\
&= v_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_1} + \frac{v_4}{v_1} + \ldots\right) \\
&= v_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2}\cdot\frac{v_2}{v_1} + \frac{v_4}{v_3}\cdot\frac{v_3}{v_2}\cdot\frac{v_2}{v_1} + \ldots\right) \\
&> v_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right) \\
&= \frac{v_1}{u_1}(u_1 + u_2 + u_3 + u_4 + \ldots)
\end{aligned}$$

Since v_1/u_1 is constant and Σu_n diverges, it follows that Σv_n diverges.

1·52. The Binomial Series.

Consider the series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

where n denotes any real number. This may be written in the form

$$\sum_{r=0}^{\infty} \binom{n}{r} x^r$$

where $\binom{n}{0}$ is interpreted as meaning unity. If u_r denote the rth term of the series,

$$u_r = \binom{n}{r-1} x^{r-1}, \quad u_{r+1} = \binom{n}{r} x^r.$$

$$\text{Thus} \quad \frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}\,x$$

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty} \left|\left(\frac{n+1}{r} - 1\right)x\right| = |x|.$$

Hence the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

Next suppose that $|x| = 1$. If $x = +1$, $\dfrac{u_{r+1}}{u_r} = \dfrac{n+1}{r} - 1$, which tends to -1 as r tends to ∞. Thus beyond a certain point the terms of the series are alternately positive and negative.

Again if $n+1$ is positive and $r > n+1$,

$$\left|\frac{u_{r+1}}{u_r}\right| = 1 - \frac{n+1}{r} < 1.$$

Thus in this case $|u_{r+1}| < |u_r|$, so that the absolute magnitudes of the terms steadily decrease.

To prove convergence it is now sufficient to show that

$$\lim_{r\to\infty} |u_r| = 0.$$

Write $k = n + 1$. Considering only the *numerical values of the terms*:

$$\frac{u_r}{u_{r+1}} = \frac{1}{1 - \frac{k}{r}}, \text{ where } k > 0,\ r > k;$$

$$\therefore \frac{u_r}{u_{r+1}} = 1 + \frac{k}{r} + \frac{k^2/r^2}{1 - k/r} > 1 + \frac{k}{r}.$$

$$\frac{u_{r+1}}{u_{r+2}} > 1 + \frac{k}{r+1},$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\frac{u_{r+p-1}}{u_{r+p}} > 1 + \frac{k}{r+p-1}.$$

Multiplying corresponding sides of the inequalities,

$$\frac{u_r}{u_{r+p}} > \left(1 + \frac{k}{r}\right)\left(1 + \frac{k}{r+1}\right) \cdots \cdots \left(1 + \frac{k}{r+p-1}\right)$$

$$> 1 + k\left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2} + \cdots\cdots + \frac{1}{r+p-1}\right).$$

Since the series $\sum\limits_{r=1}^{\infty} \frac{1}{r}$ diverges to $+\infty$, it follows that the sum $\frac{1}{r} + \frac{1}{r+1} + \cdots\cdots + \frac{1}{r+p-1}$ tends to ∞ as $p \to \infty$. Thus

$$\frac{u_r}{u_{r+p}} \to \infty \text{ as } p \to \infty.$$

Since u_r is a fixed number different from zero, it follows that this can only be true if $\lim\limits_{p \to \infty} u_{r+p} = 0$. It follows that $\lim\limits_{r \to \infty} u_r = 0$. Hence the binomial series converges for $n + 1 > 0$, when $x = 1$.

When $n + 1 = 0$, the $(r + 1)$th term becomes

$$\frac{-1 \,.\, -2 \,.\, -3 \,.\, \cdots \, -r}{r!} = (-1)^r,$$

and the series $1 - 1 + 1 - 1 + \cdots\cdots$ Thus in this case the series oscillates finitely.

If $n + 1 < 0$, $\left|\frac{u_{r+1}}{u_r}\right| = 1 + \frac{k}{r}$ when $k = -(n+1) > 0$. In this case it is clear that u_r cannot tend to zero as $r \to \infty$, so that the series will not converge.

Finally consider the case in which $x = -1$. In this case $\lim\limits_{r \to \infty} \frac{u_{r+1}}{u_r} = 1$, so that the terms are ultimately of the same sign. In the argument given below we *assume* the binomial expansion for $|x| < 1$. Thus

$$(1 + x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r.$$

The convergence of the expansion on the right has been proved above for $|x| < 1$.

We compare the series under discussion with the series $\Sigma\, 1/r^p$ which converges for $p > 1$, and use the theorem of § 1·51.

$$\frac{u_{r+1}}{u_r} = 1 - \frac{n+1}{r}.$$

Again $\frac{u_{r+1}}{u_r} < \frac{r^p}{(1+r)^p} = \frac{1}{\left(1 + \frac{1}{r}\right)^p} = \left(1 + \frac{1}{r}\right)^{-p}$,

is true provided

$$1 - \frac{n+1}{r} < 1 - \frac{p}{r} + \frac{(-p)(-p-1)}{2!} \cdot \frac{1}{r^2} + \dots.$$

$$i.e.\ n + 1 > p - \frac{p(p+1)}{2!} \frac{1}{r} + \dots.$$

Now the expansion on the right beginning with $\frac{-p(p+1)}{2!} \cdot \frac{1}{r}$ is of the form $\frac{1}{r} \sum_{0}^{\infty} \frac{a_n}{r^n}$ where $\sum_{0}^{\infty} \frac{a_n}{r^n}$ is a convergent series. If its sum is σ we have $n + 1 > p - \sigma/r$. Since $\lim\limits_{r \to \infty} \sigma/r = 0$, the condition becomes in the limit, $n + 1 > p$.

But for the convergence of $\Sigma 1/r^p$, $p > 1$. Thus the condition reduces to $n + 1 > 1$, *i.e.* $n > 0$. Hence the binomial expansion converges for $x = -1$ if $n > 0$. If $n = 0$ there is only one term in the expression.

It should be observed that there is no question of convergence in the binomial expansion if n is a positive integer as in this case there are only a finite number of terms.

1·53. The Exponential Series

The exponential expansion is the series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} x^n/n!.$$

We now prove that the expansion *converges absolutely for all values of* x. Let u_n denote the nth term of the series. Then

$$u_n = x^{n-1}/(n-1)!, \quad u_{n+1} = x^n/n!, \quad \frac{u_{n+1}}{u_n} = \frac{x}{n}.$$

Hence $$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \frac{|x|}{n} = 0.$$

Thus Σu_n converges absolutely for all finite values of x.

1·54. The Logarithmic Series

This is the series

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \ldots = \sum_{n=1}^{\infty} x^n/n.$$

We now prove that *the series converges absolutely if* $|x| < 1$, *converges conditionally when* $x = -1$, *and diverges when* $|x| > 1$ *and when* $x = 1$. If u_n denote the nth term of the series

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}/(n+1)}{x^n/n} = \frac{n}{n+1}x = x\left(1 - \frac{1}{n+1}\right).$$

$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} |x| \left(1 - \frac{1}{n+1}\right) = |x|.$$

Hence the series converges absolutely when $|x < 1$ and diverges when $|x| > 1$.

When $x = 1$, the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ which is known to be divergent. When $x = -1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ which is conditionally convergent.

1·6. Multiplication of two absolutely Convergent Series

Let $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ denote two absolutely convergent series whose sums are U and V respectively, and let U' and V' be the sums of the series $\Sigma |u_n|$, $\Sigma |v_n|$. We can represent the terms of the product in the following scheme.

$$\begin{array}{ccccccccc}
u_1v_1 & & u_1v_2 & & u_1v_3 & & u_1v_4 & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & \\
u_2v_1 & \rightarrow & u_2v_2 & & u_2v_3 & & u_2v_4 & \ldots \\
 & & & & \uparrow & & \uparrow & \\
u_3v_1 & \rightarrow & u_3v_2 & \rightarrow & u_3v_3 & & u_3v_4 & \ldots \\
 & & & & & & \uparrow & \\
u_4v_1 & \rightarrow & u_4v_2 & \rightarrow & u_4v_3 & \rightarrow & u_4v_4 & \ldots \\
\cdot & & \cdot & & \cdot & & \cdot & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot & \cdot
\end{array}$$

Consider the series formed by taking groups of terms in the order indicated by the arrows.

$u_1v_1 + (u_2v_1 + u_2v_2 + u_1v_2) + (u_3v_1 + u_3v_2 + u_3v_3 + u_2v_3 + u_1v_3) + \ldots.$

If $U_n = \sum_{r=1}^{n} u_r$, $V_n = \sum_{r=1}^{n} v_r$, then it is easily seen that the sum of this series to n terms, each group being counted as a single term, is $U_n V_n$.

Now suppose the brackets removed. Then it is clear that the sum of the finite number of terms considered is less than $U'V'$ *for all values of n.* Hence the series

$$u_1v_1 + u_2v_1 + u_2v_2 + u_1v_2 + u_3v_1 + u_3v_2 + \ldots.$$

is absolutely convergent. Thus the order of the terms may be rearranged without affecting the sum of the series. In particular we can express the series in the following way:—

$$u_1v_1 + (u_2v_1 + u_1v_2) + (u_3v_1 + u_2v_2 + u_1v_3) + \ldots.$$

Hence if $w_n = u_nv_1 + u_{n-1}v_2 + \ldots + u_2v_{n-1} + u_1v_n$

$$UV = \sum_{n=1}^{\infty} w_n.$$

APPLICATION TO POWER SERIES.—A series of the form

$$\sum_{n=0}^{\infty} a_nx^n,$$

$$i.e.\ a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots. + a_nx^n + \ldots.$$

where a_n is independent of x is called a POWER SERIES in the variable x.

Let $\sum_{n=0}^{\infty} a_nx^n$, $\sum_{n=0}^{\infty} b_nx^n$ be two power series which converge absolutely to A and B respectively for a certain range of values of x.

Consider the product $AB = (\Sigma a_nx^n)\ (\Sigma b_nx^n)$.

From the theorem on the product of absolutely convergent series it follows that the product $(\Sigma a_nx^n)\ (\Sigma b_nx^n)$ can be arranged in ascending powers of x to give a series which converges absolutely to AB. Thus

$$AB = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \ldots.$$
$$\ldots. + (a_nb_0 + a_{n-1}b_1 + \ldots. + a_1b_{n-1} + a_0b_n)x^n + \ldots$$
$$\therefore AB = \Sigma c_nx^n,$$

where $c_n = a_nb_0 + a_{n-1}b_1 + \ldots. + a_1b_{n-1} + a_0b_n$.

1·7. Expression of a given Series as the Sum or Difference of two Series

Let $\sum_{n=1}^{\infty} w_n$ *be a series which converges to* W, *and suppose that* w_n *can be written in the form* $u_n + v_n$ *where* Σu_n *and* Σv_n *both converge. Let* U *and* V *be their sums respectively. Then*

$$\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n, \text{ i.e. } W = U + V.$$

It should be observed that what we are proving here is that

$$\sum_{n=1}^{\infty} (u_n + v_n) = \sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n \quad \dots\dots\dots\dots \text{(i)}$$

If we were dealing with series which contained a finite number of terms the result would be obvious. But in the given case there are an infinite number of terms, and it is important for the student to realise that an argument which is valid for a *finite number* of quantities *may be fallacious for an infinite number.*

It will be seen that *under certain conditions* (i) *is false.*

To prove the result under the stated conditions write

$$s_n = \sum_{r=1}^{n} w_r, \quad s_n' = \sum_{r=1}^{n} u_r, \quad s_n'' = \sum_{r=1}^{n} v_r.$$

Then for all values of n we have $s_n = s_n' + s_n''$.

Since $s_n \to W$, $s_n' \to U$, $s_n'' \to V$, it follows that we can find numbers n_0, n_1, n_2 depending on ϵ, such that

$$| s_n - W | < \epsilon, \quad \text{all } n \geqslant n_0, \quad \dots\dots\dots\dots \text{(ii)}$$

$$| s_n' - U | < \epsilon, \quad \text{all } n \geqslant n_1, \quad \dots\dots\dots\dots \text{(iii)}$$

$$| s_n'' - V | < \epsilon, \quad \text{for all } n \geqslant n_2 \quad \dots\dots\dots\dots \text{(iv)}$$

Let N be the greatest of the three numbers n_0, n_1, n_2. Then the inequalities (ii), (iii), (iv) are true for all $n \geqslant N$. These inequalities may be written in the forms

$$W - \epsilon < s_n < W + \epsilon$$

$$U - \epsilon < s_n' < U + \epsilon$$

$$V - \epsilon < s_n'' < V + \epsilon.$$

Then since $s_n = s_n' + s_n''$

$$W - \epsilon < s_n = s_n' + s_n'' < U + \epsilon + V + \epsilon,$$

$$W + \epsilon > s_n = s_n' + s_n'' > U - \epsilon + V - \epsilon.$$

These two inequalities may be written in the form

$$-3\epsilon < W - U - V < 3\epsilon.$$

Since ϵ is an arbitrary number this can only be true if

$$W - U - V = 0, \text{ i.e. } W = U + V.$$

Similarly it may be proved that if Σu_n, Σv_n converge to U and V respectively, and $w_n = u_n - v_n$, then

$$\Sigma w_n = \Sigma u_n - \Sigma v_n, \text{ i.e. } W = U - V.$$

Clearly the argument will extend as follows.

Let $w_n = u_n^{(1)} + u_n^{(2)} + \ldots + u_n^{(p)}$. Then

$$\Sigma w_n = \Sigma u_n^{(1)} + \Sigma u_n^{(2)} + \ldots + \Sigma u_n^{(p)}$$

provided each of the series on the right converges.

Note.—If we consider the case in which $w_n = u_n + v_n$ it will be seen that the argument will fail if Σu_n and Σv_n *do not converge.* It is easy to arrange that Σw_n converges while Σu_n, Σv_n diverge.

Thus consider the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n(2n-1)}$.

$$\text{Now } \frac{1}{2n(2n-1)} = \frac{1}{2n-1} - \frac{1}{2n}.$$

Also the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n(2n-1)}$ is *absolutely* convergent. For compare with the series $\Sigma \dfrac{1}{n^2}$. The terms of the given series are positive and

$$\frac{1}{2n(2n-1)} < \frac{1}{n^2}.$$

But the series $\Sigma \dfrac{1}{2n-1}$, $\Sigma \dfrac{1}{2n}$, *i.e.* the series

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots; \quad \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \ldots.$$

are both divergent, as is seen by comparison with the series $\Sigma \dfrac{1}{n}$.

Thus the equation

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) = \sum_{n=1}^{\infty}\frac{1}{2n-1} - \sum_{n=1}^{\infty}\frac{1}{2n}$$

is meaningless since the right hand side is $\infty - \infty$.

This example illustrates the necessity of exercising care in decomposing one infinite series into two or more different series.

1·71. Theorem on Sum and Difference of Two Series

In the argument for the case $w_n = u_n + v_n$ we have the equation $s_n = s_n' + s_n''$. Suppose that all we know is that *two* of the three quantities s_n, s_n', s_n'' tend to limits as $n \to \infty$. *Is it possible under these conditions that the third quantity does not possess a limit?*

To make the argument precise suppose that $\lim\limits_{n \to \infty} s_n = W$, $\lim\limits_{n \to \infty} s_n' = u$, and that we know nothing about $\lim\limits_{n \to \infty} s_n''$. Then if ϵ denote an arbitrary positive number,

$$W - \epsilon < s_n < W + \epsilon, \text{ all } n \geqslant n_0$$

$$u - \epsilon < s_n' < u + \epsilon, \text{ all } n \geqslant n_1.$$

If N denote the greater of n_0, n_1, then these inequalities are both true for all $n \geqslant N$. Now

$$s_n'' = s_n - s_n';$$

$$\therefore\ s_n'' > W - \epsilon - (u + \epsilon) = W - u - 2\epsilon$$

$$\text{and } s_n'' < W + \epsilon - (u - \epsilon) = W - u + 2\epsilon;$$

$$\therefore\ -2\epsilon < s_n'' - W + u < 2\epsilon,$$

$$i.e.\ |\, s_n'' - W + u \,| < 2\epsilon \text{ for all } n \geqslant N.$$

From the definition of a limit it follows that $\lim\limits_{n \to \infty} s_n'' = W - u$. Thus the limit of s_n'' exists.

It follows that all the conditions in the statement of the theorem of § 1·7 are not necessary, and it may be stated more precisely as follows: ***Let Σw_n, Σu_n, Σv_n be three series such that $w_n = u_n \pm v_n$. Then if two of the three series are convergent, then the third is also convergent, and***

$$\Sigma w_n = \Sigma u_n \pm \Sigma v_n.$$

This theorem is used frequently in dealing with examples on expansions in series.

1·8. Some Properties of Power Series

We now give a proof of the two following important theorems:

1. ***If $\Sigma a_n x^n$ denote a power series which converges to $f(x)$ for $|\, x \,| < \lambda$ then there is an interval of x inside the range $(-\lambda, \lambda)$ in which $f(x)$ never vanishes except possibly for $x = 0$.***

2. If two power series $\Sigma a_n x^n$, $\Sigma b_n x^n$ both converge for $|x| < \lambda$ and if $\Sigma a_n x^n = \Sigma b_n x^n$ at all points satisfying $|x| < \mu$, then $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, . . . $a_n = b_n$, . . . and the series are identical. This is known as *the theorem of identical equality between power series.*

(I) Let x_0 be a value of x such that $|x_0| < \lambda$ and write $|x_0| = r_0$, $|x| = r < r_0$, and suppose that a_p is the first coefficient in the expansion $\Sigma a_n x^n$ which is not zero.

In the argument we make use of four properties of a modulus. Thus if a and b be any real numbers

$$|a+b| \leqslant |a| + |b|, \qquad |a - b| \geqslant |a| - |b|,$$

$$|ab| = |a|\,|b|, \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

The first result has been used before, and the student will easily see that the other results are true.

To prove that $f(x)$ does not vanish in a certain range it is sufficient to show that $|f(x)| > 0$ at all points of the range. The method of doing this is to arrange so that the first term which does not vanish, viz. $a_p x^p$, shall dominate over the remaining terms of the series.

Thus it is first necessary to approximate to all the terms after $a_p x^p$. Write $f(x) = a_p x^p + g(x)$, where

$$g(x) = a_{p+1}x^{p+1} + a_{p+2}x^{p+2} + \dots$$

$$|g(x)| = \left| a_{p+1}x_0^{p+1}\left(\frac{x}{x_0}\right)^{p+1} + a_{p+2}x_0^{p+2}\left(\frac{x}{x_0}\right)^{p+2} + \dots \right|$$

$$\leqslant |a_{p+1}x_0^{p+1}|\left|\frac{x}{x_0}\right|^{p+1} + |a_{p+2}x_0^{p+2}|\left|\frac{x}{x_0}\right|^{p+2} + \dots$$

Since all the terms of the series are finite and $\lim\limits_{n \to \infty} |a_n x_0^n| = 0$ for convergence, there exists a finite number A independent of n such that $|a_n x_0^n| < A$ for all values of n. Thus

$$|g(x)| \leqslant A\left(\frac{r}{r_0}\right)^{p+1} + A\left(\frac{r}{r_0}\right)^{p+2} + \dots$$

i.e. $$|g(x)| \leqslant A\left(\frac{r}{r_0}\right)^{p+1}\left\{1 + \frac{r}{r_0} + \frac{r^2}{r_0^2} + \dots\right\}$$

$$= A\left(\frac{r}{r_0}\right)^{p+1}\frac{1}{1 - \dfrac{r}{r_0}},$$

since the series in brackets is an infinite geometric progression whose common ratio lies between 0 and 1. Hence

$$|g(x)| \leqslant Ar^{p+1}/r_0^p(r_0 - r).$$

Since $f(x) = a_p x^p + g(x)$,

$$|f(x)| \geqslant |a_p x^p| - |g(x)| \geqslant |a_p| r^p - \frac{Ar^{p+1}}{r_0^p (r_0 - r)}.$$

Now assume that $x \neq 0$ so that $r > 0$. Then in order to show that $|f(x)| > 0$ it is sufficient to prove that

$$|a_p| r^p - \frac{Ar^{p+1}}{r_0^p (r_0 - r)} > 0.$$

Since $r > 0$, $r_0 > 0$, $r_0 - r > 0$, this is equivalent to showing that $|a_p| (r_0 - r) r_0^p - Ar > 0$, i.e. $r(A + |a_p| r_0^p) < |a_p| r_0^{p+1}$.

Hence if $|x| = r < |a_p| r_0^{p+1}/(A + |a_p| r_0^p)$, we have $|f(x)| > 0$, and the only possible zero is $x = 0$.

(2) Write $f(x) = \Sigma a_n x^n$, $g(x) = \Sigma b_n x^n$. Since both series converge for $|x| < \lambda$ it follows from § 1·71 that

$$f(x) - g(x) = \Sigma (a_n - b_n)x^n, \; |x| < \lambda.$$

Since $f(x) - g(x) = 0$ for $|x| < \mu$ it follows that

$$\Sigma (a_n - b_n)x^n = 0, \quad |x| < \nu,$$

where ν is the smaller of the numbers λ, μ.

Write $c_n = a_n - b_n$. Now the series $\sum_{n=0}^{\infty} c_n x^n$ is never zero in a range $-\nu_1 < x < \nu_1$ where $0 < \nu_1 < \nu$ except possibly at $x = 0$.

If $x = 0$ the series becomes c_0, and since $f(0) = g(0)$ it follows that $c_0 = 0$, i.e. $a_0 = b_0$. Suppose then that $x \neq 0$; then

$$\sum_{n=1}^{\infty} c_n x^n = 0, \; 0 < |x| < \nu_1,$$

which is a contradiction unless all the coefficients c_n are zero. Thus $a_n = b_n$ for all values of n.

Ex. *If* $\dfrac{(1 - 2x)(1 - 3x)}{(x - 2)(x - 3)} = a_0 + a_1 x + a_2 x^2 + \ldots$ *to* ∞, *where* $|x| < 1$ *prove that* $a_0^2 + a_1^2 + a_2^2 + \ldots$ *to* $\infty = 1$. [*Madras, B.A.*]

$$\frac{(1 - 2x)(1 - 3x)}{(x - 2)(x - 3)} = \Sigma a_r x^r.$$

Replace x by $\frac{1}{x}$. Then $\dfrac{\left(1 - \frac{2}{x}\right)\left(1 - \frac{3}{x}\right)}{\left(\frac{1}{x} - 2\right)\left(\frac{1}{x} - 3\right)} = \Sigma a_r x^{-r}$,

$$\text{i.e. } \frac{(x-2)(x-3)}{(1-2x)(1-3x)} = \Sigma a_r x^{-r}; \quad \therefore\ 1 \equiv (\Sigma a_r x^r)(\Sigma a_r x^{-r}).$$

Thus the constant term of the series on the R.H.S. must be unity. It is easily seen that this term is the infinite series Σa_r^2.

1·81. Odd and Even Functions

A function $f(x)$ is an *odd function* of x if $f(-x) = -f(x)$, while $f(x)$ is an *even function* if $f(-x) = f(x)$. In other words, the function is odd if the effect of the substitution of $-x$ for x changes the sign of the function, while the function is even if the substitution leaves the function unaltered. Thus x^3, $x + x^5$ are odd functions, x^2, $x^4 + x^6$ are even functions, while $x + x^2$ is neither odd nor even.*

Suppose that $f(x)$ is an odd function which can be expanded in a power series of the form

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

for some range of values of x. Since $f(-x) = -f(x)$ it follows that

$$-f(x) = a_0 - a_1x + a_2x^2 - \ldots + (-1)^n a_nx^n + \ldots,$$

$$\text{i.e. } f(x) = -a_0 + a_1x - a_2x^2 + \ldots + (-1)^{n-1}a_nx^n + \ldots.$$

From the theorem on identical equality of power series it follows that the coefficients of corresponding powers of x are equal. Hence

$$(-1)^{n-1}a_n = a_n.$$

If n is even, i.e. $n - 1$ is odd, $(-1)^{n-1} = -1$, and

$$-a_n = a_n, \quad \text{i.e. } a_n = 0.$$

Hence the expansion in ascending powers of x *contains only odd powers*. Similarly it may be shown that if $f(x)$ is an even function the expansion has the form

$$a_0 + a_2x^2 + a_4x^4 + \ldots,$$

that is, the expansion *contains only even powers* of x.

1·9. Double and Repeated Series

Suppose we have a sequence of numbers $v_1, v_2, v_3, \ldots v_p, \ldots$ each of which is defined by an infinite series,

$$v_1 = \sum_{q=1}^{\infty} u_{1q},\ v_2 = \sum_{q=1}^{\infty} u_{2q},\ \ldots,\ v_p = \sum_{q=1}^{\infty} u_{pq},\ \ldots.$$

* Observe that we must exclude the use in which $f(x)$ is identically zero. For in accordance with the definitions, $f(x) \equiv 0$ would be both an odd and even function.

If we consider the sum of the series

$$V = \sum_{p=1}^{\infty} v_p = \sum_{p=1}^{\infty} \left(\sum_{q=1}^{\infty} u_{pq} \right)$$

we have what may be called a repeated series. A question which poses itself immediately is "under what conditions can the order of summation be reversed," *i.e.* if we consider the repeated series

$$\sum_{q=1}^{\infty} \left(\sum_{p=1}^{\infty} u_{pq} \right) = \sum_{q=1}^{\infty} w_q,$$

does it converge to some number W, and if it does, is $W = V$?

We are thus led to the consideration of a lattice or rectangular array of numbers as follows:—

$$\begin{array}{l} u_{11} + u_{12} + u_{13} + \ldots + u_{1q} + \ldots \\ + u_{21} + u_{22} + u_{23} + \ldots + u_{2q} + \ldots \\ + u_{31} + u_{32} + u_{33} + \ldots + u_{3q} + \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + u_{p1} + u_{p2} + u_{p3} + \ldots + u_{pq} + \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

The first number in the suffix refers to the row, and the second to the column so that u_{pq} is the qth member of the pth row.

This array of numbers forms a double series and it is first necessary to postulate what is meant by its sum. Let s_{pq} denote the sum of the terms contained in the first p rows and the first of columns. Then, in accordance with the Pringsheim definition, the sum is s where $s = \lim. s_{pq}$, where p, q tend to infinity *independently*. Stated precisely, this means that given ϵ, we can find a number N such that for all $p \geqslant N$, $q \geqslant N$,

$$| s - s_{q} < \epsilon.$$

As in the case of ordinary series the number N will depend on ϵ.

The general condition of convergence now becomes

$$| u_{rs} - u_{pq} | < \epsilon, \text{ for all } r, s \text{ where } r > p \geqslant N, \ s > q \geqslant N.$$

In terms of the elements of the rectangular array, the sum of the elements between the two rectangles defined by r, s and p, q must be numerically less than ϵ. An immediate deduction from the definition is that a *necessary* condition for convergence is that $\lim. u_{pq} = 0$ as p, q tend to infinity independently. It does *not* follow that $\lim\limits_{p \to \infty} u_{pq} = 0$ or $\lim\limits_{q \to \infty} u_{pq} = 0$ as is easily seen by considering $u_{pq} = 1/p^2 + 1/q^2$.

In the early part of this section reference was made to repeated summation. Suppose we first form the sum of a row giving $v_p = \sum_{q=1}^{\infty} u_{pq}$ and then form $\sum_{p=1}^{\infty} v_p$ or $\sum_{p=1}^{\infty} \left(\sum_{q=1}^{\infty} u_{pq}\right) = V$. This is called *summation by rows* of the double series.

In a similar way we can first sum a column giving $w_q = \sum_{p=1}^{\infty} u_{pq}$ and then form $W = \sum_{q=1}^{\infty} w_q = \sum_{q=1}^{\infty} \left(\sum_{p=1}^{\infty} u_{pq}\right)$. This is called summation by columns of the double series. Observe that each of these summations may be represented as a repeated limit.

If the double series converges in the Pringsheim sense it does *not* necessarily follow that the sum by columns or the sum by rows exists. It has, however, been proved that *if the double series converges in the Pringsheim sense to a number s and the sums by rows and by columns are convergent*, then

$$s = \sum_{p=1}^{\infty} \left(\sum_{q=1}^{\infty} u_{pq}\right) = \sum_{q=1}^{\infty} \left(\sum_{p=1}^{\infty} u_{pq}\right)$$

If, however, the double series does not converge it does not follow that the sum by rows is equal to the sum by columns even when both are convergent. The theorem quoted proves, however, that convergence *both* by rows and columns to the *same* sum is a *necessary* condition for convergence of the double series. It is, however, not sufficient.

The following two series illustrate the points made.

(1) The series Σu_{pq} where $u_{p1} = u_{1q} = a$, $u_{p2} = u_{2q} = -a$ for $p, q > 1$, $u_{pq} = 0$ for $p, q > 2$ converges but the sums by rows and by columns are not convergent.

First set out the double series as a rectangular array.

$$\begin{array}{l} a + a + a + a + a + \ldots\ldots \\ + a - a - a - a - a - \ldots\ldots \\ + a - a + 0 + 0 + 0 + \ldots\ldots \\ + a - a + 0 + 0 + 0 + \ldots\ldots \\ + a - a + 0 + 0 + 0 + \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Now $s_{11} = a$, $s_{pq} = 2a$ for $p, q > 1$.

Hence $\lim\limits_{\substack{p \to \infty \\ q \to \infty}} s_{pq} = 2a$ and the double series converges to $2a$.

Write $v_p = \sum\limits_{q=1}^{\infty} u_{pq}$, $w_q = \sum\limits_{p=1}^{\infty} u_{pq}$. Then

$$v_1 = +\infty,\ v_2 = -\infty,\ v_p = 0 \text{ for } p > 2;$$
$$w_1 = +\infty,\ w_2 = -\infty,\ w_q = 0 \text{ for } q > 2.$$

Hence the sum by rows and the sum by columns are both divergent, although the double series converges.

(2) Consider the double series Σu_{pq} defined as follows:—
$u_{pq} = 2a$, $p = q$; $u_{pq} = -a$, $q = p \pm 2$; $u_{pq} = 0$ for other values of p, q. Then the series converges by rows and by columns and $\sum\limits_{p=1}^{\infty} \left(\sum\limits_{q=1}^{\infty} u_{pq} \right) = \sum\limits_{q=1}^{\infty} \left(\sum\limits_{p=1}^{\infty} u_{pq} \right)$ but the double series does not converge.

Setting out the series as a rectangular array we see that the elements of the principal diagonal, i.e. a_{11}, a_{22}, a_{33}, ... are each equal to $2a$, while the terms in parallel lines distant two from the principal diagonal are each equal to $-a$. All other terms are zero.

$$\begin{array}{cccccccc}
2a & +0 & -a & +0 & +0 & +\cdot & \cdot & \cdot \\
+0 & +2a & +0 & -a & +0 & +\cdot & \cdot & \cdot \\
-a & +0 & +2a & +0 & -a & +\cdot & \cdot & \cdot \\
+0 & -a & +0 & +2a & +0 & +\cdot & \cdot & \cdot \\
+0 & +0 & -a & +0 & +2a & +\cdot & \cdot & \cdot \\
+\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Writing $v_p = \sum_{q=1}^{\infty} u_{pq}$, $w_q = \sum_{p=1}^{\infty} u_{pq}$, $V = \sum_{p=1}^{\infty} v_p$,

$W = \sum_{q=1}^{\infty} w_q$, we have

$$v_1 = a,\ v_2 = a,\ v_p = 0 \text{ for } p > 2;$$
$$w_1 = a,\ w_2 = a,\ w_q = 0 \text{ for } q > 2.$$

Hence $V = 2a$, $W = 2a$, and hence $V = W$. A necessary condition for convergence of the double series is lim. $u_{pq} = 0$ as $p \to \infty$, $q \to \infty$ independently. Since $u_{pp} = 2a$, lim. $u_{pq} \neq 0$ and hence the double series does not converge, although the sum by rows and the sum by columns are equal.

1·91. Rearrangement of Terms of a Double Series

Two types of rearrangement have already been considered—summation by rows and summation by columns. Two other convenient arrangements, summation by *squares* and summation by *diagonals* are illustrated in the diagrams. In (*a*) the rearrangement can be represented by

$$u_{11} + (u_{21} + u_{22} + u_{12}) + (u_{31} + u_{32} + u_{33} + u_{23} + u_{13}) + \ldots = \sum_{n=1}^{\infty} \lambda_n$$

where $\lambda_n = u_{n1} + u_{n2} + \ldots + u_{nn} + u_{n-1,\,n} + \ldots + u_{2n} + u_{1n}$.

$$\begin{array}{|cccccccc}
\hline
u_{11} & +u_{12} & +u_{13} & +u_{14} & +u_{15} & +\cdot & \cdot & \cdot \\
+u_{21} & +u_{22} & +u_{23} & +u_{24} & +u_{25} & +\cdot & \cdot & \cdot \\
+u_{31} & +u_{32} & +u_{33} & +u_{34} & +u_{35} & +\cdot & \cdot & \cdot \\
+u_{41} & +u_{42} & +u_{43} & +u_{44} & +u_{45} & +\cdot & \cdot & \cdot \\
+u_{51} & +u_{52} & +u_{53} & +u_{54} & +u_{55} & +\cdot & \cdot & \cdot \\
+\cdot & +\cdot & +\cdot & +\cdot & +\cdot & +\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

(*a*) Summation by squares.

$$\begin{array}{llllllll}
u_{11} & +u_{12} & +u_{13} & +u_{14} & +u_{15} & +\cdot & \cdot & \cdot \\
+u_{21} & +u_{22} & +u_{23} & +u_{24} & +u_{25} & +\cdot & \cdot & \cdot \\
+u_{31} & +u_{32} & +u_{33} & +u_{34} & +u_{35} & +\cdot & \cdot & \cdot \\
+u_{41} & +u_{42} & +u_{43} & +u_{44} & +u_{45} & +\cdot & \cdot & \cdot \\
+u_{51} & +u_{52} & +u_{53} & +u_{54} & +u_{55} & +\cdot & \cdot & \cdot \\
+\cdot & +\cdot & +\cdot & +\cdot & +\cdot & +\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

(*b*) Summation by diagonals.

In (*b*) the rearranged form is

$$u_{11} + (u_{21} + u_{12}) + (u_{31} + u_{22} + u_{13}) + \ldots = \sum_{n=1}^{\infty} \mu_n \text{ where}$$

$$\mu_n = u_{n1} + u_{n-1,\,2} + u_{n-2,\,3} + \ldots + u_{2,\,n-1} + u_{1n}.$$

In both cases the double series has been converted into a single series.

Such rearrangements are of little real value unless their effect on convergence is known. In the case of a single series the terms can be rearranged in any order without affecting convergence or the sum, provided the series is absolutely convergent. A similar result holds for double series, where by definition, the series Σu_{pq} is absolutely convergent provided $\Sigma \mid u_{pq} \mid$ converges.

The following results hold for *absolutely convergent* double series, in particular, for convergent series whose terms are all positive.

I. *The sum of the series is unchanged by any rearrangement of the order of the terms.*

II. *Convergence of the double series implies convergence of all the rows and all the columns and*

$$\sum_{p} \left(\sum_{q} u_{pq}\right) = \sum_{q} \left(\sum_{p} u_{pq}\right) = \sum_{p,\,q} u_{pq}.$$

III. *If Σu_n is an absolutely convergent single series then the double series $\Sigma u_p u_q$ is absolutely convergent.*

It was proved in § 1·41 that the series $\Sigma n^{-\alpha}$ converges absolutely for $\alpha > 1$. Hence the double series $\sum\limits_{p,q} (pq)^{-\alpha}$ converges absolutely for $\alpha > 1$.

IV. *If $U = \Sigma u_n$, $V = \Sigma v_n$ are absolutely convergent single series then the double series $\Sigma u_p v_q$ is absolutely convergent and $\Sigma u_p v_q = UV$.*

Thus since the series $\Sigma n^{-\alpha}$ converges absolutely for $\alpha > 1$, the double series $\Sigma p^{-\alpha} q^{-\beta}$ converges absolutely for $\alpha > 1$, $\beta > 1$.

The multiplication of absolutely convergent single series has already been considered in § 1·6. Here the terms of the product were represented in the form of a rectangular array, forming a double series, the terms being grouped together as in the mode of summation of a double series by squares. In the case of the power series $\sum\limits_{n=0}^{\infty} a_n x^n$, $\sum\limits_{n=0}^{\infty} b_n x^n$ the product was written as $\sum\limits_{n=0}^{\infty} c_n x^n$ where $c_n = a_n b_0 + a_{n-1} b_1 + \ldots\ldots + a_1 b_{n-1} + a_0 b_n$, corresponding to diagonal summation.

A further example of a rearrangement of the terms of a double series is provided by the substitution of one power series in another power series. We first observe that if a power series converges for a non-zero value of the variable it converges absolutely for some finite range of the variable.

Suppose that $z = \sum\limits_{n=0}^{\infty} b_n y^n$ for $|y| < \mu$, and $y = \sum\limits_{n=0}^{\infty} a_n x^n$ for $|x| < \lambda$, λ and μ being positive numbers.

If we substitute the second power series in the first and rearrange the terms in powers of x we obtain a new power series of the form $\sum\limits_{r=0}^{\infty} c_r x^r$. The problem is to determine whether there is a range of values of x for which $z = \sum\limits_{r=0}^{\infty} c_r x^r$.

Consider first the separate terms of the power series in y. Then in accordance with § 1·6 we can apply repeated multiplication and write $b_n y^n = \sum\limits_{r=1}^{\infty} b_{nr} x^r$, provided $|x| < \lambda$. Hence z can be written as the *sum by rows* of the following double series.

$$
\begin{array}{l}
b_0 \\
+ b_{10} + b_{11}x + b_{12}x^2 + b_{13}x^3 + \ldots + b_{1r}x^r + \ldots \\
+ b_{20} + b_{21}x + b_{22}x^2 + b_{23}x^3 + \ldots + b_{2r}x^r + \ldots \\
+ b_{30} + b_{31}x + b_{32}x^2 + b_{33}x^3 + \ldots + b_{3r}x^r + \ldots \\
+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+ b_{n0} + b_{n1}x + b_{n2}x^2 + b_{n3}x^3 + \ldots + b_{nr}x^r + \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

If we write
$$c_0 = b_0 + b_{10} + b_{20} + \ldots,$$
$$c_r = b_{1r} + b_{2r} + b_{3r} + \ldots, \; r \geqslant 1,$$

then the series $\sum\limits_{r=0}^{\infty} c_r x^r$ is the *sum by columns* of the double series.

Hence, in order to be able to state that $z = \sum\limits_{r=0}^{\infty} c_r x^r$ it is necessary to show that the sum by rows is equal to the sum by columns. This will certainly be the case if the double series is absolutely convergent. It can be shown that provided $| a_0 | < \mu$, then there exists a positive number k, depending on λ and μ such that the double series converges absolutely for $|x| < k$. Under these conditions the transformation is justified and we can write

$$z = \sum_{r=0}^{\infty} c_r x^r, \quad | a_0 | < \mu, \quad |x| < k.$$

A case of particular interest arises when the series $\Sigma b_n y^n$ converges for all finite values of y. The transformation can then be made provided $|x| < \lambda$.

EXERCISES I

1. λ and a_1 are two positive quantities and $a_n = \lambda/(1 + a_{n-1})$. Prove that the limit of the sequence $\{a_n\}$ exists and that its value is the positive root of the equation $x^2 + x - \lambda = 0$.

2. Show that the infinite series

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots$$

is absolutely convergent for $|x| < 1$ but divergent for $|x| > 1$ for any value of m not a positive integer or zero.

Show also that

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$

is convergent, but not absolutely convergent. [*N.Sc.*]

3. Examine the behaviour of the sequences $\{a_n\}$, $\{b_n\}$ where

$$a_n = (-1)^n n/(2n-1), \quad b_n = n + (-1)^n 2n.$$

4. Prove that if the sequence $\{a_n\}$ is monotonic increasing, so also is the sequence $\{b_n\}$ where

$$b_n = (a_1 + a_2 + \ldots + a_n)/n.$$

What result can be deduced if $\{a_n\}$ is monotonic decreasing?

5. If $0 < a < 1$ show by considering the sequence $\{\lambda_n\}$ where $\lambda_n = a^{n-1}(1-a)$ that the sequence $\{\mu_n\}$ where $\mu_n = (1 - a^n)/n$, is monotonic decreasing, and deduce the inequality

$$na^{n-1}(1-a) < 1 - a^n < n(1-a).$$

Show that the following series are convergent under that stated con ditions:

6. $1 + \frac{1}{2}x + \frac{2}{3}x^2 + \frac{3}{4}x^3 + \ldots + \dfrac{n-1}{n}x^{n-1} + \ldots, \quad |x| < 1.$

7. $1 + x + \dfrac{2^2}{1.3}x^2 + \ldots + \dfrac{(n-1)^2}{(n-2)n}x^{n-1} + \ldots, \quad |x| < 1.$

8. $\sum\limits_{n=1}^{\infty} (-1)^{n-1}x^{2n-1}/(2n-1)\,!$, all finite values of x.

9. $\sum\limits_{n=0}^{\infty} x^{2n}/(2n)\,!$, all finite values of x.

10. $1 - x + \dfrac{x^2}{1^2.2^2} + \ldots + \dfrac{(-1)^n x^n}{\{(n-1)\,!\,n\,!\}^2} + \ldots$ all finite values of x

11. $1 + \dfrac{\alpha.\beta.x}{1.\gamma} + \dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}.x^2$

$$+ \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)x^n}{n\,!\,\gamma.(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}$$

provided $|x| < 1$ and γ is not a negative integer.

12. Show that the series $\sum\limits_{n=1}^{\infty} 1/(x - na)$ diverges for all finite values of x.

13. Show that if $u_n > 0$ then $\sum\limits_{n=1}^{\infty} u_n$ converges if $\lim\limits_{n\to\infty} \sqrt[n]{u_n} < 1$ and diverges if $\lim\limits_{n\to\infty} \sqrt[n]{u_n} > 1$.

[This result is frequently referred to as Cauchy's test for convergence.]

14. Prove that the sum of a million terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is less than 21.

15. Prove that a conditionally convergent series may be converted into an absolutely convergent series by the insertion of brackets.

16. Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ are absolutely convergent for all values of x, and show that their product is unity.

17. Prove that the series

$$\frac{1}{(x-a)^2} + \frac{1}{(x-2a)^2} + \frac{1}{(x-3a)^2} + \dots \frac{1}{(x-na)^2} + \dots$$

converges for all values of x except the values $x = a, 2a, 3a, 4a \dots ra \dots$ (r any positive integer).

18. If in the series $a_1x + a_2x^2 + \dots a_nx^n + \dots$

$$\operatorname*{Lt}_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{\lambda}$$

prove that the series converges *absolutely* so long as $|x| < \lambda$.

19. State precisely what is meant by saying that a series is convergent, and prove directly from your definition that $\Sigma 3^{-n}$ is convergent, and $\Sigma n^{-\frac{1}{2}}$ is not convergent.

20. Numbers $u_2, u_3, \dots$ are defined in terms of u_1 by

$$u_{r+1} = au_r^2 + 2bu_r + c.$$

Prove that if a certain condition is satisfied by a, b, c, it is possible to determine u_r by using the substitution $u_r = lx_r + m + \frac{n}{x_r}$, where $x_{r+1} = x_r^2$, and l, m, n are constants depending on a, b, c. Hence or otherwise express u_r in terms of u_1 when $a = 1, b = 2, c = 0$. [*Camb. Sch.*]

21. Show that the series

$$1 - \frac{1}{1+x} + \frac{1}{2} - \frac{1}{2+x} + \frac{1}{3} - \frac{1}{3+x} + \dots$$

is convergent, provided that x is not a negative integer. [*Camb. Sch.*]

22. A sequence of numbers $u_1, u_2, u_3, \dots$ is defined by the relations

$$u_1 = \alpha + \beta, \quad u_n = \alpha + \beta - \frac{\alpha\beta}{u_{n-1}}, \quad n > 1.$$

Show that, if $\alpha > \beta > 0$, $u_n = (\alpha^{n+1} - \beta^{n+1})/(\alpha^n - \beta^n)$, and determine the limit to which u_n tends as n tends to infinity.

If $\alpha = \beta > 0$, find a formula for u_n, and determine the limit to which u_n tends as n tends to infinity. [*M.T.*]

23. If $c, a_1, a_2, \dots > 0$, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n + c}$$

are both convergent or both divergent. [*Camb. Sch.*]

24. Show that the series $\sum_{n=1}^{\infty} \frac{n^{n-2}}{n!}$ is divergent.

25. Determine the real values of x for which the following series converge:

(i) $\sum_{1}^{\infty} n^2 (x-1)^n$, (ii) $x + x^4 + x^9 + x^{16} + \dots$,

(iii) $\sin x + \sin 2x + \sin 3x + \dots$ [*M.T.*]

26. Taking $3 \cdot 14159$ as an approximation to π, find by means of continued fractions a sequence of six rational numbers with the property that each member of the sequence is a closer approximation to π than those numbers which precede it.

27. Find the double and repeated limits of the following double sequences when p and q tend to infinity.

(i) $u_{pq} = \frac{1}{p} + \frac{1}{q}$, (ii) $u_{pq} = (-1)^{p+q}\left(\frac{1}{p} + \frac{1}{q}\right)$,

(iii) $u_{pq} = \frac{p}{p+q}$, (iv) $u_{pq} = \frac{pq}{p^2 + q^2}$.

28. Prove that the following double series cannot converge absolutely.

$$
\begin{array}{l}
\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \dots \\
-\frac{1}{2^2} - \frac{3}{2^4} + \frac{3^2}{2^6} - \frac{3^3}{2^8} - \frac{3^4}{2^{10}} - \frac{3^5}{2^{12}} - \frac{3^6}{2^{11}} - \dots \\
+\frac{1}{2^2} + \frac{3}{2^4} + \frac{3^2}{2^6} + \frac{3^3}{2^8} + \frac{3^4}{2^{10}} + \frac{3^5}{2^{12}} + \frac{3^6}{2^{14}} + \dots \\
-\frac{1}{2^3} - \frac{7}{2^6} - \frac{7^2}{2^9} - \frac{7^3}{2^{12}} - \frac{7^4}{2^{15}} - \frac{7^5}{2^{18}} - \frac{7^6}{2^{21}} - \dots \\
+\frac{1}{2^3} + \frac{7}{2^6} + \frac{7^2}{2^9} + \frac{7^3}{2^{12}} + \frac{7^4}{2^{15}} + \frac{7^5}{2^{18}} + \frac{7^6}{2^{21}} + \dots \\
-\frac{1}{2^4} - \frac{15}{2^8} - \frac{15^2}{2^{12}} - \frac{15^3}{2^{16}} - \frac{15^4}{2^{20}} - \frac{15^5}{2^{24}} - \frac{15^6}{2^{28}} - \dots \\
+\frac{1}{2^4} + \frac{15}{2^8} + \frac{15^2}{2^{12}} + \frac{15^3}{2^{16}} + \frac{15^4}{2^{20}} + \frac{15^5}{2^{24}} + \frac{15^6}{2^{28}} + \dots \\
- \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}
$$

Find the sum of the series by columns and prove that the sum by rows does not exist.

29. Prove by direct substitution that if

$$z = \sum_{r=0}^{\infty} \frac{y^r}{r!} \text{ and } y = n \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{r},$$

then
$$z = 1 + \sum_{r=1}^{\infty} \frac{n(n-1)\dots(n-r+1)}{r!} x^r, \; |x| < 1.$$

30. For what values of x are the following series convergent,

(i) $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{(n^3+1)}}$, (ii) $\sum_{n=0}^{\infty} \frac{n!\,x^n}{(2n)!}$?

[*Lond. B.Sc.*]

CHAPTER II

THEOREMS ON LIMITS AND CONTINUOUS FUNCTIONS

IN this chapter we consider some general properties of limits dealing in the first instance with a variable which becomes infinite and then with one which has a finite limiting value. From this we pass on to the consideration of continuous functions.

2·1. Infinite Limits

There are two cases to consider according as the variable tends to $+\infty$ or $-\infty$. The argument in each case is similar.

Let $\phi(n)$ denote a function of the positive integral variable n. Then we know that the following possibilities exist.

(*a*) Corresponding to the arbitrary positive number ϵ there exists a number n_0, depending on ϵ and a number l, such that

$$|\phi(n) - l| < \epsilon \text{ for all } n \geqslant n_0;$$
$$\text{then } \phi(n) \quad l \text{ as } n \to \infty.$$

(*b*) If corresponding to the arbitrary number N, however large, we can find n_0, depending on N, such that

$$\phi(n) > N, \text{ for all } n \geqslant n_0$$
$$\text{then } \phi(n) \to \infty \text{ as } n \to \infty.$$

(*c*) If corresponding to an arbitrary positive number N, which may be taken as large as we please, there exists a number n_0, depending on N, such that

$$\phi(n) < -N, \text{ for all } n \geqslant n_0$$
$$\text{then } \phi(n) \to -\infty \text{ as } n \to \infty.$$

(*d*) If as $n \to \infty$, $\phi(n)$ does not satisfy (*a*), (*b*) or (*c*) then $\phi(n)$ oscillates. The oscillation is finite if we can find a constant A such that $|\phi(n)| < A$ for all values of n. The oscillation is infinite if this condition is not satisfied.

2·11. Notation

It is sometimes convenient to extend the meaning of the symbol $\to$. If $\phi(n)$ and $\psi(n)$ denote two functions of n such that the limit of $\{\phi(n) - \psi(n)\}$ is l as n tends to infinity tnen in the sense already considered we write

$$\phi(n) - \psi(n) \to l \text{ or } \phi(n) \to \psi(n) + l,$$

the meaning being that the difference $\phi(n) - \psi(n)$ tends to a limit. $\phi(n)$ and $\psi(n)$ themselves need not tend to limits.

The $\sim$ Notation.—The symbol $\sim$ to denote the numerical difference between two quantities has now fallen into more or less complete disuse, having been replaced by modulus signs. Thus in the old notation $a \sim b$ meant the actual numerical value of the difference $a - b$. The modern way of expressing this is $| a - b |$.

In more recent years the symbol $\sim$ has been used in a different sense, in connection with the theory of limits. Thus if $\phi(n)$ and $\psi(n)$ denote two functions of n then by

$$\phi(n) \sim l\psi(n) \quad or \quad \phi \sim l\psi$$

is meant the property that $\lim_{n \to \infty} \dfrac{\phi(n)}{\psi(n)} = l$, where $l \neq 0$.

The O and o Notation.—This notation, which has proved of great value in modern theory, was introduced by Landau. It is essentially a "shorthand" notation, and care is necessary in its use until the student is quite confident that he can express any proof in which it is used in ordinary language.

Let $\phi(n)$ and $\psi(n)$ be two functions of the variable n. $\phi(n)$ is to have the following properties:

(i) $\phi(n) > 0$. (ii) $\phi(n)$ is monotonic.

Thus as $n \to \infty$, $\phi(n)$ tends to a limit, which may be zero, or to $+\infty$.

(*a*) *Then if there exists a fixed number n_0 and a positive constant A such that $| \psi(n) | < A\phi(n)$, all $n > n_0$ we write*

$$\psi = O(\phi).$$

(*b*) *And if $\dfrac{\psi}{\phi} \to 0$, as $n \to \infty$ we write*

$$\psi = o(\phi).$$

Examples.—(1) *Suppose that* $\psi(n) = \frac{1}{2}(n^3 + n^2 + n - 2)/(n^2 + 1)$.

Then $\psi(n) = \frac{1}{2}\{n(n^2 + 1) + (n^2 + 1) - 3\}/(n^2 + 1) = \frac{1}{2}n + \frac{1}{2} - \dfrac{1}{n^2 + 1}$.

$$\text{Thus} \lim_{n \to \infty} \{\psi(n) - \tfrac{1}{2}n\} = \tfrac{1}{2}.$$

$$\lim_{n \to \infty} \frac{\psi(n)}{n} = \tfrac{1}{2}, \qquad \lim_{n \to \infty} \frac{\psi(n)}{n^2} = 0.$$

Hence we have the following results:

$\psi(n) \to \frac{1}{2}n + \frac{1}{2}$............ (α) $\qquad$ $\psi(n) \sim \frac{1}{2}n$................ (β)

$\psi(n) = O(n)$.............. (γ) $\qquad$ $\psi(n) = o(n^2)$.............. (δ)

It will be observed that each statement gives less information than the one which precedes it.

(2) *If k be any constant,*

$$\text{then } kn = O(n), \qquad kn = o(n^2).$$

2·12. Properties of O and o Notation

(*a*) The symbols $O(1)$, $o(1)$ are defined as particular cases of the general result. We draw attention to their special meanings because of the frequency with which these particular cases occur Thus if we assert that

$$\psi(n) = O(1)$$

we imply that $|\psi(n)| < A$, where A is constant.

Thus $\psi(n) = O(1)$ merely asserts that $\psi(n)$ is bounded as n tends to ∞. Thus *e.g.*

$$\frac{n^2+1}{n^2-1} = O(1), \qquad (-1)^n\frac{n^3-6}{n^3-n^2} = O(1).$$

In the first example the limit of the function exists, while in the second it does not, for as $n \to \infty$ the function approaches alternately the values $+1$ and -1.

If we assert that $\psi(n) = o(1)$ we imply that $\lim\limits_{n \to \infty} \psi(n) = 0$.

(*b*) So far we have defined $O(\phi)$, $o(\phi)$ only in relation to some other function ψ. But by an obvious extension of the notation we can attach meanings to the isolated quantities $O(\phi)$, $o(\phi)$. By $O(\phi)$ we denote *any function* ψ with the property that $\psi = O(\phi)$ and similarly for $o(\phi)$. Thus *e.g.* the equation

$$O(1) + O(1) = O(1)$$

has a definite meaning. It asserts that if $\psi_1(n)$, $\psi_2(n)$ are two bounded functions then their sum is also bounded.

Again the equation $O(1) = o(n)$ is true because it asserts that a bounded function when divided by n tends to zero as $n \to \infty$.

Next suppose we have n functions $\psi_1, \psi_2, \ldots \psi_n$, each of which is $O(1)$, *i.e.* is bounded. Then

$$|\psi_1| < A_1, \; |\psi_2| < A_2, \; \ldots, \; |\psi_n| < A_n,$$

where $A_1, A_2, \ldots, A_n$ are constant. Thus

$$|\psi_1 + \psi_2 + \ldots + \psi_n| \leqslant |\psi_1| + |\psi_2| + \ldots + |\psi_n|$$
$$\leqslant A_1 + A_2 + \ldots + A_n \leqslant nA_\lambda$$

where A_λ is the greatest of the numbers $A_1, A_2, \ldots, A_n$. This may be expressed concisely in the O-notation by the equation

$$\sum_1^n O(1) = O(n).$$

(*c*) *Sum and difference of two functions.*

Let $\psi_1 = O(\phi)$, $\psi_2 = O(\phi)$. Then $\psi_1 \pm \psi_2 = O(\phi)$.

In particular $\psi_1 + \phi = O(\phi)$.

(*d*) *Product of two functions.*—The following property is obvious from the definition. If ψ_1 and ψ_2 are two functions, then

$$O(\psi_1)\, O(\psi_2) = O(\psi_1 \psi_2).$$

It is clear from the above results that it is possible to develop a series of theorems which deal with general properties of the symbols.

2·13. Application to Theorems on Convergence

Some of the earlier results in Chapter I. on series may be conveniently represented in the O-notation.

(*a*) The condition $\lim\limits_{n \to \infty} u_n = 0$ becomes $u_n = o(1)$. (1·32.)

(*b*) The theorem of 1·35 may be stated as follows: If Σu_n converges absolutely and $v_n = O\,|u_n|$ then Σv_n converges absolutely.

Again if Σu_n and Σv_n denote two series whose terms are positive, and if $v_n = O(u_n)$ then Σv_n converges or diverges according as Σu_n converges or diverges. It will be observed that if $v_n = o(u_n)$ the convergence property would follow but not the divergence property.

(*c*) Using the series of 1·41 we may say that a series Σu_n converges absolutely if $u_n = O(n^{-1-\delta})$, where $\delta > 0$. For the series $\Sigma n^{-p} = \Sigma n^{-1-\delta}$ converges for $\delta > 0$. Also if $u_n > 0$ series diverges if $u_n = O(n^{-1})$.

2·2. Some General Theorems on Limits as $n \to \infty$.

THEOREM I.—SUM AND DIFFERENCE OF TWO FUNCTIONS.—*If $\phi(n) \to a$ and $\psi(n) \to b$ then $\phi(n) \pm \psi(n) \to a \pm b$.*

Let ϵ be an arbitrary positive number. Then there exist numbers n_0, n_1, depending on ϵ such that

$$| \phi(n) - a | < \epsilon, \text{ for all } n \geqslant n_0,$$
$$| \psi(n) - b | < \epsilon, \text{ for all } n \geqslant n_1.$$

Let N be the greater of n_0, n_1. Then

$$\begin{aligned} | \phi(n) + \psi(n) - a - b | &= | \{\phi(n) - a\} + \{\psi(n) - b\} | \\ &\leqslant | \phi(n) - a | + | \psi(n) - b | \\ &< 2\epsilon, \text{ for all } n \geqslant N. \end{aligned}$$

Since ϵ is abitrary, $\lim\limits_{n \to \infty} \{\phi(n) + \psi(n)\} = a + b$. A similar argument shows that $\phi(n) - \psi(n) \to a - b$.

Subsidiary Results.—(*a*) If $\phi(n) \to a$, $\psi(n) \to \pm \infty$ or oscillates, then $\phi(n) + \psi(n)$ behaves in the same way as $\psi(n)$.

Example.—$\phi(n) = \frac{1}{n}$, $\psi(n) = (-1)^n$.

In this case $\frac{1}{n} + (-1)^n$ oscillates.

(*b*) If $\phi(n) \to \infty$, $\psi(n) \to \infty$ then $\phi(n) + \psi(n) \to \infty$.

(*c*) If $\phi(n) \to \infty$, $\psi(n) \to -\infty$, $\phi(n) + \psi(n)$ may behave in any way; it may tend to a limit, may oscillate or may diverge.

Examples.—(1) $\phi(n) = n^2$, $\psi(n) = 1 - n^2$, $\phi(n) + \psi(n) = 1$.

(2) $\phi(n) = n^2$, $\psi(n) = 1 - n$, $\phi(n) + \psi(n) = n^2 + 1 - n \to +\infty$.

(3) $\phi(n) = n$, $\psi(n) = 1 - n^3$,
$\phi(n) + \psi(n) = n + 1 - n^3 \to -\infty$.

(4) $\phi(n) = n$, $\psi(n) = (-1)^n - n$.
$\phi(n) + \psi(n) = (-1)^n$, which oscillates finitely.

(5) $\psi(n) = n^2 + n(-1)^n$, $\phi(n) = 1 - n^2$.
$\phi(n) + \psi(n) = 1 + n(-1)^n$, which oscillates infinitely.

(*d*) If $\phi(n)$ and $\psi(n)$ both oscillate finitely then $\phi(n) + \psi(n)$ either tends to a limit or oscillates finitely.

Examples.—(1) $\phi(n) = (-1)^n$, $\psi(n) = (-1)^{n+1}$.
$\phi(n) + \psi(n) = 0$.

(2) $\phi(n) = (-1)^n$, $\psi(n) = (-1)^n$.
$\phi(n) + \psi(n) = 2(-1)^n$, which oscillates between $+2$ and -2.

Other less important cases may be stipulated in which one or both of $\phi(n)$, $\psi(n)$ oscillate infinitely. It should not be difficult for

the student to discuss in any particular straightforward case the behaviour of the sum.

Example.—$\phi(n) = (-1)^n n$, $\psi(n) = n\{1 + (-1)^{n+1}\}$
$\phi(n) + \psi(n) = n \to \infty$.

The difference $\phi(n) - \psi(n)$ may be discussed in exactly the same way as $\phi(n) + \psi(n)$.

THEOREM II.—PRODUCT OF TWO FUNCTIONS.—*If* $\phi(n) \to a$, $\psi(n) \to b$ *then* $\phi(n)\psi(n) \to ab$.

Write $\phi(n) = a + \lambda(n)$, $\psi(n) = b + \mu(n)$. Then

$$\phi(n)\psi(n) = ab + a\mu(n) + b\lambda(n) + \lambda(n)\mu(n).$$

$$|\phi(n)\psi(n) - ab| \leqslant |a\mu(n)| + |b\lambda(n)| + |\lambda(n)|\,|\mu(n)|.$$

Given ϵ there exist numbers n_0, n_1 such that

$$|\lambda(n)| < \epsilon, \text{ all } n \geqslant n_0,$$
$$|\mu(n)| < \epsilon, \text{ all } n \geqslant n_1.$$

Let N be the greater of n_0, n_1. Then

$$|a\mu(n)| < |a|\,\epsilon, \quad |b\lambda(n)| < |b|\,\epsilon, \quad |\lambda(n)|\,|\mu(n)| < \epsilon^2,$$

for $n \geqslant N$. Hence for all $n \geqslant N$,

$$|\phi(n)\psi(n) - ab| < \epsilon\,\{|a| + |b| + \epsilon\}.$$

Since $|a|$, $|b|$, ϵ are finite positive numbers

$$|\phi(n)\psi(n) - ab| < A\epsilon,$$

where A is a positive constant. Since ϵ is arbitrary so also is $A\epsilon$. Hence

$$\lim_{n \to \infty} \phi(n)\psi(n) = ab.$$

In particular if $\psi(n) = k$, a constant, $k\phi(n) \to ka$.

The theorem can be extended to any finite number of functions. Thus, if $\phi_1 \to a_1$, $\phi_2 \to a_2$, ... $\phi_p \to a_p$, then

$$\phi_1\phi_2 \ldots \phi_p \to a_1a_2 \ldots a_p.$$

The student should have no difficulty in discussing the various subsidiary cases in which ϕ and ψ do not tend to limits.

THEOREM III.—QUOTIENT OF TWO FUNCTIONS.—*If* $\phi(n) \to a$, $\psi(n) \to b \neq 0$, *then* $\dfrac{\phi(n)}{\psi(n)} \to \dfrac{a}{b}$.

Write $\phi(n) = a + \lambda(n)$, $\psi(n) = b + \mu(n)$. Then

$$\frac{\phi(n)}{\psi(n)} - \frac{a}{b} = \frac{a + \lambda(n)}{b + \mu(n)} - \frac{a}{b} = \frac{b\lambda(n) - a\mu(n)}{b\{b + \mu(n)\}}.$$

Corresponding to the arbitrary positive number ϵ there exist numbers n_0, n_1 such that

$$|\lambda(n)| < \epsilon, \text{ all } n \geqslant n_0. \qquad |\mu(n)| < \epsilon, \text{ all } n \geqslant n_1.$$

Let N be the greater of the numbers n_0, n_1. Then

$$|b\{b + \mu(n)\}| = |b|\,|b + \mu(n)|,$$

Since $|b| > 0$ and $|\mu(n)| < \epsilon$, where ϵ is arbitrary, we can ensure that $|b + \mu(n)| > \frac{1}{2}|b|$. Thus

$$|b\{b + \mu(n_1)\}| > \tfrac{1}{2}|b|^2 = k, \text{ say.}$$

$$\text{Then } \left|\frac{\phi(n)}{\psi(n)} - \frac{a}{b}\right| \leqslant \frac{|b\lambda(n)| + |a\mu(n)|}{|b\{b + \mu(n)\}|} < \epsilon\{|b| + |a|\}/k$$

$$< \epsilon k_1, \qquad n \geqslant N,$$

where k_1 is a fixed positive constant. Hence

$$\lim_{n \to \infty} \frac{\phi(n)}{\psi(n)} = \frac{a}{b}.$$

In particular if $\phi(n) = A$, a constant, $\lim\limits_{n \to \infty} A/\psi(n) = A/b$.

2·21. The Behaviour of Rational Functions of n as $n \to \infty$.

Write $f(n) = \dfrac{a_0 n^p + a_1 n^{p-1} + \ldots + a_p}{b_0 n^q + b_1 n^{q-1} + \ldots + b_q}$ where $a_0 \neq 0$, $b_0 \neq 0$. Then $f(n)$ is the general rational function of n.

$$\text{Now } f(n) = n^{p-q}\left\{a_0 + \frac{a_1}{n} + \ldots + \frac{a_p}{n^p}\right\} \Big/ \left\{b_0 + \frac{b_1}{n} + \ldots + \frac{b_q}{n^q}\right\}$$

$$\text{i.e. } f(n) = n^{p-q}\phi(n)/\psi(n),$$

$$\text{where } \phi(n) = a_0 + \frac{a_1}{n} + \ldots + \frac{a_p}{n^p},$$

$$\psi(n) = b_0 + \frac{b_1}{n} + \ldots + \frac{b_q}{n^q}.$$

As $n \to \infty$, $\phi(n) \to a_0$, $\psi(n) \to b_0$, $\phi(n)/\psi(n) \to a_0/b_0$.

As $n \to \infty$, $n^{p-q} \to \infty$, if $p > q$,

$n^{p-q} \to 0$, if $p < q$,

$n^{p-q} = 1$, if $p = q$.

Hence if $p > q$, $f(n) \to +\infty$ if $a_0/b_0 > 0$,

$f(n) \to -\infty$ if $a_0/b_0 < 0$;

$$\text{if } p < q,\ f(n) \to 0;$$
$$\text{if } p = q,\ f(n) \to a_0/b_0.$$

The limit as n tends to infinity of $\left(1 + \frac{1}{n}\right)^n$. Now from the binomial theorem we know that when n is a positive integer,

$$\left(1 + \frac{1}{n}\right)^n$$
$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \dots + \frac{n(n-1)\dots(n-n+1)}{n!} \cdot \frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{n-1}{n}\right),$$

i.e. $\left(1 + \frac{1}{n}\right)^n$

$$= 1 + \sum_{r=1}^{n} \frac{1}{r!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{r-1}{n}\right).$$

Since $1 - \frac{p}{n} < 1 - \frac{p}{n+1}$, $1 \leqslant p \leqslant r - 1$, it follows that

$$\frac{1}{r!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{r-1}{n}\right)$$

increases with n. Since the expansion contains $(n + 1)$ positive terms, the number of terms also increases with n. It follows that $\left(1 + \frac{1}{n}\right)^n$ is a monotonic increasing function of n. Hence either $\left(1 + \frac{1}{n}\right)^n$ tends to a limit or to $+\infty$. Thus in order to show that the expression tends to a finite limit it is only necessary to show that it is bounded. Now

$$\left(1 + \frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$
$$< 1 + \sum_{r=0}^{\infty} \frac{1}{2^r} = 3.$$

Hence $\left(1 + \frac{1}{n}\right)^n$ is bounded and tends to a limit. This limit

is denoted by e. Thus $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Clearly $\left(1 + \frac{1}{n}\right)^n > 2$ so that $2 < e \leqslant 3$.

2·3. Limit Theorems for Sequences

The first two theorems are due to Cauchy and the third to Cesàro.

(1) *Let $u_1, u_2, u_3, \ldots u_n, \ldots$ denote a sequence of numbers such that $\lim\limits_{n \to \infty} u_n$ exists and is equal to l. Then if*

$S_n = u_1 + u_2 + \ldots + u_n$, *$\lim\limits_{n \to \infty} \frac{S_n}{n}$ also exists and is equal to l.*

Since $u_n \to l$ as $n \to \infty$, we can find a number n_0, corresponding to the arbitrary positive ϵ, such that

$$l - \epsilon < u_n < l + \epsilon$$

for all $n > n_0$. Thus if $n > n_0$,

$$(n - n_0)(l - \epsilon) < \sum_{r = n_0 + 1}^{n} u_r < (n - n_0)(l + \epsilon).$$

Adding $S_{n_0} = \sum\limits_{r=1}^{n_0} u_r$ to each term in the inequality and writing

$$S_n = \sum_{r=1}^{n} u_r, \text{ we have}$$

$$(n - n_0)(l - \epsilon) + S_{n_0} < S_n < (n - n_0)(l + \epsilon) + S_{n_0}.$$

Dividing throughout by n and rearranging,

$$\frac{S_{n_0}}{n} - \frac{n_0(l - \epsilon)}{n} - \epsilon < \frac{S_n}{n} - l < \epsilon - \frac{n_0(l + \epsilon)}{n} + \frac{S_{n_0}}{n}.$$

Observing that n_0 is a fixed number so that S_{n_0}, $n_0(l - \epsilon)$, $n_0(l + \epsilon)$ are also fixed, we can find a positive number A such that $|S_{n_0}| < A$, $|n_0(l + \epsilon)| < A$, $n_0 |(l - \epsilon)| < A$.

Hence $$-\frac{2A}{n} - \epsilon < \frac{S_n}{n} - l < \epsilon + \frac{2A}{n},\ n > n_0.$$

Since A is fixed we can find a fixed positive integer $n_1 > n_0$, depending on ϵ such that for all $n > n_1$, $2A/n < \epsilon$ and $-2A/n > -\epsilon$. The inequality now becomes

$$-2\epsilon < \frac{S_n}{n} - l < 2\epsilon \text{ or } \left|\frac{S_n}{n} - l\right| < 2\epsilon, \text{ all } n \geqslant n_1.$$

Since ϵ is arbitrary, so also is 2ϵ.

Hence $$\lim_{n\to\infty} \frac{S_n}{n} = l.$$

Observe that in the above proof we have chosen two definite positive integers n_0 and n_1, both depending on ϵ. Thus, e.g. n_0 may be 10^{10}, whereas n_1 will, in general, be a much greater number, say 10^{100}.

(2) ***If $\{u_n\}$ denote a sequence of positive numbers such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, where l is a finite positive number, then the limit as $n \to \infty$ of $u_n^{\frac{1}{n}}$ exists and is equal to l.***

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l > 0$, we can find a positive integer n_0, depending on ϵ such that

$$\left|\frac{u_{n+1}}{u_n} - l\right| < l\epsilon, \text{ or } l(1-\epsilon) < \frac{u_{n+1}}{u_n} < l(1+\epsilon), \text{ all } n \geqslant n_0.$$

In this inequality write $n = n_0, n_0 + 1, n_0 + 2, \ldots$ in succession and multiply. Then

$$l^{n-n_0}(1-\epsilon)^{n-n_0} < \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \cdots \frac{u_{n_0+1}}{u_{n_0}} < l^{n-n_0}(1+\epsilon)^{n-n_0}$$

or $$l^{n-n_0}(1-\epsilon)^{n-n_0} < \frac{u_n}{u_{n_0}} < l^{n-n_0}(1+\epsilon)^{n-n_0}.$$

Dividing throughout by l^n, we have

$$\frac{(1-\epsilon)^{n-n_0}}{l^{n_0}} . u_{n_0} < \frac{u_n}{l^n} < \frac{(1+\epsilon)^{n-n_0}}{l^{n_0}} . u_{n_0}.$$

Since $(1-\epsilon)^{n_0} < 1$ and $(1+\epsilon)^{n_0} > 1$, $(1-\epsilon)^{n-n_0} > (1-\epsilon)^n$ and $(1+\epsilon)^{n-n_0} < (1+\epsilon)^n$ so that the inequality may be written

$$A(1-\epsilon)^n < \frac{u_n}{l^n} < A(1+\epsilon)^n$$

where $A = u_{n_0}/l^{n_0}$. We observe that A is a *fixed positive* number.

Taking the nth root of each side of the inequality, we have

$$A^{\frac{1}{n}}(1-\epsilon) < \frac{u_n^{\frac{1}{n}}}{l} < A^{\frac{1}{n}}(1+\epsilon), \; n > n_0.$$

Now $\lim\limits_{n\to\infty} A^n = 1$ since A is a fixed positive number independent of n. This result can easily be seen by writing

$$A^{\frac{1}{n}} = e^{\frac{1}{n}\log A} = 1 + \frac{1}{n}\log A + O\left(\frac{1}{n^2}\right),$$

by the exponential theorem.
Hence, corresponding to ϵ, there exists an integer n_1, such that

$$1 - \epsilon < A^{\frac{1}{n}} < 1 + \epsilon, \text{ all } n \geqslant n_1.$$

Now let n_2 be a fixed positive integer greater than n_0 and n_1. Then for all $n \geqslant n_2$,

$$(1 - \epsilon)^2 < \frac{u_n^{\frac{1}{n}}}{l} < (1 + \epsilon)^2.$$

Since ϵ was an arbitrarily small positive number we can assume $\epsilon^2 < \epsilon$ so that the inequality can be written

$$1 - 3\epsilon < \frac{u_n^{\frac{1}{n}}}{l} < 1 + 3\epsilon, \text{ or } |u_n^{\frac{1}{n}} - l| < 3l\epsilon, \text{ all } n \geqslant n_2.$$

Since ϵ is arbitrary, so also is $3l\epsilon$.

Hence $$\lim_{n\to\infty} u_n^{\frac{1}{n}} = l.$$

(3) ***If the sequences $\{u_n\}$, $\{v_n\}$ are such that $\lim\limits_{n\to\infty} u_n = u$, and $\lim\limits_{n\to\infty} v_n = v$, then $\lim\limits_{n\to\infty} \frac{1}{n}(u_1v_n + u_2v_{n-1} + \ldots + u_nv_1) = uv$.***

Write $w_n = \frac{1}{n}(u_1v_n + u_2v_{n-1} + \ldots + u_nv_1)$, $u_n = u - a_n$, $v_n = v - b_n$, $|a_n| = A_n$, $|b_n| = B_n$, $|u| = U$, $|v| = V$.

Then $$w_n = \frac{1}{n}\sum_{r=1}^{n} u_r v_{n-r+1} = \frac{1}{n}\sum_{r=1}^{n}(u - a_r)(v - b_{n-r+1})$$

$$= uv - \frac{v}{n}\sum_{r=1}^{n} a_r - \frac{u}{n}\sum_{r=1}^{n} b_r + \frac{1}{n}\sum_{r=1}^{n} a_r b_{n-r+1}$$

Hence $$|w_n - uv| < \left|\frac{v}{n}\sum_{r=1}^{n} a_r\right| + \left|\frac{u}{n}\sum_{r=1}^{n} b_r\right| + \left|\frac{1}{n}\sum_{r=1}^{n} a_r b_{n-r+1}\right|$$

$$\leqslant \frac{V}{n}\sum_{r=1}^{n}|a_r| + \frac{U}{n}\sum_{r=1}^{n}|b_r| + \frac{1}{n}\sum_{r=1}^{n}|a_r|.|b_{n-r+1}|$$

$$= \frac{V}{n}\sum_{r=1}^{n} A_r + \frac{U}{n}\sum_{r=1}^{n} B_r + \frac{1}{n}\sum_{r=1}^{n} A_r B_{n-r+1}.$$

Since $A_r = |u - u_r|$, $B_r = |v - v_r|$ and $\lim\limits_{r\to\infty} u_r = u$, $\lim\limits_{r\to\infty} v_r = v$, it follows that $\lim\limits_{r\to\infty} A_r = 0$, $\lim\limits_{r\to\infty} B_r = 0$, and the sequence $\{B_r\}$ is bounded so that there exists a number B such that $B_r \leqslant B$, for all r.

From Theorem I above

$$\lim_{n\to\infty} \frac{A_1 + A_2 + \ldots + A_n}{n} = \lim_{n\to\infty} A_n = 0,$$

$$\lim_{n\to\infty} \frac{B_1 + B_2 + \ldots + B_n}{n} = \lim_{n\to\infty} B_n = 0.$$

Hence there exist positive integers n_0, n_1 such that

$$\frac{1}{n}\sum_{r=1}^{n} A_r < \frac{\epsilon}{3V}, \text{ all } n \geqslant n_0, \text{ and } \frac{1}{n}\sum_{r=1}^{n} B_r < \frac{\epsilon}{3U}, \text{ all } n \geqslant n_1.$$

Also $$\frac{1}{n}\Sigma A_r B_{n-r+1} \leqslant \frac{B}{n}\sum^{n} A_r.$$

As $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1} A_r = \lim A_n$ we can find a positive integer n_2

such that $$\frac{1}{n}\sum_{r=1} A_r < \frac{\epsilon}{3B}, \text{ for all } n \geqslant n_2.$$

Hence, if N denote the greatest value of n_0, n_1, n_2 we have

$$|w_n - uv| < V.\frac{\epsilon}{3V} + U.\frac{\epsilon}{3U} + B.\frac{\epsilon}{3B},$$

i.e. $$|w_n - uv| < \epsilon, \text{ for all } n > N.$$

Hence $$\lim w_n = uv.$$

2·4 Some Important Inequalities

Let m, p be positive rational numbers, such that $m > 1$, $0 < p < 1$. Then the following inequalities are true:

(i) $m(x-1) < x^m - 1 < mx^{m-1}(x-1)$, $x > 1$.

(ii) $m(1-x) > 1 - x^m > mx^{m-1}(1-x)$, $0 < x < 1$.

(iii) $px^{p-1}(x-1) < x^p - 1 < p(x-1)$, $x > 1$.

(iv) $px^{p-1}(1-x) > 1 - x^p > p(1-x)$, $0 < x < 1$.

First consider (i) and let m be a positive integer. Then

$$mx^m > x^{m-1} + x^{m-2} + \ldots + 1 = (x^m - 1)/(x - 1),$$

$$\text{i.e. } mx^m (x - 1) > x^m - 1.$$

Adding $m(x^m - 1)$ to both sides of the inequality,

$$m(x^{m+1} - 1) > (m + 1)(x^m - 1), \text{ i.e. } \frac{x^{m+1} - 1}{m + 1} > \frac{x^m - 1}{m}.$$

Thus the effect of changing m into $m + 1$ is to *increase* the value of $(x^m - 1)/m$. It follows that

$$\frac{x^m - 1}{m} > \frac{x^n - 1}{n}; \quad x > 1, m > n \geqslant 1,$$

m and n being positive integers.

To extend the result to positive rational m, n, write $m = q/r$, $n = s/t$, where q, r, s, t are positive integers. Since $m > n$, it follows that $qt > rs$. In order that the inequality be true for rational m, n we require

$$\frac{x^{\frac{q}{r}} - 1}{\frac{q}{r}} > \frac{x^{\frac{s}{t}} - 1}{\frac{s}{t}}.$$

Write $x = y^{rt}$. Then this inequality becomes

$$\frac{y^{qt} - 1}{qt} > \frac{y^{rs} - 1}{rs}$$

which is true since qt, rs are positive integers and $qt > rs$.

Next consider (ii). If $0 < x < 1$, and m is a positive integer,

$$mx^m < 1 + x + \ldots + x^{m-1} = (1 - x^m)/(1 - x),$$

$$\text{i.e. } m(1 - x)x^m < (1 - x^m).$$

Adding $m(1 - x^m)$ to both sides of the inequality,

$$m + 1 \qquad m$$

Arguing as in (i) it follows that if m and n are positive integers and $m > n \geqslant 1$,

$$\frac{1 - x^m}{m} < 1 - x^n, \quad 0 < x < 1.$$

To extend the result to positive rational numbers, proceed as in (i). Thus

$$\frac{1 - x^{\frac{q}{r}}}{\frac{q}{r}} < \frac{1 - x^{\frac{s}{t}}}{\frac{s}{t}} \quad \text{provided} \quad \frac{1 - x^{\frac{q}{r}}}{qt} < \frac{1 - x^{\frac{s}{t}}}{rs}$$

Writing $x = y^{rt}$ this condition becomes

$$\frac{1 - y^{qt}}{qt} < \frac{1 - y^{rs}}{rs}$$

which is true since $0 < y < 1$ and $qt > rs$.

Now write $n = 1$ in the results obtained. *Then if m is any positive rational number greater than unity*

$$x^m - 1 > m(x - 1), \; x > 1 \dots\dots\dots\dots\dots\dots (\text{i}')$$

$$1 - x^m < m(1 - x), \; 0 < x < 1 \dots\dots\dots\dots (\text{ii}')$$

The remainder of the inequalities may be deduced from these results. Thus in (i′) write $m = 1/p$, so that $0 < p < 1$. Then

$$(x^{\frac{1}{p}} - 1) > (x - 1)/p, \; x > 1.$$

Now write $x^{\frac{1}{p}} = y$, so that $y > 1$. Then

$$p(y - 1) > (y^p - 1) \dots\dots\dots\dots\dots\dots (\text{iii}')$$

and this is the inequality required with y instead of x.

In (ii′) write $m = 1/p$. Then $1 - x^{\frac{1}{p}} < (1 - x)/p$, $0 < x < 1$, $0 < p < 1$. Writing $y = x^{\frac{1}{p}}$ this inequality takes the form

$$1 - y^p > p(1 - y), \; 0 < y < 1 \dots\dots\dots\dots (\text{iv}')$$

Next in (i′) write $x = 1/y$ so that $0 < y < 1$. Then

$$\frac{1}{y^m} - 1 > m\left(\frac{1}{y} - 1\right), \; \textit{i.e.} \; 1 - y^m > my^{m-1}(1 - y).$$

Or changing the notation

$$1 - x^m > mx^{m-1}(1 - x), \; 0 < x < 1, \, m > 1 \dots\dots (\text{ii}'')$$

Similarly from (ii′) we obtain

$$x^m - 1 < mx^{m-1}(x - 1), \; x > 1, \, m > 1 \dots\dots\dots (\text{i}'')$$

Continuing these inequalities with (i′) and (ii′) we obtain

$$m\,(x-1) < x^m - 1 < mx^{m-1}(x-1),\ \ x>1,\ m>1.$$
$$m\,(1-x) > 1 - x^m > mx^{m-1}(1-x),\ \ 0<x<1,\ m>1.$$

Similarly from (iii′) and (iv′) we obtain

$$x^p - 1 > px^{p-1}(x-1),\ \ x>1,\ \ 0<p<1 \ldots\ldots \text{(iii}''\text{)}$$
$$1 - x^p < px^{p-1}\,(1-x),\ \ 0<x<1,\ 0<p<1 \ldots\ldots \text{(iv}''\text{)}$$

Combining these inequalities with (iii′) and (iv′) we obtain

$$px^{p-1}(x-1) < x^p - 1 < p\,(x-1),\ \ x>1,\ 0<p<1.$$
$$px^{p-1}(1-x) > 1 - x^p > p\,(1-x),\ \ 0<x<1,\ 0<p<1.$$

Note.—The inequalities given above are proved only for *rational* numbers. The inequalities hold strictly for irrational numbers, but the precise discussion of the passage to the limit by considering irrational numbers as the limit of rational numbers is beyond the scope of this book.

2·5. Powers of Limits

Let $f(n) \to l$ as $n \to \infty$. Then if $l>0$, $\{f(n)\}^k \to l^k$, where k is any rational number.

We first observe that if k is a positive integer the result follows immediately from the theorem on the product of limits (§ 2·2). Suppose now that k is a *positive* rational number. Then corresponding to the arbitrary positive number ϵ, there exists a number n_0 such that

$$l - \epsilon < f(n) < l + \epsilon \ \text{ for all } \ n \geqslant n_0.$$

Since l is positive we may assume that for n sufficiently great $f(n) > 0$, so that there is a positive real number defined by $f(n)^k$. Thus

$$(l-\epsilon)^k < \{f(n)\}^k < (l+\epsilon)^k.$$
$$\text{i.e. } \ l^k\,(1-\epsilon/l)^k < \{f(n)\}^k < l^k\,(1+\epsilon/l)^k.$$

Since ϵ is arbitrary and $l>0$ we may assume that

$$0 < \epsilon/l < \tfrac{1}{2}l.$$

Suppose firstly that $k<1$, then applying the inequalities

$$x^p - 1 < p\,(x-1),\ \ 0<p<1,\ x>1,$$
$$1 - x^p < px^{p-1}(1-x),\ \ 0<p<1,\ x<1,$$

$$(1 + \epsilon/l)^k < 1 + k\epsilon/l, \qquad \text{(writing } x = 1 + \epsilon/l\text{)},$$
$$(1 - \epsilon/l)^k > 1 - k\epsilon\,(1 - \epsilon/l)^{k-1}/l, \quad \text{(writing } x = 1 - \epsilon/l\text{)}.$$

Since $(1 - \epsilon/l)^{k-1}$ is finite for $0 < k < 1$ we may assume that $(1 - \epsilon/l)^k > 1 - A\epsilon$, where A is a positive constant. Hence

$$l^k\,(1 - A\epsilon) < \{f(n)\}^k < l^k\{1 + k\epsilon/l\},$$
$$\text{i.e.} \quad -\,l^kA\epsilon < \{f(n)\}^k - l^k < kl^{k-1}\epsilon.$$

Thus there exists a constant B such that $|\{f(n)\}^k - l^k| < B\epsilon$, and since ϵ is arbitrary it follows that

$$\lim_{n \to \infty} \{f(n)\}^k = l^k.$$

Similarly for the case in which $k > 1$.

Next suppose that k is negative and write $k = -h$, where $h > 0$. Then

$$\lim_{n \to \infty} \{f(n)\}^k = \lim_{n \to \infty} \{f(n)\}^{-h} = \lim_{n \to \infty} 1/\{f(n)\}^h = 1/l^h,$$

since $h > 0$,

$$\textit{i.e.} \lim_{n \to \infty} \{f(n)\}^k = l^{-h} = l^k.$$

2·61. Continuous Variable tending to Infinity

In the discussion in the earlier sections of this chapter it has been assumed that n is a variable which tends to infinity through positive integral values. Clearly this restriction can be removed.

Let $f(x)$ be a function of x, and suppose that the variable tends steadily to infinity. Then *if, corresponding to an arbitrary positive number* ϵ, *there exists a number* X *such that* $|f(x) - l| < \epsilon$ *for all* $x \geqslant X$, *then* l *is said to be the limit of* $f(x)$ *as* $x \to \infty$. Thus

$$f(x) \to l \text{ as } x \to \infty,$$

and this limiting value may be denoted by $f(\infty)$.

The notation adopted for the integral variable n extends to that of the continuous variable x. Thus *e.g.* if we say that

$$f(x) = O(x^2)$$

we mean that $|f(x)| < Ax^2$ for all $x \geqslant X$, where X is some fixed positive number, A being a constant. It must be remembered in this connection that the equation $f(x) = O(x^2)$ does not imply that $f(x)/x^2$ possesses a limit as $x \to \infty$. It does imply, however, that $f(x)/x^2$ is bounded as $x \to \infty$. If we wish to assert that $f(x)/x^2$ tends to a limit l the result could be expressed in the form $f(x) \sim lx^2$.

Example.—*Discuss the behaviour as x tends to infinity, of the function $x - [x]$, where $[x]$ denotes the integral part of x.*

The behaviour of the function $x - [x]$ is shown most clearly by its graphical representation. Write $f(x) = x - [x]$ and consider positive values of x.

Now $f(0) = 0$. For $0 \leqslant x < 1$, $[x] = 0$. Hence for the range $0 \leqslant x < 1$ the graph of $f(x)$ is the straight line $y = x$.

When $x = 1$, $[x] = 1$ so that $f(1) = 0$. Thus the function jumps down to the value zero at $x = 1$.

For $1 \leqslant x < 2$, $[x] = 1$ so that for this range of values of x the graph of $f(x)$ is the straight line $y = x - 1$.

Similarly for the range $n \leqslant x < n + 1$, where n is a positive integer. In this range the function is represented by the straight line $y = x - n$.

In Fig. 1 the dotted line does not form part of the graph but indicates the places where the function suddenly changes its values. The graph shows clearly that $f(x)$ always lies between 0 and 1, and hence $f(x)$ is bounded as

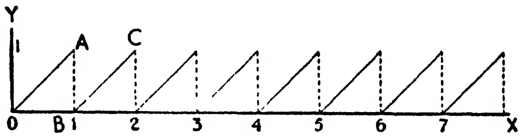


FIG. 1.

$x \to \infty$. But $\lim\limits_{x \to \infty} f(x)$ does not exist. For no matter how large x may be, it is clear that in the range $n \leqslant x < n + 1$, $f(x)$ will take all values between 0 and 1. Thus $f(x)$ oscillates finitely as $x \to \infty$.

Note.—The example considered above belongs to the class of *discontinuous* functions. The precise definitions of continuous and discontinuous functions will be considered later. See §§ 2·71, 2·73, 2·91.

The example also shows that the behaviour of a function as the variable tends to infinity depends essentially on whether that variable takes all values or only a special set of values; in particular if it takes only *integral values*.

Thus if $f(x) = x - [x]$ and x takes *only positive integral values* $f(x)$ is always zero, and the limit as $x \to \infty$ through positive integral values is zero.

But we have seen above that if x is a continuous variable the limit does not exist.

Behaviour as $x \to -\infty$.—In a similar way we may define $\lim\limits_{x \to -\infty} f(x)$. Thus if this limit exists and is equal to l, then

corresponding to the arbitrary positive number ϵ, *there exists a positive number* X *such that for all* $x \leqslant -X$,

$$|f(x) - l| < \epsilon.$$

The limit may be denoted by $f(-\infty)$.

2·62. Limits as x tends to a Finite Value

The definition has been given in Chapter I. (§ 1·11). We draw attention to one important point.

Consider $\lim_{x \to a} (x + b)$. The value is clearly $a + b$. In accordance with the definition of $\lim_{x \to a} f(x) = l$ we consider the behaviour of $f(x)$ in the neighbourhood of $x = a$. The limit is $a + b$ because corresponding to the arbitrary positive number ϵ we can find a number η such that

$$|x + b - a - b| < \epsilon$$

for all values of x satisfying $0 < |x - b| \leqslant \eta$.

In this case we can take $\eta = \epsilon$. The student may well ask "why not put $x = a$ and say immediately that the limit is $a + b$?" This is not legitimate because the statement $\lim_{x \to a} f(x) = l$ asserts a property about the values of $f(x)$ when x differs slightly from a. *It asserts nothing about the value of* $f(x)$ *when* $x = a$. The function may not even be defined at $x = a$; even when it is defined the two values may be different.

Consider the behaviour of the following function near $x = 0$: Let $f(x) = [1 - x^4]$, where $[1 - x^4]$ means the integral part of $1 - x^4$. When $x = 0$, $f(x) = [1] = 1$ so that the defined value is unity.

Next consider the value of $f(x)$ in the neighbourhood of $x = 0$. When $-1 < x < 0$ and when $0 < x < 1$, $0 < x^4 < 1$.

Hence $0 < 1 - x^4 < 1$, $[1 - x^4] = 0$. Thus $|f(x)| < \epsilon$ for all x satisfying $0 < |x| < 1$. Hence $\lim f(x) = 0$, whereas

$f(0) \quad 1.$

2·63. Theorems on Limits as $x \to a$

Corresponding to Theorems I., II., III. of § 2·2, we have the following results:

THEOREM IV.—SUM OF TWO FUNCTIONS.—*If* $f(x) \to a$, $g(x) \to \beta$ *as* $x \to a$ *then*

$$\lim_{x \to a} \{f(x) + g(x)\} = a + \beta.$$

THEOREM V.—PRODUCT OF TWO FUNCTIONS.—*If* $f(x) \to a$, $g(x) \to \beta$ *as* $x \to a$ *then*

$$\lim_{x \to a} f(x)g(x) = a\beta.$$

THEOREM VI.—QUOTIENT OF TWO FUNCTIONS.—*If* $f(x) \to a$, $g(x) \to \beta$ *as* $x \to a$, *then provided* $\beta \neq 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{a}{\beta}.$$

We give the proof of Theorem IV. By comparing the methods of § 2·2, the student should be able to obtain for himself the proofs of V. and VI.

Let ϵ be an arbitrary positive number. Then there exist numbers η_1 and η_2 such that

$$|f(x) - a| < \epsilon, \text{ for all } x \text{ satisfying } 0 < |x - a| \leqslant \eta_1,$$
$$|g(x) - \beta| < \epsilon, \text{ for all } x \text{ satisfying } 0 < |x - a| \leqslant \eta_2.$$

Let η be the smaller of η_1 and η_2. Then both inequalities hold for all values of x for which $0 < |x - a| \leqslant \eta$. Thus

$$|f(x) + g(x) - a - \beta| \leqslant |f(x) - a| + |g(x) - \beta|$$
$$< 2\epsilon, \quad 0 < (x - a) \leqslant \eta.$$

Hence $\lim. \{f(x) + g(x)\} = a + \beta.$

2·64. Function of a Function

Consider $\lim_{x \to a} f(y)$ where $y = \phi(x)$, a function of x. *Suppose that* $\lim_{x \to a} \phi(x) = a$, $\lim_{y \to a} f(y) = \beta$. Then $\lim_{x \to a} f(y) = \beta$.

Corresponding to the arbitrary positive number ϵ there exists a positive number η such that

$$|f(y) - \beta| < \epsilon$$

for all y for which $|y - a| \leqslant \eta$. Again corresponding to η, there exists a positive number η_1 such that

$$|\phi(x) - a| < \eta.$$

for all x satisfying $0 < |x - a| \leqslant \eta_1$. Thus $|f(y) - \beta| < \epsilon$ for all x satisfying $0 < |x - a| \leqslant \eta_1$. Hence

$$\lim_{x \to a} f(y) = \beta.$$

Example.—*Find* $\displaystyle \lim_{x \to a} \left(\frac{x^3 - a^3}{x - a}\right)^4$

$$\text{Now } \lim_{x \to a} \left(\frac{x^3 - a^3}{x - a}\right) = \lim. \ (x^2 + ax + a^2) = 3a^2.$$

$$\text{Write } \frac{x^3 - a^3}{x - a} = y. \quad \text{Then } \lim_{y \to 3a^2} y^4 = (3a^2)^4.$$

$$\text{Hence } \lim_{x \to a} \left(\frac{x^3 - a^3}{x - a}\right)^4 = (3a^2)^4.$$

2·641. Illustrative Examples

(1) *Find the limit to which* $(x^{-2} - a^{-2})/(x - a)$ *approaches indefinitely close, as* x *tends to the limit* a.

$$\text{Now } \frac{x^{-2} - a^{-2}}{x - a} = \frac{a^2 - x^2}{a^2x^2(x - a)}.$$

In the neighbourhood of $x = a$, $x - a$ is small but not zero. Hence we may divide throughout by $x - a$ and the expression becomes $-(a + x)/a^2x^2$.

Now $\displaystyle \lim_{x \to a} -(a + x) = -2a, \quad \lim_{x \to a} a^2x^2$

$$\lim_{x \to a} - (a + x)$$

(2) *Prove that if* k *is any positive number, the function* $(k^n - 1)/(k^n + 1)$ *tends to a limit as* n *tends to infinity, and that the value of the limit is different according as* k *is greater than, equal to, or less than, unity.* [*M.T.*]

CASE (I): $k > 1$.—In this case $\displaystyle \lim_{n \to \infty} k^n = \infty$. Now

$$\frac{k^n - 1}{k^n + 1} = \frac{1 - 1/k^n}{1 + 1/k^n}.$$

$$\text{Also } \lim_{n \to \infty} (1 - 1/k^n) = 1, \quad \lim_{n \to \infty} (1 + 1/k^n) = 1.$$

$$\text{Hence } \lim_{n \to \infty} \frac{k^n - 1}{k^n + 1} = 1.$$

CASE (II): $k = 1$.—In this case $k^n = 1$ and the expression becomes 0. Since this is true for all values of n the limit as $n \to \infty$ is 0.

CASE (III): $k < 1$.—Then $\displaystyle \lim_{n \to \infty} k^n = 0$. Thus

$$\lim_{n \to \infty} \frac{k \quad 1}{k^n + 1} = \frac{-1}{1} = -1.$$

(3) *Discuss the behaviour as x tends to zero of the rational function*

$$f(x) = (a_0x^m + a_1x^{m+1} + \ldots + a_px^{m+p})/(b_0x^n + b_1x^{n+1} + \ldots + b_qx^{n+q})$$

where a_0, b_0 are different from zero.

Now $f(x) = x^{m-n}(a_0 + a_1x + \ldots + a_px^p)/(b_0 + b_1x + \ldots + b_qx^q)$.

Also
$$\lim_{x \to 0} (a_0 + a_1x + \ldots + a_px^p) = a_0,$$
$$\lim_{x \to 0} (b_0 + b_1x + \ldots + b_qx^q) = b_0.$$
$$\lim_{x \to 0} \{(a_0 + a_1x + \ldots + a_px^p)/(b_0 + b_1x + \ldots + b_qx^q)\} = a_0/b_0.$$

CASE (I): $m - n > 0$, *i.e.* $m > n$. Then

$$\lim_{x \to 0} x^{m-n} = 0, \text{ so that } \lim_{x \to 0} f(x) = 0.$$

CASE (II): $m = n$. Then

$$x^{m-n} = 1 \text{ and } \lim f(x) = a_0/b_0.$$

CASE (III): $m - n < 0$. Write $m - n = -k$, so that $x^{m-n} = 1/x^k$ and k is a positive integer. If k is even, x^k is always positive. Hence

$$\lim_{x \to 0} 1/x^k = +\infty.$$

If $a_0/b_0 > 0$ then $\lim_{x \to 0} f(x) = +\infty$. If $a_0/b_0 < 0$ then

$$\lim_{x \to 0} f(x) = -\infty.$$

Finally if k is odd the sign of x^k will depend on the sign of x. Thus if x tends to zero through positive values $1/x^k \to +\infty$, while if x tends to zero through negative values $1/x^k \to -\infty$. Since there are both positive and negative values of x in every neighbourhood of $x = 0$ it follows that the limit does not exist.

(4) *Prove that* $\lim_{x \to 0} \dfrac{\sqrt{(1 + x)} - \sqrt{(1 + x^2)}}{\sqrt{(1 - x^2)} - \sqrt{(1 - x)}} = 1.$

$$\frac{\sqrt{(1 + x)} - \sqrt{(1 + x^2)}}{\sqrt{(1 - x^2)} - \sqrt{(1 - x)}}$$
$$= \frac{\sqrt{(1 + x)} - \sqrt{(1 + x^2)}}{\sqrt{1 - x)}\{\sqrt{(1 + x)} - 1\}}$$
$$= \frac{\{\sqrt{(1 + x)} - \sqrt{(1 + x^2)}\}\{\sqrt{(1 + x)} + 1\}}{\sqrt{(1 - x)}\,(1 + x - 1)}$$
$$= \frac{\sqrt{(1 + x)} + 1}{\sqrt{(1 - x)}} \cdot \frac{\sqrt{1 + x)} - \sqrt{(1 + x^2)}}{x}$$

Now $\lim_{x \to 0} \dfrac{\sqrt{(1 + x)} + 1}{\sqrt{(1 - x)}} = 2.$

Again $\dfrac{\sqrt{(1+x)} - \sqrt{(1+x^2)}}{x}$

$$= \frac{\{\sqrt{(1+x)} - \sqrt{(1+x^2)}\}\{\sqrt{(1+x)} + \sqrt{(1+x^2)}\}}{x\{\sqrt{(1+x)} + \sqrt{(1+x^2)}\}}$$

$$= \frac{x(1-x)}{x\{\sqrt{(1+x)} + \sqrt{(1+x^2)}\}} = \frac{1-x}{\sqrt{(1+x)} + \sqrt{(1+x^2)}}.$$

The limit of this expression as $x \to 0$ is $\frac{1}{2}$. Thus the result is proved.

Note.—It is legitimate to divide numerator and denominator by x in the last part of the proof since x is *different from zero*, the limit being concerned with the behaviour of the function *near* $x = 0$ not *at* $x = 0$.

(5) *Prove if* $\lim\limits_{x \to a} f(x) = l$ *then* $\lim\limits_{x \to a} |f(x)| = |l|$ *and that if* $l \neq 0$ *the converse is false.*

To prove that $|f(x)| \to |l|$ it is necessary to show that corresponding to the arbitrary positive number ϵ, there exists a number η such that

$$\big|\,|f(x)| - |l|\,\big| < \epsilon$$

for all x for which $0 < |x - a| \leqslant \eta$.

Since $f(x) \to l$, we have $|f(x) - l| < \epsilon$ for all x satisfying

$$0 < |x - a| \leqslant \eta,$$

where ϵ is an arbitrary positive number and η depends on ϵ. Hence

$$l - \epsilon < f(x) < l + \epsilon \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

If $l = 0$ it is clear that $|f(x)| < \epsilon$ implies $-\epsilon < f(x) < \epsilon$ and conversely.

Hence if $\lim\limits_{x \to a} f(x) = 0$, $\lim\limits_{x \to a} |f(x)| = 0$ and conversely.

Suppose now that $l > 0$, so that $|l| = l$ and $l - \epsilon$ may be assumed to be positive, ϵ being arbitrary. Then if (i) is satisfied

$$l - \epsilon < |f(x)| < l + \epsilon.$$

$$\text{i.e. } |l| - \epsilon < |f(x)| < |l| + \epsilon, \ 0 < |x - a| \leqslant \eta.$$

$$\text{Hence } \lim_{x \to a} |f(x)| = |l|.$$

Next suppose that $l < 0$. Write $l = -k$ so that $k > 0$ and $|l| = k$. Then from (i) $-k - \epsilon < f(x) < -k + \epsilon$.

$$\text{i.e. } k + \epsilon > -f(x) > k - \epsilon, \ 0 < |x - a| \leqslant \eta.$$

Arguing as in the previous case and observing that $|-f(x)| = |f(x)|$, it follows that

$$\lim_{x \to a} |f(x)| = k = |l|.$$

Now to prove that a theorem is not true the simplest method is to produce an example in which the result is false. Let $f(x)$ be a function of x defined as follows, in the interval $-1 \leqslant x \leqslant 1$.

$$f(x) = 1 \text{ if } x = \frac{1}{n}, \text{ } n \text{ denoting a positive or negative integer.}$$

$$f(x) = -1, \text{ for other values of } x \text{ in the range.}$$

Then $|f(x)| = 1$ at *all* points of the interval $(-1, 1)$.

$$\text{Thus } \lim_{x \to 0} |f(x)| = 1.$$

Next consider $\lim\limits_{x \to 0} f(x)$. It will be observed that in every neighbourhood of x o there are values which are of the form $\frac{1}{n}$ and also values which are not of this form. Thus no matter how small η may be we can always find an integer n which is *not a perfect square* such that $\frac{1}{n}$, $\frac{1}{\sqrt{n}}$ both lie in the range $0 < |x| < \eta$. Also when $x = \frac{1}{n}$, $f(x) = 1$, while when $x = \frac{1}{\sqrt{n}}$, $f(x) = -1$. Hence $\lim\limits_{x \to 0} f(x)$ does not exist.

Note.—An example such as the above, which is produced in order to disprove an assertion, is sometimes called in higher mathematics by the single German word *Gegenbeispiel.*

2·65. One-sided Limits

In the definition of the existence of the limit as $x \to a$, use is made of the inequality $0 < |x - a| \leqslant \eta$, i.e. x is a point which lies in one or other of the intervals $0 < a - x \leqslant \eta$, $0 < x - a \leqslant \eta$.

Suppose now that *we restrict x to lie in one interval only.* Thus with the usual convention if $0 < x - a \leqslant \eta$, x lies always to the *right of a,* since $x > a$. Then if corresponding to the arbitrary positive ϵ there exists a number η such that

$$|f(x) - l| < \epsilon$$

for all x satisfying $0 < x - a \leqslant \eta$, then l is said to be the *right-hand limit,* or *the limit on the right,* as $x \to a$. The notation adopted is

$$\lim_{x \to a+0} f(x) = l.$$

If in the above definition the inequality $0 < x - a \leqslant \eta$ is replaced by $0 < a - x \leqslant \eta$, we obtain the definition of a *left-hand limit,* or *limit on the left.* The notation adopted is

$$\lim_{x \to a-0} f(x).$$

It is obvious from the definitions that if $\lim\limits_{x \to a} f(x) = l$ then $\lim\limits_{x \to a+0} f(x)$, $\lim\limits_{x \to a-0} f(x)$ *must both exist and be equal to l.* That the converse, viz., that the existence of each of the one-sided limits

implies the existence of the ordinary limit, is not true as is shown by the following *gegenbeispiel*:

Consider the behaviour of $f(x) = x - [x]$ near $x = 1$. (See Ex. § 2·61 and the corresponding figure). Suppose that $x \to 1$ from below, *i.e.* x increases to the value unity. From the figure it is seen that $f(x)$ approaches the point A along the line OA and hence $\lim\limits_{x \to 1 - 0} f(x) = 1$. On the other hand if $x \to 1$ from above, i.e. x decreases to the value unity then $f(x)$ approaches the point B along the line CB and $\lim\limits_{x \to 1 + 0} f(x) = 0$.

Clearly $\lim\limits_{x \to 1} f(x)$ cannot exist for a necessary condition for this is that the right-hand and left-hand limits be equal.

2·71. The Idea of Continuity

Suppose we have a single-valued function $f(x)$ which is defined at all points of an interval (α, β) and let a be a point in the interior of the interval. The *defined value* of $f(x)$ at $x = a$ is denoted by $f(a)$. Then if *the limit as x tends to a of $f(x)$ is equal to $f(a)$*, i.e.

$$\lim_{x \to a} f(x) = f(a)$$

the function $f(x)$ is said to be continuous at $x = a$. If $\lim\limits_{x \to a} f(x)$ is different from $f(a)$, or the limit does not exist, $f(x)$ is said to be *discontinuous* at $x = a$. *A function which is continuous at all points of an interval is said to be continuous in the interval.*

The end points α, β of the interval must be regarded as being slightly different from interior points. For if a *be an interior point* we can find *a neighbourhood of* a *which lies entirely inside the given interval.*

α a β

FIG. 2.

But this property is not true for α or β. We can take a "left-hand" neighbourhood of β and a "right-hand" neighbourhood of α. As $f(x)$ is not defined outside (α, β) it is necessary then to consider at α, β not the limit as x tends to these values but the one-sided limits.

Thus $f(x)$ is continuous at $x = \beta$ if $\lim\limits_{x \to \beta - 0} f(x) = f(\beta)$.

Also $f(x)$ is continuous at $x = \alpha$ if $\lim\limits_{x \to \alpha + 0} f(x) = f(\alpha)$.

Because of the special behaviour at the end points of a finite interval it is usual to distinguish between *open* and *closed* intervals:

(*a*) The **open interval** (α, β) consists of all points of the interval (α, β) with the exception of the end points, *i.e.* the points $\alpha < x < \beta$.

(*b*) The **closed interval** (α, β) consists of the open interval (α, β) together with the end points α, β, *i.e.* the points $\alpha \leqslant x \leqslant \beta$.

(*c*) A **half-open interval** is one in which one of the end points is included, *i.e.* either $\alpha \leqslant x < \beta$ or $\alpha < x \leqslant \beta$.

It should be observed that properties which are valid for a closed interval are essentially different from those for open intervals. Thus, *e.g.*, the function $f(x) = 1/x$ is continuous at every point of the interval $(0, 1)$ except at $x = 0$.

For $\lim\limits_{x \to +0} 1/x = +\infty$, so that the limit does not exist.

We can say that $f(x)$ is continuous in the half-open interval $0 < x \leqslant 1$ meaning thereby that $f(x)$ is continuous in every closed interval $\eta \leqslant x \leqslant 1$, $\eta > 0$ no matter how small η may be

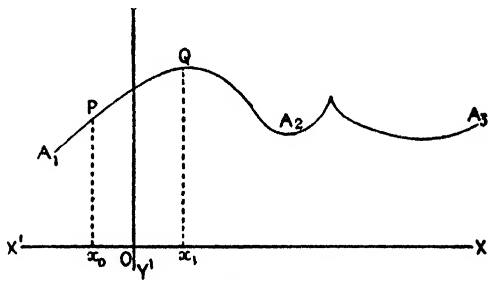


FIG. 3.

2·72. Properties of Continuous Functions

From the definition of $\lim\limits_{x \to a} f(x)$ it follows that a *function $f(x)$ is continuous at $x = a$ if corresponding to the arbitrary positive number ϵ, there exists a positive number η such that*

$$|f(x) - f(a)| < \epsilon,$$

for all x satisfying $|x - a| \leqslant \eta$.

The modification of the statement necessary in the case of end points of an interval should be clear.

From the theorems on limits (see §§ 2·63, 2·64) the following properties are true:

THEOREM I.—The sum of two continuous functions is continuous.

THEOREM II.—The product of two continuous functions is continuous.

THEOREM III.—The quotient of two continuous functions is continuous provided the denominator does not vanish.

THEOREM IV.—A continuous function of a continuous function is continuous.

Example.—(*a*) The polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

is continuous for all finite values of x.

(*b*) The rational function

$$R(x) = P(x)/Q(x)$$

where $P(x)$ and $Q(x)$ are polynomials, is continuous in an interval which does not include values of x for which $Q(x)$ is zero.

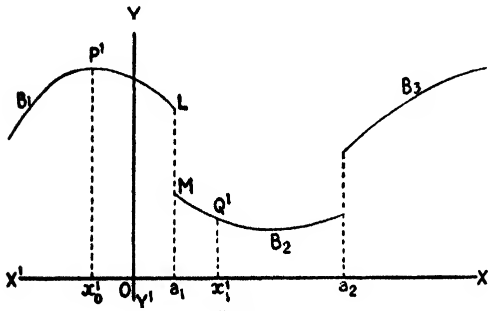


FIG. 4.

THEOREM V.—*If* $f(x)$ *is continuous at* $x = a$ *and* $f(a) > 0$ *then there exists a neighbourhood of* $x = a$ *in which* $f(x) > 0$.

Write $f(a) = k > 0$. Since $f(x)$ is continuous at $x = a$,

$$|f(x) - k| < \epsilon, \text{ where } |x - a| \leqslant \eta,$$

ϵ being arbitrary. Thus $k - \epsilon < f(x) < k + \epsilon$, for $|x - a| \leqslant \eta$. Write $\epsilon = \frac{1}{2}k$; then $\frac{1}{2}k < f(x) < \frac{3}{2}k$.

in the range $a - \eta \leqslant x \leqslant a + \eta$.

It will be observed that η depends upon ϵ and if we write $\epsilon = \frac{1}{2}k$, η will depend on k.

A similar theorem holds if $f(a) < 0$. In this case there is a neighbourhood in which $f(x)$ is negative.

2·73. The Fundamental Property of a Continuous Function

From graphical considerations we can obtain some idea as to the properties which would be expected of a continuous function. Suppose that the notion of a *continuous wave* is understood. Thus in Fig. 3 the wave $A_1A_2A_3$ is a continuous wave, while in Fig. 4

the wave $B_1\ B_2\ B_3$ is *discontinuous* at $x = a_1$ and $x = a_2$. We regard the latter wave as discontinuous because there are *sudden changes* in the values of the plotted points near $x = a_1$ and $x = a_2$.

Suppose that P and Q are two points on a continuous wave whose equation is $y = f(x)$, the corresponding values of x, y being x_0, y_0 and x_1, y_1 respectively. To make the argument more precise suppose that Q is above P so that $y_1 = f(x_1) > f(x_0) = y_0$. Then as x moves from x_0 to x_1 along the X-axis the point tracing the wave will move from P to Q. If the wave is continuous we would expect *y to take all values between y_0 and y_1*.

Next if P', Q' be two points on a discontinuous curve, the corresponding values of x, y being x_0', y_0' and x_1', y_1' respectively then it is clear that as x moves from x_0' to x_1' along the X-axis y need not take all values between y_0' and y_1'. Thus in Fig. 4 there is a gap in the values represented by LM.

The discussion given above makes it clear that if we can represent a continuous function $f(x)$ by a continuous wave in the sense we have indicated, the continuous function will have the following property:

THEOREM VI.—*If x_0, x_1 are two points in an interval of continuity of $f(x)$, then as x ranges from x_0 to x_1, $f(x)$ will take at least once all values between* $f(x_0)$ and $f(x_1)$.*

2·8. Bounds of a Function

A function $f(x)$ is said to be *bounded above* in the interval $\alpha \leqslant x \leqslant \beta$ if for all values of x in the interval $f(x) < A$, where A is a constant. Similarly $f(x)$ is *bounded below* if there exists a constant B such that for all values of x in the interval $f(x) > B$. The function is said to be *bounded* if it is bounded *above and below*.

The upper bound of $f(x)$ in $\alpha \leqslant x \leqslant \beta$ is a number M with the following properties:

(i) $f(x) \leqslant M$ at all points of $\alpha \leqslant x \leqslant \beta$.

(ii) There exists a value of x in (α, β) such that $f(x) > M - \epsilon$, where ϵ is an arbitrary positive number.

Similarly m is the lower bound if $f(x) \geqslant m$ for all points and $f(x) < m + \epsilon$ for some point, in the range.

* For a rigid proof of this theorem reference may be made to Hardy, *Course of Pure Mathematics*.

THEOREM VII. *If $f(x)$ is continuous in a closed interval* (α, β) *then $f(x)$ is bounded in* (α, β), *i.e* there exist numbers m, M such that $m \leqslant f(x) \leqslant M$ for all values of x in (α, β).*

THEOREM VIII.—*If $f(x)$ is continuous in the closed interval* (α, β) *and m and M are its lower and upper bounds respectively then there exist values a, b of x, lying in the interval* (α, β) *such that* $f(a) = m$, $f(b) = M$.

Let $\epsilon > 0$ be chosen. Then since M is the upper bound of $f(x)$ in (a, b) there exist a value of x in (a, b) such that

$$f(x) > M - \epsilon, \text{ i.e. } M - f(x) < \epsilon.$$

Suppose that there is no value of x in (α, β) such $f(x) = M$. Thus $M - f(x) > 0$ and since $\epsilon > 0$ we have

$$1/\{M - f(x)\} > 1/\epsilon.$$

Since ϵ is arbitrary it follows that $1/\{M - f(x)\}$ is unbounded. From Theorem VII it follows that $1/\{M - f(x)\}$ is not continuous.

Since $f(x)$ is continuous and M is a constant it follows that $M - f(x)$ is continuous (Theorem I). Hence provided $M - f(x) \neq 0$, $1/\{M - f(x)\}$ is continuous (Theorem III). Thus we have a contradiction and hence there is a value b of x such that $f(b) = M$.

Similarly by using the property that there exists a value of x such that $f(x) < m + \epsilon$ we can show that there exists a value a of x lying in (α, β) such that $f(a) = m$.

Definition.—The number $M - m$ is called the *oscillation* of $f(x)$ in (α, β). Clearly this number is a function of the interval (α, β).

2·9. Monotonic Functions

Let $f(x)$ be a function of x defined in an interval (α, β), x_1, x_2 any two values of x such that $\alpha \leqslant x_1 < x_2 \leqslant \beta$. Then if

$$f(x_1) \leqslant f(x_2) \text{ for all } x_1, x_2$$

then $f(x)$ is said to be *monotonic increasing* in (α, β).

Similarly if $f(x_1) \geqslant f(x_2)$ at all points such that

$$\alpha \leqslant x_1 < x_2 \leqslant \beta$$

then $f(x)$ in *monotonic decreasing* in (α, β).

Some of the properties of monotonic functions may be inferred by comparison with those of monotonic sequences. Thus, *e.g.*, if $f(x)$ is monotonic increasing and is bounded above then $f(x)$ is bounded. For it is bounded below by $f(\alpha)$.

* For a rigid proof of this theorem reference may be made to Hardy, *Course of Pure Mathematics*.

As in the case of continuity the monotonic property may hold in an open or closed interval. Thus, *e.g.*, it is not difficult to show that if $f(x)$ *is bounded and monotonic in the open interval* (α, β) then $\lim\limits_{x \to \alpha + 0} f(x)$, $\lim\limits_{x \to \beta - 0} f(x)$ both exist. Further if $|f(x)| < A$ in *the open interval* (α, β)

$$\lim_{x \to \alpha + 0} f(x) \geqslant -A, \qquad \lim_{x \to \beta - 0} f(x) \leqslant A.$$

2·91. Discontinuous Functions

A function which is not continuous at a point is said to be *discontinuous* at the point. Discontinuities may be divided into two classes. We consider the function $f(x)$ and suppose that $x = a$ is a point of discontinuity.

(1) $|f(x)| < A$ at all points of a neighbourhood of a, A being a fixed positive number.

(2) There is no number A with the property defined in (1).

The former are called *ordinary* discontinuities and the latter *infinite* discontinuities.

The following are types which may occur in class (1):

(*a*) $\lim\limits_{x \to a + 0} f(x) = \lim\limits_{x \to a - 0} f(x) \neq f(a)$, or $f(a)$ not defined.

Examples.—$f(x) = x \sin \dfrac{1}{x}$, and $\begin{matrix} x > 0 \\ \text{or } x < 0. \end{matrix}$

$$f(0) = 1.$$

It is easily seen that as $x \to 0$ both right- and left-hand limits exist and are equal to zero.

(*b*) $\lim\limits_{x \to a + 0} f(x) \neq \lim\limits_{x \to a - 0} f(x)$.

Example.—$f(x) = x - [x]$ as $x \to 1$. See § 2·65.

(*c*) One or both of the one-sided limits does not exist.

Example.—$f(x) = \cos \dfrac{1}{x}$, at $x = 0$.

The following types occur in class (2):

(*a*) $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$.

Example.—$f(x) = 1/x^2$ at $x = 0$.

(*b*) $\lim_{x \to a+0} f(x) = \pm \infty$, $\lim_{x \to a-0} f(x) = \mp \infty$, unlike signs corresponding.

Example.—$f(x) = \tan x$, at $x = \frac{1}{2}\pi$.

$$\lim_{x \to \frac{1}{2}\pi - 0} \tan x = +\infty, \quad \lim_{x \to \frac{1}{2}\pi + 0} \tan x = -\infty.$$

In cases (*a*), (*b*) of this class the function $f(x)$ is said to *become infinite* at $x = a$.

(*c*) One or both of the one-sided limits does not exist.

Example.—$f(x) = \dfrac{1}{x} \sin \dfrac{1}{x}$, $x \neq 0$.

Clearly neither of the one-sided limits exists as $x \to 0$ and the function *oscillates infinitely* at $x = 0$.

2.92 Sequences whose Terms are Functions of a Variable

In the infinite sequences $\{u_n\}$ so far considered we have assumed that each member u_n has been defined with reference to n, *i.e.* to one variable only. Suppose now that u_n depends on a variable x so that we can write the sequence as $\{u_n(x)\}$. For each value of n, $u_n(x)$ is assumed to be defined for some range of values of x. The limit function as $n \to \infty$ will thus depend on x and we write it as $u(x)$, i.e. $u(x) = \lim_{n \to \infty} u_n(x)$. Consider carefully what is meant by this limit. The variable x may have *any* value in the given range and this value must have been inserted *before* we allow n to tend to ∞. The following five sequences will serve as examples.

(i) $u_n(x) = n/(n + x)$, $0 \leqslant x \leqslant k$, k being any fixed positive number.

(ii) $u_n(x) = nx/(1 + n^2x^2)$, $-k \leqslant x \leqslant k$, k as in (i).

(iii) $u_n(x) = n/(1 + nx)$, $0 \leqslant x \leqslant k$, k as in (i).

(iv) $u_n(x) = x^n$, $0 \leqslant x \leqslant 1$.

(v) $u_n(x) = 1/(1 + nx)$, $0 \leqslant x \leqslant k$, k as in (i).

The general principle of convergence tells us that corresponding to the arbitrary positive number ϵ there exists a number n_0 such that
$$|u_n(x) - u(x)| < \epsilon, \; n \geqslant n_0.$$

We consider each sequence in turn, finding first the value of $u(x)$ and then determining a suitable value of n_0.

(i) $u_n(x) = n/(n + x)$, $0 \leqslant x \leqslant k$.

$$u(x) = \lim_{n \to \infty} u_n(x) = 1, \text{ the result being true for } 0 \leqslant x \leqslant k.$$

The inequality $| u_n(x) - u(x) | < \epsilon$ is equivalent to

$$\left| \frac{n}{n + x} - 1 \right| < \epsilon, \text{ or } x < n\epsilon + x\epsilon.$$

Hence, $$n > x(1 - \epsilon)/\epsilon.$$

Thus we can take n_0 to be the first positive integer greater than $x(1 - \epsilon)/\epsilon$.

Observe that n_0 depends on x as well as on ϵ. But in this case we can choose our n_0 so that it is *independent of* x. $x(1 - \epsilon)/\epsilon$ will have its maximum value when $x = k$. Hence if we take n_0 to be the first integer greater than $k(1 - \epsilon)/\epsilon$ we have obtained a number n_0 which is the *same for all values of* x in the given interval.

(ii) $u_n(x) = nx/(1 + n^2x^2)$, $-k \leqslant x \leqslant k$.

If $x = 0$, $u_n(x) = 0$, giving $u(x) = 0$.

If $x \neq 0$, $u_n(x) = \dfrac{1}{\dfrac{1}{nx} + nx}$.

If $x > 0$, $nx \to \infty$, $u_n(x) \to 0$.

If $x < 0$, $nx \to -\infty$, $u_n(x) \to 0$.

Thus $u(x) = 0$, $-k \leqslant x \leqslant k$.

The condition $| u_n(x) - u(x) | < \epsilon$ then gives

$$\frac{nx}{1 + n^2x^2} < \epsilon, \text{ i.e. } \frac{ny}{1 + n^2y^2} \quad \epsilon, \text{ where } y = | x |.$$

Suppose $x \neq 0$ so that $0 < y \leqslant k$.

Then $$\frac{ny}{1 + n^2y^2} = \frac{1}{ny} \frac{1}{1 + \frac{1}{n^2y^2}} < \frac{1}{ny} < \epsilon,$$

provided $$n > 1/\epsilon y, \text{ i.e. } n > 1/\epsilon | x |.$$

For any particular value of $x = x_0 \neq 0$ we can choose n_0 such that $\left| \dfrac{nx}{1 + n^2x^2} \right| < \epsilon$. In particular we can take n_0 to be the first integer greater than $1/\epsilon | x_0 |$.

If $x = 0$ the original inequality is true for all values of n since $u_n(0) = u(0)$.

Thus we have found for each value of x in the range $-k \leqslant x \leqslant k$ a suitable value of n_0. As in Example (i) n_0 depends on x. But there is a fundamental difference between the n_0 of Example (i) and that of Example (ii). Since $1/\epsilon\,|x| \to \infty$ as $|x| \to 0$ it is *not possible* to choose in (ii) a value of n_0 which will serve for *all values of* x in the given interval. Observe that this is true even though the limit function $u(x)$ is the same, in this case, for all values of x in the given interval.

(iii) $$u_n(x) = n/(1 + nx),\ 0 \leqslant x \leqslant k.$$

Now $$u_n(0) = n \to \infty \text{ with } n.$$

Hence $$\lim_{n \to \infty} u_n(0) \text{ does not exist.}$$

When $$x > 0,\ u_n(x) = \frac{1}{\frac{1}{n} + x} \to x \text{ as } n \to \infty.$$

Thus $$u_n(x) = 1/x,\ x > 0.$$

The condition $\left|\dfrac{n}{1 + nx} - \dfrac{1}{x}\right| < \epsilon,\ x > 0$ is equivalent to $1 < \epsilon x(1 + nx)$, i.e. $n > \dfrac{1}{x}\left(\dfrac{1}{\epsilon x} - 1\right)$.

Thus we can choose n_0 to be the first integer greater than $\dfrac{1}{x}\left(\dfrac{1}{\epsilon x} - 1\right)$. As $x \to 0$, this expression tends to infinity as might be expected since the sequence does not converge for $x = 0$. But if t is any positive number as small as we please, then corresponding to the closed interval $t \leqslant x \leqslant k$ we can find an n_0 which is *independent* of x, *i.e.* the same n_0 will serve for all values of x in the interval. For we can take n_0 to be the first integer greater than $\dfrac{1}{t}\left(\dfrac{1}{t\epsilon} - 1\right)$, $t \leqslant x \leqslant k$.

(iv) $u_n(x) = x^n,\ 0 \leqslant x \leqslant 1$.

For $$0 \leqslant x < 1,\ \lim_{n \to \infty} x^n = 0, \text{ while } u_n(1) = 1.$$

Hence $$\begin{aligned} u(x) &= 0,\ 0 \leqslant x < 1, \\ &= 1,\ x = 1. \end{aligned}$$

We now have a new type of limit function. It exists for all values of x in the given range and has an ordinary discontinuity at $x = 1$.

The condition $u_n(x) - u(x) \mid < \epsilon$ requires

(*a*) $x^n \quad \epsilon$ for $0 \leqslant x < 1$, and (*b*) $0 < \epsilon$ for $x = 1$. The latter is true for all values of n.

For (*a*) we require $n > \log(1/\epsilon)/\log(1/x)$ and as $x \to 1$, $\log(1/x) \to 0$, $n \to \infty$. Hence it is impossible to choose an n_0 which is independent of x for x lying in the range $0 \leqslant x \leqslant 1$. Perhaps this might be expected since the limit function is discontinuous.

(v) $u_n(x) = 1/(1 + nx)$, $0 \leqslant x \leqslant k$.

Now $u_n(0) = 1$, so that $u(0) = 1$. When $x > 0$, $nx \to \infty$ as $n \to \infty$ so that $u_n(x) = 0$. The limit function is again discontinuous and as in Example (iv) it is easily seen that we cannot find a number n_0 independent of x such that $\mid u_n(x) - u(x) \mid < \epsilon$, $n \geqslant n_0$.

In Examples (i), (iii), (iv), (v) the sequences converge for each value of x in the given range. A suitable number n_0 corresponding to the arbitrary ϵ was found and was a function of x as well as of ϵ. In Example (i) and (i) only it was possible to take the analysis further and assert that n_0 can be chosen *independent* of x, *i.e.* the *same* n_0 can serve *for all values of x in the given range.* Where this additional property exists the convergence is said to be **uniform.** Thus we have shown that in Example (i) convergence is uniform whereas in Examples (iii), (iv), (v) convergence is non-uniform.

Note 1.—In Examples (ii)-(v) it is possible to choose sub-intervals of the given range of values of x in which the sequences are uniformly convergent. In each case the new intervals are closed, *i.e.* the end points of the intervals are included. These can be as follows:—

	Given Interval	*Interval of Uniform Convergence*
Ex. (ii)	$-k \leqslant x \leqslant k$, $k > 0$	$\left.\begin{matrix} t \leqslant x \leqslant k \\ -k \leqslant x \leqslant -t \end{matrix}\right\} 0 < t < k.$
Ex. (iii)	$0 \leqslant x \leqslant k$, $k > 0$	$t \leqslant x \leqslant k$, $0 < t < k$.
Ex. (iv)	$0 \leqslant x \leqslant 1$	$0 \leqslant x \leqslant s$, $0 < s < 1$.
Ex. (v)	$0 \leqslant x \leqslant k$	$t \leqslant x \leqslant k$, $0 < t < k$.

Note 2.—Each u_n is a continuous function of x in the range of x given. The limit function $u(x)$ is continuous in Examples (i), (ii), does not exist at all points in the case of Example (iii) and is discontinuous in the case of Examples (iv), (v). Thus the limit of a sequence of continuous functions need not exist, and where it does exist, need not be continuous. If we take a sub-interval in which $\{u_n(x)\}$ is uniformly convergent the limit functions all become continuous. This suggests that if $u_n(x)$ is continuous then uniform convergence is a sufficient condition to ensure that $u(x)$ is continuous. But the condition is not necessary. For in Example (ii) $u(x)$ is continuous in the interval $-k \leqslant x \leqslant k$ but $u_n(x)$ is not uniformly convergent in any interval which includes $x = 0$.

Uniform convergence will be considered in detail in Chap. IV, where there will be references to some of the above examples in connection with convergence of series.

EXERCISES II

1. Prove that, if $\phi(n) \to a$ and $\psi(n) \to b$ as $n \to \infty$, then

$$\phi(n)\ \psi(n) \to ab.$$

Prove that, as n tends to infinity, the function $(n+1)^n/n^{n+1}$ tends to a limit and state the value of the limit. [*M.T.*]

2. Find the limit as x tends to zero through positive values of

$$(ax^{\frac{3}{2}} + px^{\frac{5}{2}} + qx^3)/(lx^{\frac{3}{2}} + mx^{\frac{5}{2}})$$

3. Prove that $\lim\limits_{x \to 0} \dfrac{(1+x)^3 - 3(1+x)^2 + 2(1+x)}{(1+x)^4 - (1-x)^3} = -\frac{1}{7}$.

4. Prove that if p, q, r, s are positive integers, $\lim\limits_{x \to 1} \dfrac{x^p - x^q}{[illegible]} = \dfrac{p}{[illegible]}$

5. Find the limiting values of the following expressions as h tends to the limit zero:

(i) $\{(x+h)^3 - x^3\}/h$;

(ii) $\left\{\dfrac{1}{x} - \dfrac{1}{x+h}\right\} \Big/ h$.

6. Find the limiting value of the expression

$$f(x) = (x^3 - 3x^2 + 2)/(x^2 - 2x + 1),$$

(i) when x tends to unity, (ii) when x tends to infinity.

7. Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{(a+x)} - \sqrt{(a-x)}}{\sqrt{(a^2 - x^2)} + x - a}$.

8. Prove that if n is a positive integer,

$$\lim_{x \to 1} \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} = \tfrac{1}{2}n(n+1).$$

9. Prove that if r be a positive integer,

$$\lim_{n \to \infty} \left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right) \dots \left(x - \frac{r}{n}\right) = x^r.$$

Show also that the expression lies between the values x^r and $x^r - r(r+1)x^{r-1}/2n$ for positive n.

10. If $u_n = f(n)x^n/n!$, $v_n = f(n)x^n$, where $f(n)$ is a polynomial in n, prove (i) that Σu_n converges for all values of x; (ii) that Σv_n converges if $|x| < 1$.

11. If $f(n) \to l$, $l > 0$, as $n \to \infty$, prove that $\{f(n)\}^r \to l^r$, where r denotes a positive or negative integer.

12. Prove that if m and n are positive integers, $m > n$, then the following inequalities are true:

$$\text{(i)} \quad \frac{x^m - 1}{m} > \frac{x^n - 1}{n}, \quad x > 1;$$

$$\text{(ii)} \quad \frac{1 - x^m}{m} > \frac{1 - x^n}{n}, \quad 0 < x < 1.$$

Show that the inequalities are also true when m and n are *positive rational numbers*.

13. Apply Ex. 12 to prove that $f(n) = n(x^{\frac{1}{n}} - 1)$ is a monotonic function of n, $x > 0$, and deduce that $\lim_{n \to \infty} f(n)$ exists.

Prove also that if $\phi(x) = \lim_{n \to \infty} n(x^{\frac{1}{n}} - 1)$ then $\phi(x)$ satisfies the functional relations

$$\text{(i)} \quad \phi(xy) = \phi(x) + \phi(y);$$

$$\text{(ii)} \quad \phi(x) = -\phi\left(\frac{1}{x}\right).$$

14. Discuss the behaviour of the following functions as x tends to infinity:

(i) $1 + x^{-2}$;

(ii) $x - [x]$, where $[x]$ denotes the integral part of x;

(iii) $\sin x\pi$;

(iv) $(\sin x\pi)/x$.

15. Prove that $\sqrt{x}$ is continuous for all finite values of $x > 0$.

16. Show that the rational function $(x^2 + 3)/(x^2 - 4x + 3)$ is discontinuous in the range $0 \leqslant x \leqslant 4$.

At what points is the function discontinuous?

17. Discuss the continuity of the function $\{(x - a)/(b - x)\}$.

18. Discuss the continuity of the function $\log(1 + x) \sin \frac{1}{x}$ in the interval (0, 1). [*Camb. Sch.*]

19. Evaluate the following limits:

$$\text{(i)} \lim_{x \to a} \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{2}} - a^{\frac{1}{2}}}, \quad \text{(ii)} \lim_{x \to 1} \frac{4x^5 - 5x^4 + 1}{(x - 1)^2}.$$ [*Camb. Sch.*]

20. Find the limiting value of $\dfrac{2x^3 - 7ax^2 + 4a^2x + 4a^3}{x^3 - 3ax^2 + 4a^3}$ when $x \to 2a$.

21. (i) Find the values of

$$\lim. \frac{3x^2 - 7x + 4}{x^2 - 5x + 3}. \quad \lim. \frac{(1 + x)^{\frac{1}{2}} - (1 - x)^{\frac{1}{2}}}{x}.$$

[*N.Sc.*]

22. Prove that if

$$\text{(i) } s_n = \sum_{r=1} a_r \to l, \text{ as } n \to \infty,$$

$$\text{then (ii) } \frac{1}{n}(s_1 + s_2 + \ldots + s_n) \to l, \text{ as } n \to \infty.$$

If (iii) $na_n \cdot\ 0$, as $n \to \infty$,
prove that (ii) implies (i). [*Camb. Sch.*]

23. The sequence $\{u_n(x)\}$ is defined by $u_n(x) = x^{n-1} - x^n$, $0 < x < 1$. Prove that the sequence converges at all points in $0 < x < 1$ but not uniformly.

CHAPTER III

THE BINOMIAL THEOREM FOR A RATIONAL INDEX

IN this chapter we first consider the binomial theorem for a rational index and then proceed to discuss properties of the coefficients. The determination of a binomial function from the expansion is treated at a later stage in the chapter.

3·1. The Binomial Expansion

The binomial expansion:

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1}{r!}x^r \ldots$$

$$= \Sigma \binom{n}{r} x^r$$

was considered in Vol. I, Chaps. XXV and XXVI. The expansion represents the function $(1 + x)^n$ and the proof for n a positive integer, when the series has only a finite number of terms, is given in Chap. XXV. When n is not a positive integer the expansion has an infinite number of terms and the conditions for convergence have been stated in Chap. XXVI. It was proved in Vol. II, Chap. I, § 1·52 that the series converges absolutely provided $|x| < 1$ and diverges for $|x| > 1$. We now prove that when $|x| < 1$ the sum function is $(1 + x)^n$, where n is any rational number.

Write $f(n) = \sum\limits_{r=0}^{\infty} \binom{n}{r} x^r$, $f(m) = \sum\limits_{r=0}^{\infty} \binom{m}{r} x^r$. We first prove that

$$f(m + n) = f(m) f(n).$$

Since each series converges absolutely for $|x| < 1$, the infinite series obtained by considering the product of the two series also converges absolutely for $|x| < 1$. [Chap. I., § 1·6.] Hence:

$$f(m) f(n) = \sum^{\infty} c_r x^r$$

where $c_r = \binom{n}{r} + \binom{n}{r-1}\binom{m}{1} + \binom{n}{r-2}\binom{m}{2} + \ldots + \binom{m}{r}$

$$= \sum_{p=0}^{r} \binom{n}{r-p}\binom{m}{p}.$$

Now if m and n are positive integers, $f(m) = (1+x)^m$,

$$f(n) = (1+x)^n,\ f(m)\,f(n) = (1+x)^{m+n} = \Sigma\binom{m+n}{r}x^r.$$

Hence if m and n are positive integers $> r$, $c_r = \binom{m+n}{r}$.

Now the equation

$$\sum_{p=0}^{r}\binom{n}{r-p}\binom{m}{p} = \binom{m+n}{r}^{*} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

is of the rth degree in m and n. Again the equation is satisfied by more than r values of m and similarly for n. For clearly we can take *more than* r positive integral values, each greater than r, for m or n and for each value (i) is true.

Now an equation of the rth degree which is satisfied by more than r values of the variable must be an *identity*. Thus (1) is true for *all values of m and n*. Hence

$$c_r = \binom{m+n}{r} \text{ for all values of } m \text{ and } n.$$

$$\text{and } f(m)\,f(n) = \sum_{r=0}^{\infty}\binom{m+n}{r}x^r = f(m+n).$$

Similarly

$$f(m) \times f(n) \times f(u) \times \ldots = f(m+n+u+\ldots)\ldots\text{(ii)}$$

Now let $n = q/s$ denote any positive rational number, q and s being positive integers. In (ii) write $m = n = u \ldots = q/s$, s values in all being considered. Then

$$f\left(\frac{q}{s}\right) \times f\left(\frac{q}{s}\right) \times \ldots \text{ to } s \text{ factors} = f\left(\frac{q}{s}+\frac{q}{s}+\ldots \text{ to } s \text{ terms}\right),$$

$$\text{i.e. } \left\{f\left(\frac{q}{s}\right)\right\}^{s} = f(q) = (1+x)^q,$$

* This result expressed by this equation is known as Vandemonde's Theorem.

since q is a positive integer. Taking the sth root of both sides of the equation

$$f\left(\frac{q}{s}\right) = (1 + x)^{\frac{q}{s}},$$

$$\text{i.e.}\quad (1 + x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r, \quad |x| < 1,$$

where n is any positive rational number.

It remains to extend the result to negative rational numbers. Since $f(m + n) = f(m) \times f(n)$ for all values of m, n it follows that

$$f(n) \times f(-n) = f(0) = 1;$$

$$\therefore \quad f(n) = 1/f(-n).$$

Now let n be a negative rational number, $= -q/s$, where q and s are positive integers. Then

$$f\left(-\frac{q}{s}\right) = 1/f\left(\frac{q}{s}\right) = 1/(1 + x)^{\frac{q}{s}}, \text{ since } \frac{q}{s} > 0,$$

$$\text{i.e.}\quad f\left(-\frac{q}{s}\right) = (1 + x)^{-\frac{q}{s}}. \quad \text{Hence } (1 + x)^n = f(n).$$

i.e. $(1 + x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r$, $|x| < 1$ where n is any rational number.

3·11. Some Properties of the Expansion

There are a few points about the expansion which are worth emphasising.

(*a*) The proof depends essentially on the fact that the series $\Sigma \binom{n}{r} x^r$ *converges absolutely* for $|x| < 1$.

(*b*) The theorem can be extended in certain cases when $|x| = 1$. The convergence conditions for the expansion in this case are discussed in Vol. II., Chap. I., § 1·52.

(*c*) If we wish to expand a binomial of the form $(a + b)^n$, where n is not a positive integer we must first write it in the form $(1 + x)^n$ where $|x| < 1$. Thus

$$(a + b)^n = a^n \left(1 + \frac{b}{a}\right)^n = a^n \sum_{r=0}^{\infty} \binom{n}{r} \left(\frac{b}{a}\right)^r,$$

provided $\left|\frac{b}{a}\right| < 1$, i.e. $|a| > |b|$. Again

$$(a+b)^n = b^n\left(1+\frac{a}{b}\right)^n = b^n \sum_{r=0}^{\infty} \binom{n}{r}\left(\frac{a}{b}\right)^r$$

provided $\left|\frac{a}{b}\right| < 1$, i.e. $|a| < |b|$.

(*d*) The $(r+1)$th term of the expansion is

$$\binom{n}{r} x^r \quad \text{i.e.} \quad \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^r,$$

the same form as is obtained in the case in which n is a positive integer. This $(r+1)$th term may be regarded as the general term.

3·12. Particular Values of n.

The special form of the binomial in the case of $(1-x)^{-n}$, when n and x are *positive*, is worth special note. Thus

$$(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r}(-x)^r, \quad |x| < 1.$$

$$\text{Now } (-1)^r\binom{-n}{r} = (-1)^r \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!}$$

$$= (-1)^{2r}\frac{n(n+1)\ldots(n+r-1)}{r!}$$

$$= \frac{n(n+1)\ldots(n+r-1)}{r!}.$$

Thus if *n is positive all the coefficients are positive.*

The expansions in the cases in which $n = \frac{1}{2}, \frac{1}{3}, 1, 2, 3$ are given below. Thus

(*a*) $(1-x)^{-\frac{1}{2}} = \Sigma \dfrac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots(\frac{1}{2}+r-1)}{r!} x^r$

$$= \Sigma \frac{1\cdot3\cdot5\ldots(2r-1)}{2^r r!} x^r, \quad |x| < 1.$$

(*b*) $(1-x)^{-\frac{1}{3}} = \Sigma \dfrac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdot\ldots(\frac{1}{3}+r-1)}{r!} x^r$

$$= \Sigma \frac{1\cdot4\cdot7\ldots(3r-2)}{3^r r!} x^r, \quad |x| < 1.$$

(*c*) $(1-x)^{-1} = \Sigma x^r, \quad |x| < 1.$

(This form is already familiar from the theory of geometric progressions.)

$$(d)\quad (1-x)^{-2} = \Sigma \frac{2.3 \ldots (r+1)}{r!} x^r$$

$$= \Sigma (r+1) x^r, \qquad |x| < 1.$$

$$(e)\quad (1-x)^{-3} = \Sigma \frac{3.4 \ldots (r+1)(r+2)}{r!} x^r$$

$$= \Sigma \tfrac{1}{2} (r+1)(r+2) x^r, \qquad |x| < 1.$$

It is important that the student should be able to recognise a binomial expansion when it occurs. A study of the form of the expansions given above should be of assistance in this connection. In particular if all the terms of a given series are positive and a binomial series is suspected, the form $(1-x)^{-n}$ should be first examined. [*Cf.* § 3·61.]

3·2. Signs of the Coefficients

Next consider the signs of the coefficients in the expansions of $(1+x)^n, x > 0$. Since $\binom{n}{r}$ is obtained from $\binom{n}{r-1}$ by multiplying by the factor $(n-r+1)/r$ it is clear that as soon as $r > n+1$ the factor will be negative. Thus for $1 \leqslant r < n+1$, the coefficient $\binom{n}{r}$ will be positive, while for $r > n+1$, the coefficients will be alternately negative and positive. If $n < 0$ so that there are *no values of r* satisfying $1 \leqslant r < n+1$ the first term is unity and the coefficients of $x, x^2, \ldots$ are alternately negative and positive.

If $n > 0$ there will be a set of values of r satisfying

$$1 \leqslant r < n+1$$

so that in this case there will be a group of terms at the beginning of the expansion which are all positive. Once we pass beyond this group the terms will be alternately positive and negative. Thus, *e.g.*

$$(1+x)^{\frac{3}{2}} = 1 + \tfrac{3}{2} x + \frac{\frac{3}{2}\cdot\frac{1}{2}}{2!} x^2 + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot -\frac{1}{2}}{3!} x^3 + \ldots$$

The first three terms are positive and beyond that point are alternately negative and positive.

Finally consider the form $(1-x)^n$ where $x > 0$, $n > 0$. The case $n < 0$ has been considered in § 3·12. Thus

$$(1-x)^n = \Sigma \binom{n}{r} (-x)^r$$

so that the coefficient of x^r is $(-1)^r \binom{n}{r}$

The coefficient of x^r is obtained from the coefficient of x^{r-1} by multiplying by the factor $-(n-r+1)/r$. For $r > n+1$ this factor is positive. Hence *beyond a certain point in the series the coefficients of the powers of x all have the same sign.* Whether that sign is positive or negative depends on the value of n. Thus

$$(1-x)^{\frac{3}{2}} = 1 + \tfrac{3}{2}(-x) + \frac{\frac{3}{2}\cdot\frac{1}{2}}{2!}(-x)^2 + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot -\frac{1}{2}}{3!}(-x)^3$$
$$+ \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{4!}(-x)^4 + \ldots$$

$$= 1 - \tfrac{3}{2}x + \frac{3.1}{2^2.2!}x^2 + \frac{3.1.1}{2^3.3!}x^3 + \frac{3.1.1.3}{2^4.4!}x^4 + \ldots$$

In this case the coefficients of x^2 and all higher powers are *positive.*

Again $(1-x)^{\frac{5}{2}}$

$$= 1 + \tfrac{5}{2}(-x) + \frac{\frac{5}{2}\cdot\frac{3}{2}}{2!}(-x)^2 + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}{3!}(-x)^3$$
$$+ \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot -\frac{1}{2}}{4!}(-x)^4 + \ldots$$

$$= 1 - \tfrac{5}{2}x + \frac{5.3}{2^2.2!}x^2 - \frac{5.3.1}{2^3.3!}x^3 - \frac{5.3.1.1}{2^4.4!}x^4 - \ldots$$

In this case the coefficients of x^3 and higher powers of x are all *negative.*

Again *in the finite group of terms* which precedes the infinite group in which the coefficients of powers of x all have the same sign, it will be observed that signs of the coefficients of the powers of x *must alternate.* The expansions of $(1-x)^{\frac{3}{2}}$, $(1-x)^{\frac{5}{2}}$ illustrate this property.

3·21. Illustrative Examples

(1) *Obtain the binomial expansion of* $(1-x)^{\frac{1}{2}}$ *as far as the term in* x^4. *Prove that, if* a_n *denote the coefficient of* x^n, *and* $x < 1$,

$$8n(n+1)a_{n+1} - 12n(n-1)a_n + (2n-3)(2n-4)a_{n-1} = 0.$$

[*Lond. B.A.*]

$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{2!}x^2 - \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{3!}x^3 + \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}}{4!}x^4 + \dots$$

$$= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \dots$$

The coefficient of x^n is

$$(-1)^n \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\cdot \dots (\frac{1}{2} - n + 1)}{n!} = -\frac{1\cdot 3\cdot 5 \cdot \dots (2n-3)}{2^n\,(n!)}$$

Hence $a_n = -\dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5 \dots (2n-4)\,(2n-3)}{2^n\,(n!)\,.\,2\,.\,4\,.\,6 \dots (2n-4)} = \dfrac{-(2n-3)!}{2^{2n-2}\,(n!)\,(n-2)!}$.

Thus $a_{n+1} = \dfrac{-(2n-1)!}{2^{2n}\,(n+1)!\,(n-1)!}$, $a_{n-1} = \dfrac{-(2n-5)!}{2^{2n-4}\,(n-1)!\,(n-3)!}$;

$$\therefore\ 8n\,(n+1)a_{n+1} - 12n\,(n-1)a_n + (2n-3)\,(2n-4)a_{n-1}$$

$$= -\frac{(2n-1)!}{2^{2n-3}\,(n-1)!\,(n-1)!} + \frac{3\,(2n-3)!}{2^{2n-4}\,(n-2)!\,(n-2)!} - \frac{(2n-3)!}{2^{2n-4}\,(n-1)!\,(n-3)!}$$

$$= -\frac{(2n-3)!}{2^{2n-4}\,(n-2)!\,(n-3)!}\left\{\frac{(2n-1)\,(2n-2)}{2\,(n-1)^2\,(n-2)} - \frac{3}{n-2} + \frac{1}{n-1}\right\}$$

$$= -\frac{(2n-3)!}{2^{2n-4}\,(n-2)!\,(n-3)!}\left\{\frac{2n-1}{(n-1)\,(n-2)} - \frac{3}{n-2} + \frac{1}{n-1}\right\}$$

$= 0$, since the final expression in brackets is zero.

(2) *Find the first term with a negative coefficient in the expansion of* $(1+x)^{\frac{11}{3}}$.

The $(r+1)$th term in the expansion of $(1+x)^{\frac{11}{3}}$ is

$$\frac{\frac{11}{3}\,.\,(\frac{11}{3}-1)\dots(\frac{11}{3}-r+1)}{r!}x^r.$$

The first negative term will occur for the least value of r such that

$$\tfrac{11}{3} - r + 1 < 0,\ \text{i.e.}\ r > 4\tfrac{2}{3}.$$

Thus the first negative term is obtained by taking $r = 5$ and its value is

$$\frac{\frac{11}{3}\,.\,\frac{8}{3}\,.\,\frac{5}{3}\,.\,\frac{2}{3}\,.\,-\frac{1}{3}}{5!}x^5 = -\frac{11\,.\,8\,.\,5\,.\,2\,.\,1}{3^5\,.\,5!}x^5.$$

(3) *Find the general term in the expansion of* $(1+x)^{-\frac{p}{q}}$.

The $(r+1)$th term is $\dfrac{-\frac{p}{q}\cdot\left(-\frac{p}{q}-1\right)\left(-\frac{p}{q}-2\right)\dots\left(-\frac{p}{q}-r+1\right)x^r}{r!}$

$$= (-1)^r\frac{p\,(p+q)\,(p+2q)\dots\{p+(r-1)q\}}{q^r\,.\,r!}x^r.$$

It will be observed that in this case the coefficients are always alternately positive and negative if $p > 0$, $q > 0$.

(4) *If* $x > 2$, *prove that*

$$1 + \frac{n}{x} + \frac{n\,(n+1)}{2!\,x^2} + \dots = 1 + \frac{n}{x-1} + \frac{n\,(n-1)}{2!\,(x-1)^2} + \dots$$

and deduce that

$$n + \frac{n(n-1)}{2!}\cdot\frac{(r-1)}{1!} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{(r-1)(r-2)}{2!} + \dots$$
$$= \frac{n(n+1)\dots(n+r-1)}{r!}. \qquad [M.T.]$$

$$1 + \frac{n}{x} + \frac{n(n+1)}{2!\,x^2} + \dots = \left(1 - \frac{1}{x}\right)^{-n} \text{ provided } |x| > 1.$$

$$\text{Now } \left(1 - \frac{1}{x}\right)^{-n} = \left(\frac{x-1}{x}\right)^{-n} = \left(\frac{x}{x-1}\right)^{n} = \left(1 + \frac{1}{x-1}\right)^{n}$$
$$= 1 + \frac{n}{x-1} + \frac{n(n-1)}{2!\,(x-1)^2} + \dots \text{ provided } |x-1| > 1.$$

This condition is satisfied if $x > 2$.

Now $\frac{n(n+1)\dots n+r-1)}{r!}$ is the coefficient of $1/x^r$ in the expansion of $\left(1 - \frac{1}{x}\right)^{-n}$. We require the coefficient of $1/x^r$ in the other expansion.

$$\text{Now } \frac{n}{x-1} = \frac{n}{x}\left(1 - \frac{1}{x}\right)^{-1} = \frac{n}{x}\sum_{s=0}^{\infty}\left(\frac{1}{x}\right)^s, \ |x| > 1,$$

and thus the coefficient of $1/x^r$ is n

Again $\frac{1}{(x-1)^2} = \frac{1}{x^2}\left(1 - \frac{1}{x}\right)^{-2} = \frac{1}{x^2}\sum^{\infty}(s+1)\frac{1}{x^s}$. Thus the coefficient of $1/x^r$ in $\frac{n(n-1)}{2!\,(x-1)^2}$ is $\frac{n(n-1)}{2!}\cdot\frac{(r-1)}{1!}$. Similarly for succeeding terms.

(5) *Expand* $\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$ *in ascending powers of* x.

Rationalising the denominator, we have

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \frac{(1+x) + (1-x) - 2\sqrt{1-x^2}}{(1+x) - (1-x)}$$
$$= \frac{2\{1 - (1-x^2)^{\frac{1}{2}}\}}{2x} = x^{-1}\{1 - (1-x^2)^{\frac{1}{2}}\}$$
$$= x^{-1} - x^{-1}(1-x^2)^{\frac{1}{2}}.$$

$$\text{Now } (1-x^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x^2 + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{2!}(-x^2)^2 + \dots$$
$$+ \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot \dots (\frac{1}{2} - r + 1)}{r!}(-x^2)^r + \dots$$

The $(r+1)$th term in the expansion is

$$(-1)^{r-1}\frac{1.1.3.\dots(2r-3)}{2^r.r!}.(-1)^r x^{2r}$$
$$= -\frac{1.3.\dots(2r-3)}{2^r.r!}x^{2r}, \text{ since } (-1)^{2r-1} = -1.$$

Hence the required expansion is

$$x^{-1} - x^{-1}\left(1 - \tfrac{1}{2}x^2 - \ldots - \frac{1.3.5.\ldots(2r-3)}{2^r.r!}x^{2r} - \ldots\right)$$

$$= \tfrac{1}{2}x + \sum_{r=2}^{\infty} \frac{1.3.5.\ldots(2r-3)}{2^r.r!}x^{2r-1}.$$

(6) *Prove that the series*

$$1 - \tfrac{1}{4} + \tfrac{1}{4}\cdot\tfrac{3}{8} - \tfrac{1}{4}\cdot\tfrac{3}{8}\cdot\tfrac{5}{12} + \ldots$$

is a binomial series; examine its convergence and find its value.

If u_n denote the nth term of the series

$$u_{n+1} = (-1)^n\frac{1.3.5.\ldots(2n-1)}{4.8.12.\ldots 4n} = (-1)^n\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}}{n!}\left(\tfrac{1}{2}\right)^n.$$

Now the binomial expansion of $(1+x)^{-\frac{1}{2}}$ is

$$\Sigma\,(-1)^n\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}}{n!}x^n$$

which is convergent provided $|x| < 1$. Writing $x = \frac{1}{2}$ we obtain the given series, which must therefore converge. Its sum is $(1+\frac{1}{2})^{-\frac{1}{2}} = \sqrt{\frac{2}{3}}$.

(7) *If B and C are independent of x, find their values in order that the expansion of*

$$\frac{1}{1-x} + \frac{B}{(1-2x)^{\frac{1}{2}}} + \frac{C}{(1-3x)^{\frac{1}{3}}}$$

in ascending powers of x may begin at the term in x^3, and obtain this term when B and C have the values found.

Now $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + \ldots$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$(1-2x)^{-\frac{1}{2}} = 1 + x + \frac{1.3}{2!}x^2 + \frac{1.3.5}{3!}x^3 + \ldots$$

$$(1-3x)^{-\frac{1}{3}} = 1 + x + \frac{1.4}{2!}x^2 + \frac{1.4.7}{3!}x^3 + \ldots$$

Hence

$$\therefore \frac{1}{1-x} + \frac{B}{(1-2x)^{\frac{1}{2}}} + \frac{C}{(1-3x)^{\frac{1}{3}}} = 1 + B + C + x(1+B+C)$$
$$+ x^2(1 + \tfrac{3}{2}B + \tfrac{4}{2}C) + x^3(1 + \tfrac{15}{6}B + \tfrac{28}{6}C) + \ldots$$

If the expansion is to begin with x^3, then

$$1 + B + C = 0, \text{ and } 1 + \tfrac{3}{2}B + 2C = 0.$$

Solving these equations we obtain $B = -2$, $C = 1$. The term in x^3 is

$$x^3(1 - \tfrac{15}{3} + \tfrac{14}{3}), \text{ i.e. } \tfrac{2}{3}x^3.$$

3·3. The Numerically Greatest Term in a Binomial Extension

Since we are concerned with *numerical* values only x may be assumed to be *positive*. Also it is clear that unless n is a positive integer we must restrict x to lie in the interval $-1<x<1$. For if $|x|>1$ the expansion of $(1+x)^n$ in ascending powers is not convergent. The case in which n is a positive integer is discussed in Vol. I., Chap. XXV., § 25·4. For convenience we restate the results proved there.

Write $p=(n+1)x/(x+1)$, $q=[p]$ where $[p]$ denotes the integral part of p. If $p=q$, i.e. p is an integer there are two greatest terms which are equal to one another, the qth and the $(q+1)$th terms. If $p \neq q$, the $(q+1)$th term is the greatest.

We now consider the case in which n is not a positive integer.

If $u_{r+1}=\binom{n}{r}x^r$ then u_{r+1} is obtained from u_r by multiplying by the factor $(n-r+1)x/r$. Hence

$$u_{r+1} \geqslant u_r \text{ numerically as } |n-r+1|\,x \geqslant r \quad \text{.......(i)}$$

$$\text{If } r<n+1,\ |n-r+1|=n-r+1,$$

$$\text{If } r>n+1,\ |n-r+1|=r-n-1.$$

In the former case the inequality (i) gives

$$r \leqslant (n+1)x/(x+1) \quad \text{..................(ii)}$$

In the latter case the inequality (i) reduces to

$$(r-n-1)x \geqslant r, \text{ i.e. } r(1-x) \leqslant -(n+1)x,$$

and since $1-x>0$ this is equivalent to

$$r \leqslant -(n+1)x/(1-x). \quad \text{..............(iii)}$$

If $n+1>0$ there will be no values of r satisfying (iii), so that it is only necessary to consider (ii). Write

$$p=(n+1)x/(x+1),\ q=[p].$$

If $p=q$, there are two greatest terms, *viz.* the qth and the $(q+1)$th, while if $p \neq q$, the $(q+1)$th term is the greatest.

In particular if in addition $n<0$, i.e. $1>n+1>0$, it follows that $0<\dfrac{(n+1)x}{x+1}<1$, since $0<x/(x+1)<1$. Hence $q=0$. In this case the first term is the greatest and the terms then steadily decrease.

If $n+1<0$, it is only necessary to consider (iii). Write $p_1=-(n+1)x/(1-x)>0$, $q_1=[p_1]$. If $p_1=q_1$, i.e. p_1 is an integer, the q_1th and (q_1+1)th terms are equal and are the greatest

terms. If $p_1 \neq q_1$, the greatest term is the $(q_1 + 1)$th term. Finally if $n + 1 = 0$, i.e. $n = -1$, $|u_r| = 1$ and the greatest term is the first, *i.e.* the terms steadily decrease.

Examples.—(1) *Find the numerically greatest term in the expansion of* $(1 + \frac{4}{7})^{-7}$.

Let u_r denote the rth term in the expansion of $(1 + \frac{4}{7})^{-7}$ Then

$$u_{r+1} = \frac{-7(-7-1)(-7-2)\dots(-7-r+1)}{r!}\left(\tfrac{4}{7}\right)^r$$

$$= (-1)^r\frac{7.8.9\dots(r+6)}{r!}\left(\tfrac{4}{7}\right)^r$$

$$u_r = (-1)^{r-1}\frac{7.8.9\dots(r+5)}{(r-1)!}\left(\tfrac{4}{7}\right)^{r-1}.$$

$$\frac{u_{r+1}}{u_r} = -\frac{r+6}{r}.\tfrac{4}{7}.$$

As we are considering the *numerically* greatest the negative sign need not be considered.

$$r + 6$$

$$\frac{u_{r+1}}{u_r} > 1 \text{ provided } 4r + 24 > 7r, \text{ i.e. } r < 8.$$

Hence u_8 and u_9 are numerically equal, both being the greatest. The numerical value of each term is

$$\frac{7.8.9.10.11.12.13.14}{8!}\left(\tfrac{4}{7}\right)^8$$

(2) *Find the greatest term in the expansion of* $(1 + x)^{-n}$, when $x = \frac{1}{5}$ and $n = 60$.

$$(1 + x)^{-n} = \sum_{r=0}^{\infty}\binom{-n}{r}x^r = \sum_{r=0}^{\infty}(-1)^r\frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r.$$

Denoting the $(r + 1)$th term by u_{r+1},

$$\left|\frac{u_{r+1}}{u_r}\right| = \frac{n+r-1}{r}|x| \quad \frac{59+r}{r}.\tfrac{1}{5}.$$

Hence $\left|\frac{u_{r+1}}{u_r}\right| > 1$ according as $59 + r > 5r$, i.e. $r < 14\frac{3}{4}$.

Hence the greatest term is the 15th.

(3) *Show that the coefficients of* x^n *in the expansions of* $(1 + x)^{\frac{5}{3}}$ and $(1 + ax)^{-\frac{1}{3}}$ *will be the same provided that* $a^n = 10/[(3n - 2)(3n - 5)]$.

Find a *when* $n = 5$, *and for this value of* a *calculate the ratio of the coefficients of* x^{10} *in the two expansions.*

$$(1 + x)^{\frac{5}{3}} = \Sigma\frac{\frac{5}{3}\cdot\frac{2}{3}\cdot-\frac{1}{3}\dots(\frac{5}{3}-r+1)}{r!}x^r.$$

$$(1 + ax)^{-\frac{1}{3}} = \Sigma\frac{-\frac{1}{3}\cdot-\frac{4}{3}\cdot\dots(-\frac{1}{3}-r+1)}{r!}a^r x^r.$$

Hence the coefficients of x^n are equal provided

$$a^n\left(-\tfrac{1}{3}-n+1\right)\left(\tfrac{2}{3}-n+1\right)=\tfrac{5}{3}\cdot\tfrac{2}{3}, \text{ i.e. } a^n=10/(3n-2)(3n-5).$$

When $n=5$, we have $a^5=\frac{10}{130}=\frac{1}{13}$

$$\text{Coefficient of } x^{10} \text{ in } (1+x)^{\frac{5}{3}} \text{ is } \tfrac{5}{3}\cdot\tfrac{2}{3}\cdot-\tfrac{1}{3}\cdot\ldots\cdot-\tfrac{22}{3}/10!$$

Coefficient of x^{10} in $(1+ax)^{-\frac{1}{3}}$ is

$$\tfrac{}{3}\cdot \qquad -\tfrac{22}{}\cdot-\tfrac{25}{3}\cdot-\tfrac{28}{3}\cdot a^{10}/10!$$

The ratio of the coefficients is

$$\frac{\frac{5}{3}\cdot\frac{2}{3}}{\frac{25}{3}\cdot\frac{28}{3}}\cdot\frac{1}{a^{10}}=\frac{169}{70}.$$

EXERCISES III

1. Show that if x be positive, and n unrestricted in value, the terms of $(1+x)^n$ will be alternately positive and negative from the term containing x^r, where r is the least integer greater than n.

Hence deduce that the terms of $(1+x)^{-n}$, where x and n are positive, are alternately positive and negative from the first term forwards.

2. Show that in the expansion of $(1-x)^n$, where x is positive and n a positive fraction, the terms will have the same sign from and after the term containing x^r, r being the least integer which exceeds n; and show that this sign will be positive or negative according as r is even or odd.

Expand each of the following expressions to four terms:

3. $(1-\frac{1}{2}x)^{\frac{1}{2}}$. 4. $(1+x)^{\frac{2}{3}}$. 5. $(a^2-x^2)^{-1}$. 6. $(a^2-ax)^{-\frac{3}{10}}$.

7. $(a-a^{-1}x^{-1})^{-3}$. $x^2)^{\frac{}{4}}$. 9. $\dfrac{1}{(\sqrt[3]{a}-\sqrt[3]{x})^6}$. 10. $(a^5-x^5)^{-\frac{1}{5}}$.

11. Expand $(1-x^2)^{\frac{5}{2}}/(1+x)^4$ in ascending powers of x as far as the term containing x^3.

12. Show that the coefficient of x^r where $r>4$ in the expansion of $(1+x)^5/(1-x)^2$ in ascending powers of x is $16(2r-3)$.

13. Find the coefficient of x^{4r} in $(1-x^2)^{-3}$.

14. If p and q be the rth terms of $(1\cdot x)^{-\frac{1}{2}}$ and $(1-x)^{-\frac{3}{2}}$, then $q=(2r-1)p$.

15. Show that all the coefficients of $(1+x)^{-n}$ are integers, if n be an integer.

What terms in the following expansions are first negative, x being positive?

16. $(1-x)^{\frac{7}{2}}$. 17. $(1+x)^{\frac{171}{10}}$. 18. $(1+x)^{\frac{20}{3}}$. 19. $(1+x)^{-3\frac{1}{3}}$

20. Find the ratio of the coefficient of the 11th term to that of the 10th term in the expansion of $(2+3x)^n$ in a series of ascending powers of x by the Binomial Theorem.

For what value of n is this ratio equal to $\frac{4}{5}$? When n has the value just obtained, if the rth term is the first term with a negative coefficient, what is r?

21. Show that

$$x = \tfrac{1}{2}\frac{2x}{1+x^2} + \frac{1}{2.4}\left(\frac{2x}{1+x^2}\right)^3 + \frac{1.3}{2.4.6}\left(\frac{2x}{1+x^2}\right)^5 + \dots,$$

if $-1 < x < 1$. What is the sum of the series when x is numerically greater than 1?

22. If x is positive and less than 1, prove that, if r and n are positive integers and $r < n$, the coefficient of x^n in the expansion of $x^r/(1+x)^r$ is $(-1)^{n-r}\,{}_{n-1}C_{r-1}$. Show that the coefficient of x^n in

$$1 + \frac{}{1+x} + \frac{}{(1+x)^2} + \frac{}{(1+x)^3} +$$

when each term is expanded, is zero except when $n = 1$.

23. From what term in each of the following expansions are all the terms of the same sign?

$$(1-x)^{\frac{40}{3}};\ (1-x)^{\frac{51}{2}};\ (1-x)^{-3}.$$

Find the greatest term in the following expansions:

24. $(1+x)^{\frac{3}{2}}$, when $x = \frac{5}{6}$.   25. $(1+\frac{2}{3})^{\frac{31}{5}}$.

26. $(1 - x/a)^{-\frac{4}{3}}$, when $x = 12$ and $a = 13$.

27. Show that the greatest term of $(\frac{1}{4}+\frac{1}{5})^{\frac{11}{3}}$ is $\frac{11}{900}\sqrt[3]{4}$.

3·4. Expansion of a Polynomial

Consider the expansion of

$$\{a_0 + a_1x + a_2x^2 + \dots + a_px^p\}^n$$

where n is any rational number. We assume in the first instance that $a_0 \neq 0$. Then

$$\{a_0 + a_1x + a_2x^2 + \dots + a_px^p\}^n$$

$$= a_0^n\left\{1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \dots + \frac{a_p}{a_0}x^p\right\}^n$$

$$= a_0^n\{1 + b_1x + b_2x^2 + \dots + b_px^p\}^n$$

where $b_s = a_s/a_0$, $s = 1, 2, \dots, p$. Write $y = \sum\limits_{s=1}^{p} b_sx^s$. Then provided $|y| < 1$, we may expand $(1+y)^n$ by the binomial theorem. Thus

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \dots$$
$$+ \frac{n(n-1)\dots(n-r+1)}{r!}y^r + \dots$$

The general term of the expansion is

$$\binom{n}{r}\{b_1x + b_2x^2 + \ldots + b_px^p\}^r$$

$$= \binom{n}{r} x^r \{b_1 + b_2x + \ldots + b_px^{p-1}\}^r.$$

Since r is a positive integer we may expand

$$\{b_1 + b_2x + \ldots + b_px^{p-1}\}^r$$

by the binomial theorem for positive integral index and obtain a series with a finite number of terms. When this is done for $r = 1, 2, 3 \ldots$ and the series arranged in powers of x we obtain a power series in x. Thus

$$\{a_0 + a_1x + a_2x^2 + \ldots + a_px^p\}^n = \Sigma c_r x^r.$$

If $a_0 = 0$ let a_q be the first coefficient which does not vanish. Then

$$\{a_qx^q + a_{q+1}x^{q+1} + \ldots + a_px^p\}^n$$

$$= a_q{}^n x^{qn}\left\{1 + \frac{a_{q+1}}{a_q}x + \ldots + \frac{a_p}{a_q}x^{p-q}\right\}^n$$

and we may proceed as before.

The above method is one way of expanding the polynomial. It is not the only way since the polynomial may be expressed in more than one form before expansion. See Example (1) below.

Examples.—(1) *Write down the first five terms in the expansion of* $(1 - 2x + 3x^2)^{-3}$ *in ascending powers of* x *and show that there is a range of values of* x *for which the expansion is valid.*

Write $y = 2x - 3x^2 = x(2 - 3x)$ so that the expansion is

$$(1 - y)^{-3} = \Sigma \tfrac{1}{2}(r + 1)(r + 2)y^r, \ |y| < 1,$$

$$= 1 + 3y + 6y^2 + 10y^3 + 15y^4 + \ldots$$

$$= 1 + 3x(2 - 3x) + 6x^2(2 - 3x)^2 + 10x^3(2 - 3x)^3 + \ldots$$

$$\begin{aligned} &= 1 + 6x - 9x^2 \\ &\qquad + 24x^2 - 72x^3 + 54x^4 \\ &\qquad\qquad + 80x^3 - 360x^4 + \ldots \\ &\qquad\qquad\qquad + 240x^4 + \ldots \end{aligned}$$

$$= 1 + 6x + 15x^2 + 8x^3 - 66x^4 \ldots$$

Now the function $y = 2x - 3x^2$ has a maximum equal to $\frac{1}{3}$ when $x = \frac{1}{3}$ and ranges between -1 and $+1$ as x ranges from $-\frac{1}{3}$ to $+1$. Hence the series converges absolutely for $|x| < \frac{1}{3}$ and for this range of values of x the expansion is valid. Observe that we have not proved that $|x| < \frac{1}{3}$ is the best possible range of convergence.

A second method of expansion is as follows:—

Write $$1 - 2x + 3x^2 = \tfrac{1}{3}[2 + (3x - 1)^2] = \tfrac{2}{3}\left[1 + \frac{(3x-1)^2}{2}\right].$$

Then $$(1 - 2x + 3x^2)^{-3} = \tfrac{27}{8}\,\Sigma\,(-1)^r\left(\frac{(r+1)(r+2)}{2}\,\frac{(3x-1)^{2r}}{2^r}\right).$$

This expansion in powers of $(3x - 1)$ is absolutely convergent provided $\dfrac{|3x-1|}{\sqrt{2}} < 1$, *i.e.* $|3x - 1| < \sqrt{2}$. Hence there is a range of values of x for which we can rearrange the expansion as a power series in x. This second method illustrates the point that the form of the expansion may be varied by rearranging the polynomial. An expansion can also be obtained by using series whose terms are complex numbers. Such series will be considered in a subsequent chapter and it will be sufficient to note at this stage the method and the assumptions made. Write $1 - 2x + 3x^2 = \left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)$ where α, β are the roots of the equation $1 \quad 2x + 3x^2 = 0$, *i.e.* $\alpha = \frac{1}{3}(1 + i\sqrt{2})$, $\beta = \frac{1}{3}(1 - i\sqrt{2})$.

Then $$(1 - 2x + 3x^2)^{-3} = \left(1 - \frac{x}{\alpha}\right)^{-3}\left(1 - \frac{x}{\beta}\right)^{-3}$$
$$= \left(\sum_{r=0}^{\infty} a_r x^r\right)\left(\sum_{s=0}^{\infty} b_s x^s\right),$$

where $$a_r = \frac{(r+1)(r+2)}{2\alpha^r},\quad b^r = \frac{(s+1)(s+2)}{2\beta^s}.$$

This step assumes the binomial theorem for complex numbers with index real, and that conditions for convergence are satisfied. As will be seen subsequently the conditions for absolute convergence of the binomial expansions are $\quad < 1$ and $\quad < 1$.

Next, assuming that the result for multiplication of absolutely convergent series of real terms (§ 1·6) applies also to complex series, then

$$\left(\sum_{r=0}^{\infty} a_r x^r\right)\left(\sum_{s=0}^{\infty} b_s x^s\right) = \sum_{n=1}^{\infty}$$

where $$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0.$$

Thus $(1 - 2x + 3x^2)^{-3}$ has been expanded as a power series in x. The coefficient c_n, although expressed in terms of complex numbers, must on simplification give a real number.

(2) *Prove that, when* $-1 < x < 1$,

$$\text{(i)}\quad (1 - x + x^2 - x^3)^{-1} = \sum_{n=0}^{\infty} (x^{4n} + x^{4n+1}),$$

$$\text{(ii)}\quad (1 + 2x + 3x^2 + 2x^3 + x^4)^{-1} = \sum_{n=0}^{\infty} (n + 1)(x^{3n} - 2x^{3n+1} + x^{3n+2})$$

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(i) Now $\sum_{n=0}^{\infty} (x^{4n} + x^{4n+1}) = (1 + x) \sum_{n=0}^{\infty} x^{4n} = (1 + x)(1 - x^4)^{-1}$,

provided $|x| < 1$.

$$= (1 + x)(1 - x)^{-1}(1 + x)^{-1}(1 + x^2)^{-1} = \{(1 - x)(1 + x^2)\}^{-1}$$

$$= \{1 - x + x^2 - x^3\}^{-1}.$$

(ii) $\sum_{n=0}^{\infty} (n + 1)(x^{3n} - 2x^{3n+1} + x^{3n+2}) = \sum_{n=0}^{\infty} (n + 1)x^{3n}(1 - 2x + x^2)$

$= (1 - x)^2 \sum_{n=0}^{\infty} (n + 1)x^{3n} = (1 - x)^2(1 - x^3)^{-2}$, provided $|x| < 1$,

$$= (1 - x)^2(1 - x)^{-2}(1 + x + x^2)^{-2} = (1 + 2x + 3x^2 + 2x^3 + x^4)^{-1}.$$

3·5. Irrational Index

Consider, *e.g.*, how we can attach a meaning to the binomial theorem in the case of $(1 + x)^{\sqrt{3}}$. Since we require all algebraic functions to obey the fundamental index laws $\{(1 + x)^{\sqrt{3}}\}^{\sqrt{3}}$ must be equal to

$$(1 + x)^{\sqrt{3} \times \sqrt{3}} = (1 + x)^3.$$

This equation may be taken to define $(1 + x)^{\sqrt{3}}$, *i.e.* the process of raising a quantity to the $(\sqrt{3})$th power is such that when the process is repeated on the resulting quantity, the final result is the cube of the original quantity.

Such a complex definition is of course not very satisfactory and it is better to proceed to the approximation of the irrational number $\sqrt{3}$ by rational numbers.

Now $\sqrt{3} = 1{\cdot}73205 \ldots$ The numbers $1{\cdot}7$, $1{\cdot}73$, $1{\cdot}732$, $1{\cdot}73205 \ldots$ form a series of rational numbers, each member of the series being a closer approximation to $\sqrt{3}$ than any of the members which precede it. Hence the series of values

$$(1 + x)^{1\cdot7},\ (1 + x)^{1\cdot73},\ (1 + x)^{1\cdot732},\ (1 + x)^{1\cdot73205}, \ldots$$

are such that each member is a closer approximation to $(1 + x)^{\sqrt{3}}$ than any member which precedes.

Now each member has a precise value, *e.g.*

$$(1+x)^{1\cdot 73} = (1+x)^{\frac{173}{100}} = \sqrt[100]{(1+x)^{173}},$$

$$(1+x)^{1\cdot 73205} = (1+x)^{\frac{173205}{100000}} = \sqrt[100000]{(1+x)^{173205}}$$

and so on. Also if $|x| < 1$ binomial expansion is valid for each of the members since each index is a rational number.

It follows that the binomial theorem will hold good for $(1+x)^{\sqrt{3}}$ as closely as we please if we substitute for $\sqrt{3}$ throughout a rational number which is sufficiently near to it for the purpose.

We consider now a formal method of obtaining the binomial coefficients. Suppose that

$$(1+x)^{\sqrt{3}} \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + \dots, \quad |x| < 1.$$

Putting $x = 0$ in both sides of the identity, it follows that $a_0 = 1$. Thus

$$(1+x)^{\sqrt{3}} \equiv (1+y) \text{ where } y = a_1x + a_2x^2 + a_3x^3 + \dots$$

$$\text{Again } (1+y)^{\sqrt{3}} = 1 + a_1y + a_2y^2 + \dots$$

Now $(1+y)^{\sqrt{3}} = \{(1+x)^{\sqrt{3}}\}^{\sqrt{3}} = (1+x)^3$. Hence

$$\begin{aligned}
1 + 3x + 3x^2 + x^3 &\\
&= 1 + a_1y + a_2y^2 + a_3y^3 + a_4y^4 + \dots\\
&= 1 + a_1(a_1x + a_2x^2 + \dots + a_nx^n + \dots)\\
&\quad + a_2(a_1x + a_2x^2 + \dots + a_nx^n + \dots)^2\\
&\quad + a_3(a_1x + a_2x^2 + \dots + a_nx^n + \dots)^3\\
&\quad + a_4(a_1x + a_2x^2 + \dots + a_nx^n + \dots)^4\\
&\quad + \dots\\
&= 1 + a_1^2x + (a_1a_2 + a_2a_1^2)x^2\\
&\quad + (a_1a_3 + 2a_1a_2^2 + a_3a_1^3)x^3\\
&\quad + (a_1a_4 + a_2^3 + 2a_1a_2a_3 + 3a_1^2a_2a_3 + a_4a_1^4)x^4\\
&\quad + \dots
\end{aligned}$$

Equating coefficients of corresponding powers of x

$$3 = a_1^2 \quad \dots\dots\dots\dots \text{(i)}$$

$$3 = a_1a_2(1 + a_1) \quad \dots\dots\dots\dots \text{(ii)}$$

$$1 = a_1(a_3 + 2a_2^2 + a_3a_1^2) \quad \dots\dots\dots\dots \text{(iii)}$$

$$0 = a_1a_4 + a_2^3 + 2a_1a_2a_3 + 3a_1^2a_2a_3 + a_4a_1^4 \quad \dots\dots\dots\dots \text{(iv)}$$

$$\text{From (i), } a_1 = \sqrt{3}.$$

From (ii), $\sqrt{3} = a_2(1 + \sqrt{3})$,

$$a_2 = \sqrt{3}/(1 + \sqrt{3}) = \sqrt{3}(\sqrt{3} - 1)/2.$$

From (iii), $a_3\,(a_1 + a_1^3) = -2\,a_1 a_2^2 + 1$

$$= 1 - 2\sqrt{3}\,(\sqrt{3})^2\,(\sqrt{3} - 1)^2/4$$
$$= 1 - 3\sqrt{3}\,(2 - \sqrt{3}) = 10 - 6\sqrt{3}.$$

Hence $a_3 = (10 - 6\sqrt{3})/4\sqrt{3}$

$$= (5\sqrt{3} - 9)/6 = \sqrt{3}\,(\sqrt{3} - 1)\,(\sqrt{3} - 2)/6.$$

Again $a_2^3 = \dfrac{\sqrt{3}\cdot(\sqrt{3} - 1)}{2} \cdot \dfrac{3(\sqrt{3} - 1)^2}{4}$

$$= \tfrac{3}{4}\sqrt{3}\,(\sqrt{3} - 1)\,(2 - \sqrt{3}) = -\tfrac{9}{2}a_3;$$

$\therefore\ a_2^3 + 2\,a_1 a_2 a_3 + 3a_1^2 a_2 a_3$

$$= a_3\left\{-\tfrac{9}{2} + 2\sqrt{3}.\frac{\sqrt{3}\,(\sqrt{3} - 1)}{2} + 9.\frac{\sqrt{3}\,(\sqrt{3} - 1)}{2}\right\}$$
$$= \tfrac{1}{2}a_3\,\{12 - 3\sqrt{3}\} = \tfrac{3}{2}a_3\,\{4 - \sqrt{3}\}$$
$$= \tfrac{3}{4}a_3\,(1 + 3\sqrt{3})\,(\sqrt{3} - 1).$$

From (iv), $\sqrt{3}a_4\,(1 + 3\sqrt{3}) = -\tfrac{3}{4}a_3\,(1 + 3\sqrt{3})\,(\sqrt{3} - 1)$,

$$\text{i.e. } a_4 = -\frac{\sqrt{3}}{4}a_3\,(\sqrt{3} - 1) = \tfrac{1}{4}\,(\sqrt{3} - 3)a_3;$$

$$\therefore\ a_4 = \sqrt{3}\,(\sqrt{3} - 1)\,(\sqrt{3} - 2)\,(\sqrt{3} - 3)/4!.$$

In a similar way further coefficients would be calculated. Thus the expansion of $(1 + x)^{\sqrt{3}}$ is

$$1 + \sqrt{3}x + \frac{\sqrt{3}\,(\sqrt{3} - 1)}{2!}x^2 + \frac{\sqrt{3}\,(\sqrt{3} - 1)\,(\sqrt{3} - 2)}{3!}x^3$$
$$+ \frac{\sqrt{3}\,(\sqrt{3} - 1)\,(\sqrt{3} - 2)\,(\sqrt{3} - 3)}{4!}x^4 + \ldots$$

3·51. Further Examples

In this section we give further examples of expansions in series, other than direct binomial expansions.

(1) *Prove that* $\left(\dfrac{1 + x}{1 - x}\right)^3 = 1 + \displaystyle\sum_{n=1}^{\infty} (4n^2 + 2)\,x^n.$ [*Lond. B.Sc.*]

Now $(1 - x)^{-3} = 1 + 3x + \dfrac{3\cdot4}{2}x^2 + \dfrac{4\cdot5}{2}x^3$

$$+ \ldots + \frac{(n + 1)\,(n + 2)}{2}x^n + \ldots,\ |x| < 1.$$

Thus $\left(\frac{1+x}{1-x}\right)^3 = \{1 + 3x + 3x^2 + x^3\}\left\{1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2}x^n\right\}.$

In this product the coefficient of x^n is

$$\frac{(n+1)(n+2)}{2} + \frac{3n(n+1)}{2} + \frac{3(n-1)n}{2} + \frac{(n-2)(n-1)}{2}$$
$$= \tfrac{1}{2}(8n^2 + 4).$$

$$\text{Thus } \left(\frac{1+x}{1-x}\right)^3 = 1 + \sum_{n=1}^{\infty} (4n^2 + 2)\,x^n.$$

(2) *Expand* $(1 + x)^4/(1 - x^2)^3$ *in a series of ascending powers of* x.

$$\frac{(1+x)^4}{(1-x^2)^3} = \frac{(1+x)(1+x)^3}{(1-x)^3(1+x)^3} \quad (1+x)(1-x)^{-3}$$

$$= (1+x) \sum_{r=0}^{\infty} a_r x^r, \ |x| < 1,$$

where $a_r = \frac{1}{2}(r+1)(r+2)$. The coefficient of x^r in the product is $a_r + a_{r-1}$, i.e. $\frac{1}{2}(r+1)(r+2) + \frac{1}{2}r(r+1) = (r+1)^2$.

$$\text{Hence } (1+x)^4/(1-x^2)^3 = \sum_{r=0}^{\infty} (r+1)^2 x^r.$$

(3) *Find the coefficient of* x^{100} *and* x^{101} *in the expansion of* $(1 + x + x^2)^{-2}$ *in ascending powers of* x.

$$\frac{1}{(1+x+x^2)^2} = \frac{(1-x)^2}{(1-x^3)^2} = (1 - 2x + x^2)\sum_{r=0}^{\infty}(r+1)x^{3r}, \qquad |x| < 1$$

$$= \sum_{r=0}^{\infty} (r+1)(x^{3r} - 2x^{3r+1} + x^{3r+2}).$$

The terms involving x^{100}, x^{101} are obtained by writing $r = 33$ so that $x^{3r+1} = x^{100}$, $x^{3r+2} = x^{101}$. Hence the coefficient of x^{100} is $-2 \times 34 = -68$ and the coefficient of $x^{101} = 34$.

(4) *Show that the coefficient of* x^{11} *in the expansion of* $\sqrt{\frac{(1+x)^3}{(1-x)^5}}$ *in ascending powers of* x *is* $\frac{7 \cdot 9 \cdot 11 \cdot 23}{8\,(4!)}$.

$$\text{The expression} \quad \frac{(1+x)^{\frac{3}{2}}}{(1-x)^{\frac{5}{2}}}$$

To escape the difficulty of determining the required coefficient by dealing with two infinite series, multiply the numerator and the denominator of the expression by $(1 + x)^{\frac{5}{2}}$ which operation will give us an integral power of $1 + x$ for the numerator, leaving the denominator still a binomial.

Thus the expression

$$= \frac{(1+x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}}{(1+x)^{\frac{5}{2}}(1-x)^{\frac{5}{2}}} = \frac{(1+x)^4}{(1-x^2)^{\frac{5}{2}}} = (1+x)^4(1-x^2)^{-\frac{5}{2}}$$

$$= (1 + 4x + 6x^2 + 4x^3 + x^4)\left(1 + \frac{5}{2}x^2 + \ldots\right), \qquad |x| < 1.$$

Since even powers of x alone appear in the expansion of $(1 - x^2)^{-\frac{5}{2}}$, the coefficient of x^{11} required in the question can be got by multiplying the coefficients of $(x^2)^4$ and $(x^2)^5$ in this expansion by the coefficients of x^3 and x respectively in the expansion of $(1 + x)^4$, and therefore the coefficient of x^{11}

$$= 4 \cdot \frac{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}}{1.2.3.4} + 4 \cdot \frac{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{1.2.3.4.5}$$

$$= \frac{7.9.11}{(4!)}\cdot\left(\frac{5}{4} + \frac{13}{8}\right) = \frac{7.9.11.23}{8\,(4!)}.$$

(5) *Write down in its simplest form the coefficient of x^n in the expansion in ascending powers of x, of*

$$(a + bx + cx^2 + dx^3)\,(1 - x)^{-4}, \quad -1 < x < 1,$$

and determine the coefficients a, b, c, d so that this expansion reduces to $\sum_{n=1}^{\infty} n^3x^n$.

Prove that $\sum_{n=1}^{\infty} \frac{n^3}{2^n} = 26.$ [*Lond. B.Sc.*]

$$(1 - x)^{-4} = 1 + \sum_{r=1}^{\infty} \frac{(-4)(-5)\ldots(-4-r+1)}{r\,!}(-x)^r$$

i.e.
$$(1 - x)^{-4} = 1 + \sum_{r=1}^{\infty} \frac{(r+1)(r+2)(r+3)}{3\,!}x^r.$$

$$= \sum_{r=0}^{\infty} \frac{(r+1)(r+2)(r+3)}{3\,!}x^r.$$

The coefficient of x^n in $(a + bx + cx^2 + dx^3) \sum_{n=0}^{\infty} \frac{(r+1)(r+2)(r+3)}{3!}x^r$ is

$$\frac{1}{3\,!}\{a(n+1)(n+2)(n+3) + bn(n+1)(n+2)$$
$$+ c(n-1)n(n+1) + d(n-2)(n-1)n\}.$$

This expression is denoted by u_n. Then

$$(a + bx + cx^2 + dx^3)\,(1 - x)^{-4} = \sum_{n=0}^{\infty} u_nx^n.$$

Suppose now that $u_n \equiv n^3$. Since u_n is of the third degree in n it is possible to choose the constants a, b, c, d so that this condition is satisfied.

In the identity write $n = -2, -1, 0, 1$ in succession. Then $-8 = -c - 4d$, $-1 = -d$, $0 = a$, $1 = 4a + b$. Hence $a = 0, b = 1$, $c = 4$, $d = 1$.

Since $u_0 = a$ it follows that $\sum_{n=0}^{\infty} u_nx^n$ becomes $\sum_{n=1}^{\infty} n^3x^n$, when $a_0 = 0$, $b = 1$, $c = 4$, $d = 1$. Thus

$$\sum_{n=1}^{\infty} n^3x^n = (x + 4x^2 + x^3)\,(1 - x)^{-4}, \quad -1 < x < 1.$$

In this equation write $x = \frac{1}{2}$. Then

$$\sum_{n=1}^{\infty} \frac{n^3}{2^n} = (\tfrac{1}{2} + 1 + \tfrac{1}{8})(\tfrac{1}{2})^{-4} = 26.$$

3·61. Determination of a Binomial Function from its Expansion

So far we have been concerned mainly with the problem of determining the expansion from the given function. We now consider some examples of the converse problem, *i.e.* **given a binomial series to determine the function from which it is derived.**

If we are given a *power series in x* we can determine by a study of the coefficients whether or not the series is a binomial expansion. It must be remembered that the complete expansion may not be given and further that we may not be dealing with a direct binomial expansion of the form $(1 + x)^n$. The student must be guided by his previous experience gained in dealing with particular binomial expansions.

If we are given *a numerical series* the problem is in general less simple than the previous case. For now another unknown factor has been introduced, *viz.* the particular numerical value of x which has been substituted in the power series to give the particular numerical series.

Examples of both types of series are given below. In the first example we indicate the details of the analysis which enables us to decide the function from which the expansion is derived.

Examples.—(1) *Sum the following series to infinity:*

$$\frac{1}{2^3(3\,!)} - \frac{1.3}{2^4(4\,!)}x + \frac{1.3.5}{2^5\,(5\,!)}x^2 - \ldots, \text{ where } |x| < 1.$$

The general form of the coefficients suggests a binomial expansion. First we observe that the factors in the numerators of successive coefficients increase by 2. This suggests that the denominator of the index of the power of the binomial expression is 2. This is confirmed by the presence of the powers of 2 in the denominators of the coefficients.

Next we observe that the terms are alternately positive and negative and that the powers of x increase by unity. This indicates that the expansion will be of the form $(1 + x)^n$ as distinct from $(1 - x)^n$. The fact that the terms are alternately positive and negative from the beginning does not indicate precisely whether n is a positive or negative. For some of the terms of the complete binomial expansion may be missing from the beginning of the given expansion.

The next point to be observed is that the "factorial" terms in the denominators of the coefficients begin with $3\,!$. This shows clearly that three terms of the series are missing and that $1/2^3(3\,!)$ is the coefficient of the 4th term

of the complete binomial expansion. Thus we must multiply each term of the given expansion by x^3 and divide the resulting sum by x^3.

Again if $1/2^3(3\,!)$ is the coefficient of the 4th term there must be three factors corresponding to $n(n-1)(n-2)$. There are *three factors* $\frac{1}{2}$ and this indicates that two of the required factors are $-\frac{1}{2}$, $\frac{1}{2}$, for those numbers differ by unity. The factor which is missing must either by $-\frac{3}{2}$ or $+\frac{3}{2}$. This point is settled by the next coefficient which is obtained from the first by multiplying by $-\frac{3}{2}$ and changing $3\,!$ into $4\,!$. Hence we must take the value $+\frac{3}{2}$ and the given expansion must form part of the binomial expansion of $(1+x)^{\frac{3}{2}}$.

The precise setting out of the working is as follows.

$$\frac{1}{2^3(3\,!)} - \frac{1.3}{2^4(4\,!)}x + \frac{1.3.5}{2^5(5\,!)}x^2 - \ldots$$

$$= \frac{1}{3x^3}\left\{\frac{3.1.1}{2^3(3\,!)}x^3 - \frac{3.1.1.3}{2^4(4\,!)}x^4 + \frac{3.1.1.3.5}{2^5(5\,!)}x^5 - \ldots\right\}$$

$$= \frac{1}{3x^3}\left\{\frac{\frac{3}{2}.\frac{1}{2}.\frac{1}{2}}{3\,!}x^3 - \frac{\frac{3}{2}.\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{4\,!}x^4 + \frac{\frac{3}{2}.\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{5\,!}x^5 - \ldots\right\}$$

$$= \frac{1}{3x^3}\left\{-\frac{\frac{3}{2}.\frac{1}{2}.-\frac{1}{2}}{3\,!}x^3 - \frac{\frac{3}{2}.\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}}{4\,!}x^4 - \frac{\frac{3}{2}.\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}}{5\,!}x^5 - \ldots\right\}$$

$$= -\frac{1}{3x^3}\left\{(1+x)^{\frac{3}{2}} - 1 - \frac{3}{2}x - \frac{3}{2}.\frac{1}{2}.\frac{x^2}{2\,!}\right\}$$

$$= \frac{1}{3x^3}\{(1+\tfrac{3}{2}x+\tfrac{3}{8}x^2-(1+x)^{\frac{3}{2}})\}.$$

(2) *Evaluate* $\quad 1 + \sum_{n=1}^{\infty} \dfrac{m(m-2)(m-4)\ldots(m-2n+2)}{n\,!\,2^{2n}}.$

[*Lond. B.Sc.*]

$$\frac{m(m-2)(m-4)\ldots(m-2n+2)}{n\,!\,2^{2n}}$$

$$= \frac{\frac{m}{2}\left(\frac{m}{2}-1\right)\left(\frac{m}{2}-2\right)\ldots\left(\frac{m}{2}-n+1\right)}{n\,!}.\frac{1}{2^n}.$$

$$= \binom{\frac{1}{2}m}{n}x^n, \text{ where } x = \tfrac{1}{2}.$$

Hence $1 + \displaystyle\sum_{n=1}^{\infty}\frac{m(m-2)(m-4)\ldots(m-2n+2)}{n\,!\,2^{2n}}$

$$= \sum_{n=0}^{\infty}\binom{\frac{1}{2}m}{n}x^n$$

$$= (1+x)^{\frac{1}{2}m} = (\tfrac{3}{2})^{\frac{1}{2}m}.$$

(3) *Sum the infinite series:*

$$1 + 2.\frac{1}{3^2} + \frac{2.5}{1.2}.\frac{1}{3^4} + \frac{2.5.8}{1.2.3}.\frac{1}{3^6} + \ldots$$

Taking the expansion of $(1-x)^{-n}$, we have

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \ldots + \frac{n(n+1)\ldots(n+r-1)}{1.2.3\ldots r}x^r + \ldots$$

Here (1) the indices of the powers of x increase regularly by unity, the index of x in any term being the number of the factors in the numerator or denominator of its coefficient; (2) the numerator of the coefficient is the product of an A.P. of which the first term is n, and the common difference is 1, and (3) the denominator of the coefficient is also the product of an A.P. of which the first term is 1, and the common difference is 1.

In the series given for summation, the first and second conditions do not appear to be satisfied. We therefore modify it, by dividing by 3 each factor of the numerator of every coefficient, the requisite number of 3's for this division being taken from the power of $\frac{1}{3}$. Thus the series after modification

$$= 1 + \tfrac{2}{3}\cdot\tfrac{1}{3} + \frac{\frac{2}{3}\cdot\frac{5}{3}}{1.2}\cdot\left(\frac{1}{3}\right)^2 + \frac{\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}}{1.2.3}\cdot\left(\frac{1}{3}\right)^3 + \ldots$$

$$= (1 - \tfrac{1}{3})^{-\frac{2}{3}} = (\tfrac{2}{3})^{-\frac{2}{3}} = (\tfrac{3}{2})^{\frac{2}{3}} = \tfrac{1}{2}\sqrt[3]{18}.$$

(4) *Sum to infinity:*

$$2\tfrac{1}{5} + \frac{6.2}{5.10} - \frac{6.2.2}{5.10.15} + \frac{6.2.2.6}{5.10.15.20} - \ldots$$

$$\text{The series} = 2\tfrac{1}{5} + \frac{6.2}{1.2}\cdot(\tfrac{1}{5})^2 - \frac{6.2.2}{1.2.3}\cdot(\tfrac{1}{5})^3 + \frac{6.2.2.6}{1.2.3.4}\cdot(\tfrac{1}{5})^4 - \ldots$$

$$= 1 + \tfrac{6}{5} + \frac{\frac{6}{4}\cdot\frac{2}{4}}{1.2}\cdot(\tfrac{4}{5})^2 + \frac{\frac{6}{4}\cdot\frac{2}{4}\cdot(-\frac{2}{4})}{1.2.3}\cdot(\tfrac{4}{5})^3$$

$$+ \frac{\frac{6}{4}\cdot\frac{2}{4}\cdot(-\frac{2}{4})\,(-\frac{6}{4})}{1.2.3.4}\cdot(\tfrac{4}{5})^4 + \cdots$$

$$= 1 + \tfrac{3}{2}\,(\tfrac{4}{5}) + \frac{\frac{3}{2}\cdot\frac{1}{2}}{1.2}\cdot(\tfrac{4}{5})^2 + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot(-\frac{1}{2})}{1.2.3}\cdot(\tfrac{4}{5})^3 + \ldots$$

$$= (1 + \tfrac{4}{5})^{\frac{3}{2}} = (\tfrac{9}{5}) \times (\tfrac{9}{5})^{\frac{1}{2}} = \tfrac{27}{25}\sqrt{5}.$$

(5) *Sum to infinity:*

$$\frac{1}{12} - \frac{1.4}{12.18} + \frac{1.4.7}{12.18.24} - \ldots$$

$$\text{The series} = \frac{\frac{1}{3}}{4} - \frac{\frac{1}{3}\cdot\frac{4}{3}}{4.6} + \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{4.6.8} - \ldots$$

$$= \frac{\frac{1}{3}}{2}\cdot(\tfrac{1}{2}) - \frac{\frac{1}{3}\cdot\frac{4}{3}}{2.3}\cdot(\tfrac{1}{2})^2 + \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{2.3.4}\cdot(\tfrac{1}{2})^3 - \ldots$$

The factors of the denominators are in A.P.; but they do not begin with 1. Hence one additional factor, namely unity, has to be introduced into the denominator of each coefficient; and as the number of factors in the numerator is the same as that of the factors in the denominator, we have to introduce an additional factor in the numerator also, which factor is clearly, $-\frac{2}{3}$. Let S denote the series. Now

$$-\tfrac{2}{3}S = \frac{(-\frac{2}{3})\cdot\frac{1}{3}}{1.2}\cdot\tfrac{1}{2} - \frac{(-\frac{2}{3})\cdot\frac{1}{3}\cdot\frac{4}{3}}{1.2.3}\cdot(\tfrac{1}{2})^2 + \frac{(-\frac{2}{3})\cdot\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{1.2.3.4}(\tfrac{1}{2})^3 - \ldots$$

Again, since the index of x in every term must be the same as the number of factors in the numerator or denominator of the coefficient, we have

$$-\tfrac{2}{3}S \times \tfrac{1}{2} = \frac{\frac{2}{3}\cdot(-\frac{1}{3})}{1.2}\cdot(\tfrac{1}{2})^2 + \frac{\frac{2}{3}\cdot(-\frac{1}{3})\,(-\frac{4}{3})}{1.2.3}\cdot(\tfrac{1}{2})^3 + \ldots$$

Here the first two terms are evidently wanting. When we supply these, we find that

$$-\tfrac{2}{3}S \times \tfrac{1}{2} + 1 + \tfrac{2}{3}(\tfrac{1}{2}) = 1 + \tfrac{2}{3}(\tfrac{1}{2}) + \frac{\tfrac{2}{3}(\tfrac{2}{3} - 1)}{1.2}(\tfrac{1}{2})^2 + \ldots$$

$$\therefore\ -\tfrac{1}{3}S + 1\tfrac{1}{3} = (1 + \tfrac{1}{2})^{\frac{2}{3}} = (\tfrac{3}{2})^{\frac{2}{3}} = \tfrac{1}{2}\sqrt[3]{18}.$$

$$\therefore\ S = 3(1\tfrac{1}{3} - \tfrac{1}{2}\sqrt[3]{18}) = 4 - \tfrac{3}{2}\sqrt[3]{18}.$$

3·62. Properties of the Coefficients of Binomial Expansions

We first prove that *the sum of the first* $(r + 1)$ *coefficients in the expansion of* $(1 - x)^n$ is

$$(-1)^r (n-1)(n-2)(n-3)\ldots(n-r)/r!$$

$$i.e.\quad \sum_{s=0}^{r} (-1)^s \binom{n}{s} = (-1)^r \binom{n-1}{r}$$

$$\text{Now}\quad (1-x)^n = \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} x^r, \qquad |x| < 1,$$

$$(1-x)^{-1} = \sum_{r=0} x^r, \qquad |x| < 1.$$

Since the series converge absolutely the product series also converges absolutely for $|x| < 1$. Thus

$$(1-x)^n (1-x)^{-1} = \sum_{r=0}^{\infty} a_r x^r \quad \text{[Chap. I., § 1·6]}$$

$$\text{where}\quad a_r = 1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^r \binom{n}{r}$$

$$\text{Now}\quad (1-x)^n (1-x)^{-1} = (1-x)^{n-1} = \sum_{x=0}^{\infty} b_r x^r,$$

$$\text{where}\quad b_r = (-1)^r \binom{n-1}{r}.$$

Hence $a_r = b_r$ (Chap. I., § 1·8).

$$\text{Thus}\quad \sum_{s=0}^{r} (-1)^s \binom{n}{s} = (-1)^r \binom{n-1}{r}.$$

The remainder of the section is devoted to examples which indicate how the properties of binomial coefficients may be obtained and used.

Examples.—(1) *Find the sum of the coefficients of the first* $(r + 1)$ *terms in the expansion of* $(1 - x)^{-3}$.

$$(1-x)^{-3} = \sum_{r=0}^{\infty} \tfrac{1}{2}(r+1)(r+2)x^r = \Sigma a_r x^r \text{ and } (1-x)^{-1} = \sum_{r=0}^{\infty} x^r.$$

Using the product theorem for power series

$$(1 - x)^{-4} = \Sigma b_r x^r \quad \text{where} \quad a_r = b_0 + b_1 + \ldots + b_r.$$

$$\text{Now} \quad (1 - x)^{-4} = \sum^{\infty} \tfrac{1}{6}(r + 1)(r + 2)(r + 3)x^r.$$

$$\therefore \quad \sum_{s=0} \tfrac{1}{2}(s + 1)(s + 2) = \tfrac{1}{6}(r + 1)(r + 2)(r + 3).$$

(2) *If c_r denote the coefficient of x^r in the binomial expansion*

$$1 + nx + \frac{n(n-1)}{2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots,$$

where $1 > x > -1$, *use the identity* $(1 - x)^n(1 + x)^n = (1 - x^2)^n$ *to prove that*

$$c_0c_{2r} - c_1c_{2r-1} + \ldots + (-1)^s c_sc_{2r-s} + \ldots$$
$$+ (-1)^{r-1}c_{r-1}c_{r+1} = \tfrac{1}{2}(-1)^r c_r(1 - c_r).$$

[*Lond. B.Sc.*]

$$(1 - x)^n = \sum_{r=0}^{\infty} (-1)^r c_r x^r, \text{ and } (1 + x)^n = \sum_{r=0}^{\infty} c_r x^r.$$

Since each series converges absolutely for $|x| < 1$

$$(1 - x)^n(1 + x)^n = \sum_{r=0}^{\infty} a_r x^r,$$

where $a_r = c_rc_0 - c_{r-1}c_1 + c_{r-2}c_2 + \ldots$
$$+ (-1)^s c_{r-s}c_s + \ldots + (-1)^r c_rc_0.$$

Since this is true for all values of r,

$$a_{2r} = c_0c_{2r} - c_1c_{2r-1} + \ldots + (-1)^s c_sc_{2r-s} + \ldots + (-1)^{2r} c_{2r}c_0.$$

$$\text{Now} \quad (1 - x)^n(1 + x)^n = (1 - x^2)^n = \sum_{r=0}^{\infty} (-1)^r c_r x^{2r}.$$

Comparing the two expansions, the coefficients of x^{2r} must be the same in each case. Hence

$$c_0c_{2r} - c_1c_{2r-1} + \ldots + (-1)^s c_sc_{2r-s} + \ldots + c_{2r}c_0 = (-1)^r c_r,$$

$$\text{i.e.} \quad \sum_{s=0}^{2r} (-1)^s c_sc_{2r-s} = (-1)^r c_r.$$

In the series on the left there are $2r + 1$ terms and the terms equidistant from the end and beginning are equal. The middle term is obtained by writing $s = r$ and is $(-1)^r c_r^2$. Hence

$$2\sum_{s=0}^{r-1} (-1)^s c_sc_{2r-s} + (-1)^r c_r^2 = (-1)^r c_r,$$

$$\text{i.e.} \quad \sum_{s=0}^{r-1} (-1)^s c_sc_{2r-s} = \tfrac{1}{2}(-1)^r c_r(1 - c_r).$$

(3) *Write down the expansion of* $(1 - x^2)^{-\frac{1}{2}}$ *in a series of positive integral powers of x. If the coefficient of x^{2n} in this expansion is a_n, show that*

$$a_0a_n + a_1a_{n-1} + \ldots + a_na_0 = 1$$

[*Lond. B.Sc.*]

$$(1 - x^2)^{-\frac{1}{2}} = \Sigma \frac{-\frac{1}{2} \cdot -\frac{3}{2} \cdot \ldots (-\frac{1}{2} - r + 1)}{r\,!} (-x^2)^r, \qquad |x| < 1$$

$$= \Sigma \frac{1 \cdot 3 \cdot 5 \cdot (2r-1)}{2^r\, r\,!} x^{2r} = \Sigma \frac{(2r-1)\,!}{2^{2r-1}\, r\,!\,(r-1)\,!} x^{2r}.$$

$$\therefore \; a_r = \frac{(2r-1)\,!}{2^{2r-1}\, r\,!\,(r-1)\,!}.$$

Next consider the square of the series $\Sigma a_r x^{2r}$. Since the series converges absolutely for $|x| < 1$ the series obtained by considering the square will also be absolutely convergent. Thus

$$\{(1 - x^2)^{-\frac{1}{2}}\}^2 = \Sigma b_r x^{2r} \quad \text{where} \quad b_r = a_0 a_r + a_1 a_{r-1} + a_2 a_{r-2} + \ldots + a_r a_0.$$

$$\text{Now} \quad \{(1 - x^2)^{-\frac{1}{2}}\}^2 = (1 - x^2)^{-1} = \Sigma x^{2r}.$$

Hence $b_r = 1$. Writing $r = n$;

$$1 = a_0 a_n + a_1 a_{n-1} + \ldots + a_n a_0.$$

(4) *Find the sum to n terms of the series whose rth term is*

$$r(r+1)(r+2)(r+3).$$

Write $u_r = r(r+1)(r+2)(r+3)$.

$$\text{Then} \quad u_r = 4\,!\,.\frac{r(r+1)(r+2)(r+3)}{4\,!}$$

$$\therefore \; u_r = 4\,!\,.\frac{5 \cdot 6 \cdot 7 \cdot \ldots \cdot r(r+1)(r+2)(r+3)}{(r-1)\,!}$$

$$\text{Now} \quad (1 - x)^{-5} = \sum_{r=0}^{\infty} \binom{-5}{r} (-1)^r x^r$$

$$= \sum_{r=0}^{\infty} \frac{5 \cdot 6 \cdot 7 \cdot \ldots \cdot r(r+1)(r+2)(r+3)(r+4)}{r\,!} x^r,$$

$$\text{i.e.} \quad (1 - x)^{-5} = \frac{1}{4\,!} \sum_{r=0}^{\infty} u_{r+1} x^r.$$

$$\text{The sum required is} \quad \sum_{r=1}^{n} u_r = \sum_{r=0}^{n-1} u_{r+1}.$$

If s denote the sum of the given series it follows that

$$s = 4\,! \times \text{the sum of the first } n \text{ coefficients of} \quad (1 - x)^{-5}$$

$$= 4\,! \times \text{coefficient of } x^{n-1} \text{ in } (1 - x)^{-6}$$

$$\text{i.e.} \quad s = 4\,! \times \frac{n(n+1)(n+2)(n+3)(n+4)}{5\,!}$$

$$= \tfrac{1}{5} n(n+1)(n+2)(n+3)(n+4).$$

3·63. Application of the Binomial Theorem to Approximations

The binomial expansion $\sum_{r=0}^{\infty} \binom{n}{r} x^r$ is a series which converges absolutely to $(1 + x)^n$ provided $|x| < 1$. In particular from the properties of convergence

$$\lim_{n \to \infty} \binom{n}{r} x^r = 0. \quad \text{(Chap. I., § 1·32.)}$$

Also we know from the properties of the coefficients that there is a numerically greatest term in the series, which may in particular be the first term, beyond which the terms steadily decrease.

Further, it is clear that *the smaller the value of* $|x|$ *the more rapidly will* $\Sigma \binom{n}{r} x^r$ *converge to its sum function* $(1 + x)^n$. In other words, if $|x|$ is small enough we can obtain good approximations to the value of $(1 + x)^n$ by taking comparatively few terms of the expansion.

Consider the problem of finding the nth root of any number. Let N denote the number. Express N in the form $a^n + b$ where a^n is as close to N as is convenient, while b is a positive or negative number which is small compared with a^n.

Thus, *e.g.* if we require the fourth root of 626 we should write $626 = 625 + 1 = 5^4 + 1$. In general.

$$N = a^n + b, \quad \therefore N^{\frac{1}{n}} = (a^n + b)^{\frac{1}{n}} = a\left(1 + \frac{b}{a^n}\right)^{\frac{1}{n}};$$

$$\therefore N^{\frac{1}{n}} = a(1 + x)^{\frac{1}{n}}, \text{ where } x = b/a^n \text{ and } |x| \text{ is small.}$$

It is clear that the number of terms which are retained will depend on the degree of accuracy required, since clearly the more terms we take the more closely do we approach to the precise value of the root.

Examples.—(1) *Extract by means of the binomial theorem the fourth root of* 624 *correct to six places of decimals.* [*N.Sc.*]

$$(624)^{\frac{1}{4}} = (625 - 1)^{\frac{1}{4}} = 5\left(1 - \frac{1}{5^4}\right)^{\frac{1}{4}}$$

$$= 5\left(1 - \tfrac{1}{4}\cdot\frac{1}{5^4} + \frac{\tfrac{1}{4}\cdot -\tfrac{3}{4}}{2!}\frac{1}{5^8} - \frac{\tfrac{1}{4}\cdot -\tfrac{3}{4} - \tfrac{7}{4}}{3!}\frac{1}{5^{12}} + \dots\right)$$

$$= 5\left(1 - \frac{1}{2500} - \frac{3}{125 \times 10^6} - \dots\right) = 5\left(1 - \frac{4}{10^4} - \frac{24}{10^8} - \dots\right)$$

$$= 5(1 - \cdot0004 - \cdot00000024 - \dots) = 5 \times \cdot99959976;$$

$$\therefore (624)^{\frac{1}{4}} = 4\cdot997999 \text{ correct to 6 places of decimals.}$$

(2) *Show that the error in taking* $\frac{1 + x}{2 + x} + \frac{2 + x}{4}$ *as an approximation to* $\sqrt{(1 + x)}$ *is approximately equal to* $x^4/2^7$ *when* x *is small.*

By taking $x = \frac{1}{2}$, *find a rational approximation to* $\sqrt{6}$ *and state to what number of figures the result is correct.*

$$\text{Now } \sqrt{1 + x} = (1 + x)^{\frac{1}{2}}$$

$$(1 + x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \dots$$

$$\text{Now}\quad \frac{2+x}{4}+\frac{1+x}{2+x}=\frac{2+x}{4}+\frac{1+x}{2}.(1+\tfrac{1}{2}x)^{-1}$$

$$=\frac{2+x}{4}+\frac{1+x}{2}\,(1-\tfrac{1}{2}x+\tfrac{1}{4}x^2-\tfrac{1}{8}x^3+\tfrac{1}{16}x^4-\ldots)$$

$$=\tfrac{1}{2}+\tfrac{1}{4}x+\tfrac{1}{2}-\tfrac{1}{4}x+\tfrac{1}{8}x^2-\tfrac{1}{16}x^3+\tfrac{1}{32}x^4+\tfrac{1}{2}x$$
$$-\tfrac{1}{4}x^2+\tfrac{1}{8}x^3-\tfrac{1}{16}x^4$$

$$=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{1}{32}x^4\ldots$$

Neglecting x^5 and higher powers the difference between the two expressions is

$$\tfrac{5}{128}x^4-\tfrac{1}{32}x^4=\tfrac{1}{128}x^4=x^4/2^7.$$

$$\text{When}\quad x=\tfrac{1}{2},\quad \sqrt{1+x}=\sqrt{\tfrac{3}{2}}=\tfrac{1}{2}\sqrt{6}.$$

$$\text{Also}\quad \frac{1+x}{2+x}+\frac{2+x}{4}\quad \frac{1+\frac{1}{2}}{2+\frac{1}{2}}+\frac{2+\frac{1}{2}}{4}=\tfrac{3}{5}+\tfrac{5}{8}=1{\cdot}225.$$

Hence adopting the approximation we obtain $\sqrt{6}=2{\cdot}45$.

The error term is $x^4/2^7=1/2^{11}={\cdot}00049$ approx. Thus we would certainly expect the approximation to be correct to 2 decimal places and perhaps to 3 places. It is easily verified that $\sqrt{6}=2{\cdot}4495\ldots$ so that the result is just correct to 3 places of decimals.

When $|x|$ is *large* we may approximate in a similar way by considering an expansion in ascending powers of $1/x$.

(3) *If x is very large, find, correct to the term in $1/x^2$, the expansion in powers of $1/x$ of* $\sqrt[3]{(x+3)}\times\sqrt[6]{x}/\sqrt{(x-2)}$.

$$\frac{\sqrt[3]{(x+3)}\,\sqrt[6]{x}}{\sqrt{(x-2)}}=\frac{(x+3)^{\frac{1}{3}}x^{\frac{1}{6}}}{(x-2)^{\frac{1}{2}}}$$

Dividing numerator and denominator by $x^{\frac{1}{2}}$ the expansion becomes

$$\left(1+\frac{3}{x}\right)^{\frac{1}{3}}\left(1-\frac{2}{x}\right)^{-\frac{1}{2}}=\left\{1+\tfrac{1}{3}.\frac{3}{x}+\frac{\frac{1}{3}.(-\frac{2}{3})}{2\,!}\left(\frac{3}{x}\right)^2+\ldots\right\}$$
$$\left\{1-\tfrac{1}{2}\left(-\frac{2}{x}\right)+\frac{(-\frac{1}{2})\,(-\frac{3}{2})}{2\,!}.\left(-\frac{2}{x}\right)^2+\ldots\right\}$$
$$=\left\{1+\frac{1}{x}-\frac{1}{x^2}+\ldots\right\}\left\{1+\frac{1}{x}+\tfrac{3}{2}\frac{1}{x^2}+\ \ldots\right\}=1+\frac{2}{x}+\tfrac{3}{2}\frac{1}{x^2}.$$

(4) *Prove that*

$$\sqrt{(x^2+a^2)\,(x^2+b^2)}-\sqrt{(x^2+c^2)\,(x^2+d^2)}=\tfrac{1}{2}\,(a^2+b^2-c^2-d^2),$$

when x is very great.

$$\text{The expression}\quad =(x^2+a^2)^{\frac{1}{2}}\,(x^2+b^2)^{\frac{1}{2}}-(x^2+c^2)^{\frac{1}{2}}\,(x^2+d^2)^{\frac{1}{2}},$$
$$=x^2\,(1+a^2/x^2)^{\frac{1}{2}}\,(1+b^2/x^2)^{\frac{1}{2}}-x^2\,(1+c^2/x^2)^{\frac{1}{2}}\,(1+d^2/x^2)^{\frac{1}{2}}$$
$$=x^2\left(1+\tfrac{1}{2}.\frac{a^2}{x^2}+\ldots\right)\left(1+\tfrac{1}{2}.\frac{d^2}{x^2}+\ldots\right)$$
$$-x^2\left(1+\tfrac{1}{2}.\frac{c^2}{x^2}+\ldots\right)\left(1+\tfrac{1}{2}.\frac{d^2}{x^2}+\ldots\right)$$
$$=x^2\left(1+\frac{a^2}{2x^2}+\frac{b^2}{2x^2}\right)-x^2\left(1+\frac{c^2}{2x^2}+\frac{d^2}{2x^2}\right);$$

reciprocals of powers of x higher than x^2 being neglected, since x is very great,

$$= x^2 - x^2 + \tfrac{1}{2}x^2\left(\frac{a^2 + b^2 - c^2 - d^2}{x^2}\right) \text{ approximately}$$

$$= \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2) \text{ approximately.}$$

(5) *If x be nearly equal to unity, show that* $\dfrac{mx^m - nx^n}{m - n} = x^{m+n}$ approximately.

Let $x = 1 + y$, where y must be small, since x is nearly equal to 1;

$$\therefore \text{ the expression } = \frac{m(1 + y)^m - n(1 + y)^n}{m - n}$$

$$= \frac{m(1 + my + \ldots) - n(1 + ny + \ldots)}{m - n}$$

$$= \frac{m - n + y(m^2 - n^2)}{m - n};$$

$$= 1 + (m + n)y,$$

y^2 and higher powers of y being omitted.

Also $x^{m+n} = (1 + y)^{m+n} = 1 + (m + n)y$, under the same assumptions, which prove the equation.

The binomial theorem may sometimes be used to find approximately the root of a given equation.

(6) *Given that a is small compared with b and c, find an approximation to numerically smaller root of the equation $ax^2 + 2bx + c = 0$.*

The roots of the equation are $x = \{-b \pm \sqrt{b^2 - ac}\}/a$.

Since a is small the root required is $\{-b + \sqrt{b^2 - ac}\}/a$.

It is clear that if in the quadratic equation we suppose that $a \to 0$ the equation would become

$$0.x^2 + 2bx + c = 0$$

whose roots are $-c/2b$ and ∞, with the usual convention on the meaning of the symbol infinity.

It follows that the first term in the required approximation will be $-c/2b$. Now

$$(b^2 - ac)^{\frac{1}{2}} = b\left(1 - \frac{ac}{b^2}\right)^{\frac{1}{2}} = b\left(1 - \tfrac{1}{2}\frac{ac}{b^2} - \tfrac{1}{8}\frac{a^2c^2}{b^4} - \ldots\right)$$

$$= b - \tfrac{1}{2}\frac{ac}{b} - \tfrac{1}{8}\frac{a^2c^2}{b^3} - \ldots$$

Thus the required root is $-\tfrac{1}{2}\dfrac{c}{b} - \tfrac{1}{8}\dfrac{ac^2}{b^3}$ where a^2/b^2 and higher powers are neglected.

(7) *Find an approximation to the small positive value of x which satisfies the equation*

$$(1 + x)^7(1 - x)^{-4} = 1{\cdot}05.$$

$$(1 + x)^7 (1 - x)^{-4} = (1 + 7x + 21x^2 + \ldots)(1 + 4x + 10x^2 + \ldots)$$
$$= 1 + 11x + 59x^2 \text{ (higher powers of } x \text{ neglected).}$$

Thus the equation becomes $\cdot 05 = 11x + 59x^2$.

Solving this as a quadratic equation in x, we have

$$x = \{-11 \pm \sqrt{(121 + 4 \times \cdot 05 \times 59)}\}/118.$$

The required root is clearly $x = \dfrac{-11 + \sqrt{132\cdot 8}}{118}$

From tables $\sqrt{132\cdot 8} = 11\cdot 522$ so that $x = \cdot 522/118 = \cdot 004424$. If we had neglected x^2 the approximation to the root would have been $x = \cdot 05/11 = \cdot 004545$.

The binomial theorem may sometimes be used to determine approximately the *error term* in equations which represent relations between physical quantities, due to small variations in the quantities. Problems involving determination of small errors are, however, usually solved most readily by calculus methods.

(8) *The pressure and volume of a given mass of gas are connected by the equation $pv^{1\cdot 4} = c$. If the volume is increased by 2 per cent., find the percentage change in pressure.*

Now $pv^{1\cdot 4} = c$, c being a constant. Suppose that when v changes into $v + h$, p changes into $p + k$, h and k being small. Then

$$(p + k)(v + h)^{1\cdot 4} = c = pv^{1\cdot 4},$$

$$\text{i.e. } \frac{p + k}{p} = \left(\frac{v}{v + h}\right)^{1\cdot 4} = \left(1 + \frac{h}{v}\right)^{-1\cdot 4};$$

$$\therefore\ 1 + \frac{k}{p} = 1 - 1\cdot 4\frac{h}{v} + \frac{(1\cdot 4)(2\cdot 4)}{2!}\frac{h^2}{v^2}\ldots$$

Now $\frac{h}{v} = \frac{1}{50}$ since v increases by 2 per cent. Neglecting $\frac{h^2}{v^2}$ and higher powers, $\frac{\ }{p} \quad -\frac{1\cdot 4}{50}$. Now k/p is fractional increase in p so that percentage increase in p is $100\ k/p$. Thus percentage increase in p is $-2\cdot 8$ per cent. The negative sign indicates that pressure is *decreased*.

Observe that logarithmic differentiation would have given this result immediately.

3.64. Approximation to a Function by a Series

In approximating to any function by means of a finite number of terms of an infinite series, which is the expansion of the given function, the following points should be observed.

The approximation will be required to some stated *degree of accuracy*, say r decimal places. Then it will be necessary to

ensure that all terms which can affect the rth decimal place are included. Further, all the terms which are included should each be calculated to at least $(r + 2)$ decimal places. This is to ensure that the accumulation of errors will not affect the rth decimal place.

Again, in determining the number of terms which must be included it is necessary to know the *greatest term* of the expansion.

Suppose, further, that it has been necessary to take p terms. Then we require some estimate of *the remainder after p terms*. In the case of the Binomial Theorem there is no single elementary formula which is applicable in all cases.

The ultimate behaviour of the terms of the binomial expansion as regards sign has been discussed in §§ 3·12, 3·2. There are two cases. (i) Terms alternately positive and negative. (ii) Terms all of the same sign. To (i) belong the forms $(1 + x)^n$, $(1 + x)^{-n}$ where $n > 0$, $x > 0$, while to (ii) belong $(1 - x)^{-n}$, $(1 - x)^n$ where $n > 0$, $x > 0$.

We have supposed that p terms are taken and we wish to estimate the magnitude of R_p, the remainder after p terms.

CASE (i). We take p greater than the order of the numerically greatest term, and also greater than n. This second condition is to ensure that the terms have reached the "alternately positive and negative" stage. Under these conditions

$$R_p = u_{p+1} - u_{p+2} + u_{p+3} - u_{p+4} + \dots$$

$$\text{or } R_p = - u_{p+1} + u_{p+2} - u_{p+3} + u_{p+4} - \dots$$

where u_t denotes the absolute value of the tth term. The former equation applies if the first term in R_p is positive, the latter if the first term is negative.

Now the sequence $\{u_t\}$ is monotonic decreasing, as once we pass beyond the numerically greatest term, the terms steadily decrease in absolute value.

Taking the first equation

$$\begin{aligned} R_p &= u_{p+1} - (u_{p+2} - u_{p+3}) - (u_{p+4} - u_{p+5}) - \dots \\ &= (u_{p+1} - u_{p+2}) + (u_{p+3} - u_{p+4}) + \dots \end{aligned}$$

$$\text{Hence } u_{p+1} - u_{p+2} < R_p < u_{p+1}.$$

Taking the second equation,

$$\begin{aligned} R_p &= - u_{p+1} + (u_{p+2} - u_{p+3}) + (u_{p+4} - u_{p+5}) + \dots \\ &= - (u_{p+1} - u_{p+2}) - (u_{p+3} - u_{p+4}) - \dots \end{aligned}$$

$$\text{Thus } - (u_{p+1} - u_{p+2}) > R_p > - u_{p+1}.$$

Observing that in the first inequality R_p is positive, while in the second it is negative, it follows that in both cases

$$u_{p+1} - u_{p+2} < | R_p | < u_{p+1}.$$

In other words, if *we stop the expansion at the pth term the error in the approximation is less than the absolute value of the* $(p + 1)$*th term*, and greater than the absolute value of the difference of the $(p + 1)$*th* and $(p + 2)$*th* terms.

CASE (ii). All terms ultimately of the same sign. Consider first $(1 - x)^n$, and suppose $p > n$. This will ensure that the greatest term is passed. Also

$$\frac{u_{r+1}}{u_r} < x \text{ all } r > p. \text{ Hence}$$

$$\begin{aligned} R_p &= u_{p+1} + u_{p+2} + u_{p+3} + \dots \\ &= u_{p+1}\left(1 + \frac{u_{p+2}}{u_{p+1}} + \frac{u_{p+3}}{u_{p+1}} + \dots\right) \\ &= u_{p+1}\left(1 + \frac{u_{p+2}}{u_{p+1}} + \frac{u_{p+3}}{u_{p+2}} \cdot \frac{u_{p+2}}{u_{p+1}} + \dots\right) \\ &< u_{p+1}(1 + x + x^2 + \dots) = u_{p+1}/(1 - x). \end{aligned}$$

Similarly we may consider $(1 - x)^{-n}$, $x > 0$, $n > 0$ in which case all the terms are positive. As before

$$| R_p | < u_{p+1}/(1 - x).$$

Thus in Example (1) of § 3·63 the error in stopping at the third term is less than

$$\frac{\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{7}{4}}{3!}\,\frac{1}{5^{12}}\Big/\left(1 - \frac{1}{5^4}\right) < \frac{7}{4^3 \times 5^{12}} = \frac{7 \times 2^6}{10^{12}} < \frac{5}{10^{10}}.$$

This error cannot affect the 6th decimal place.

3·7. Limit of a Quotient

Suppose we require to determine the limit as $x \to a$ of the quotient of the functions $f(x)$ and $\phi(x)$ where $f(x)$ and $\phi(x)$ are continuous. We suppose first that a is finite. Then since the functions are continuous

$$\lim_{x \to a} f(x) = f(a), \qquad \lim_{x \to a} \phi(x) = \phi(a),$$

$f(a)$, $\phi(a)$ denoting the defined values at $x = a$. Also provided $\phi(a) \neq 0$, $f(x)/\phi(x)$ is continuous at $x = a$ and

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{f(a)}{\phi(a)}.$$

Now suppose $\phi(a) = 0$. Then there are two possibilities. Either $f(a) \neq 0$ or $f(a) = 0$.

In the former case it is clear that the limit does not exist, for

$$\frac{f(x)}{\phi(x)} \to \pm\infty \text{ as } x \to a.$$

In the second case $f(x)/\phi(x)$ has the indeterminate form $0/0$ at $x = a$. The equations

$$f(x) = 0, \quad \phi(x) = 0$$

have a *common root* at $x = a$. We can say that $x = a$ *is a zero* of both $f(x)$ and $\phi(x)$, i.e. *the functions possess a common zero.* It does *not* follow that $f(x)$ and $\phi(x)$ have a common factor of the form $x - a$. *This would only be the case if* $f(x)$ *and* $\phi(x)$ *were polynomials.*

To discuss $\lim\limits_{x \to a} \dfrac{f(x)}{\phi(x)}$ in the case of the indeterminate form change the variable from x to h where $x - a = h$. Thus $h \to 0$ as $x \to a$. Then

$$\frac{f(x)}{\phi(x)} = \frac{f(a+h)}{\phi(a+h)}.$$

If we can expand the functions $f(a+h)$, $\phi(a+h)$ as a series in powers of h, h being small, we can proceed further in the determination of the limit.

Thus suppose that by means of the binomial theorem, or by any other method we can obtain expansions of the form

$$f(a+h) = a_0 + a_1h + a_2h^2 + \ldots$$
$$\phi(a+h) = b_0 + b_1h + b_2h^2 + \ldots$$

Since a_0 is the value of $f(a+h)$ when $h = 0$ and $f(a) = 0$ it follows that $a_0 = 0$. Similarly $b_0 = 0$. Thus

$$\frac{f(a+h)}{\phi(a+h)} = \frac{a_1h + a_2h^2 + \ldots}{b_1h + b_2h^2 + \ldots} = \frac{a_1 + a_2h + \ldots}{b_1 + b_2h + \ldots}, \text{ if } h \neq 0;$$

$$\therefore \lim_{h \to 0} \frac{f(a+h)}{\phi(a+h)} = \frac{a_1}{b_1}, \text{ provided } b_1 \neq 0.$$

If $b_1 = 0$ and $a_1 \neq 0$ then clearly the limit does not exist for the ratio $f(a+h)/\phi(a+h)$ becomes infinite as $h \to 0$. If $b_1 = 0$, $a_1 = 0$ we obtain

$$\frac{f(a+h)}{\phi(a+h)} = \frac{a_2 + a_3h + \ldots}{b_2 + b_3h + \ldots}$$

and we may now argue as before. Thus

$$\lim_{h \to 0} \frac{f(a+h)}{\phi(a+h)} = \frac{a_2}{b_2}, \text{ provided } b_2 \neq 0, \text{ and so on.}$$

Next suppose that a is infinite, *i.e.* we require $\lim_{x \to \infty} \frac{f(x)}{\phi(x)}$. This may be reduced to the consideration of a finite limit by the substitution $x = 1/y$. As $x \to +\infty$, $y \to 0$ through *positive values*. Thus

$$\lim_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim \frac{f\left(\frac{1}{y}\right)}{\phi\left(\frac{1}{y}\right)},$$

as y tends to zero through *positive* values only. Similarly

$$\lim_{x \to -\infty} \frac{f(x)}{\phi(x)} = \lim \frac{f\left(\frac{1}{y}\right)}{\phi\left(\frac{1}{y}\right)}.$$

as y tends to zero through *negative* values only.

3·71. Application of the Binomial Expansion to Limits

We give a number of examples illustrating the use of the binomial expansion in the determination of limits of quotients.

Examples.—(1) *Find* $\lim_{h \to 0} \{(x+h)^n - x^n\}/h$, *where n denotes any rational number.*

$$(x+h)^n = x^n \left(1 + \frac{h}{x}\right)^n, \; x \neq 0.$$

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)}{2} hx^{n-2} + \begin{Bmatrix}\text{terms in } h^2 \text{ and}\\ \text{higher powers}\end{Bmatrix};$$

$$\lim \frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Note.—The student will observe that what has been proved in the evaluation of this limit is that *if $f(x) = x^n$, where n is any rational number, then the first differential coefficient $f^1(x)$ is nx^{n-1}.*

(2) *Prove that* $\lim\limits_{x \to 0} \dfrac{\sqrt{(1+x)} - \sqrt{(1+x^2)}}{\sqrt{(1-x^2)} - \sqrt{(1-x)}} = 1.$

See § 2·641, Ex. 4.

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \ldots, \qquad (1+x^2)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \ldots$$

$$(1-x^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 + \ldots, \qquad (1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \ldots;$$

$$\therefore \frac{\sqrt{(1+x)} - \sqrt{(1+x^2)}}{\sqrt{(1-x^2)} - \sqrt{(1-x)}} = \frac{\tfrac{1}{2}x - \tfrac{5}{8}x^2 + \text{higher powers of } x}{\tfrac{1}{2}x - \tfrac{3}{8}x^2 + \text{higher powers of } x}$$

$$= \frac{1 - \tfrac{5}{4}x + \text{powers of } x}{1 - \tfrac{3}{4}x + \text{powers of } x}.$$

$$\text{Thus} \quad \lim_{x \to 0} \frac{\sqrt{(1+x)} - \sqrt{(1+x^2)}}{\sqrt{(1-x^2)} - \sqrt{(1-x)}} = 1$$

(3) *Prove that* $\lim\limits_{x \to 1} \dfrac{(1 - m + mx)^n - (1 - n + nx)^m}{(1-x)^2} = \tfrac{1}{2}\, mn\, (n - m).$

Write $x - 1 = h$. Then

$$(1 - m + mx)^n = \{1 - m(1 - x)\}^n = (1 + mh)^n$$

$$= 1 + mnh + \frac{n(n-1)}{2!} m^2h^2 - \ldots, \quad |\, mh \,| < 1$$

$$(1 - n + nx)^m = \{1 - n(1 - x)\}^m = (1 + nh)^m$$

$$= 1 + mnh + \frac{m(m-1)}{2!} n^2h^2 \quad \ldots |\, nh \,| < 1.$$

The given quotient becomes

$$\frac{1}{h^2}\left[\left\{\frac{n(n-1)}{2} m^2 - \frac{m(m-1)}{2} n^2\right\} h^2 + \text{higher powers of } h\right]$$

$$= \tfrac{1}{2}\, mn\, (n - m) + \text{powers of } h$$

Letting $h \to 0$, the limit is $\tfrac{1}{2}\, mn\, (n - m)$.

(4) *Evaluate* $\lim\limits_{x \to \infty} \left[x^{n-m} \cdot \dfrac{(x+b)^m - (x-b)^m}{(x+b)^n - (x-b)^n}\right]$

$$x^{n-m} \cdot \frac{(x+b)^m - (x-b)^m}{(x+b)^n - (x-b)^n} = x^{n-m} \cdot \frac{x^m\left\{\left(1 + \frac{b}{x}\right)^m - \left(1 - \frac{b}{x}\right)^m\right\}}{x^n\left\{\left(1 + \frac{b}{x}\right)^n - \left(1 - \frac{b}{x}\right)^n\right\}}$$

$$= \frac{\left(1 + \frac{b}{x}\right)^m - \left(1 - \frac{b}{x}\right)^m}{\left(1 + \frac{b}{x}\right)^n - \left(1 - \frac{b}{x}\right)^n}$$

Since x is large we may assume that $\left|\frac{b}{x}\right| < 1$. Then

$$\left(1+\frac{b}{x}\right)^m - \left(1-\frac{b}{x}\right)^m = 1 + \frac{mb}{x} + \frac{m(m-1)}{2!}\frac{b^2}{x^2} + \ldots$$
$$- 1 + \frac{mb}{x} - \frac{m(m-1)}{2!}\frac{b^2}{x^2} - \ldots$$
$$= \frac{2mb}{x} + \text{terms in } \frac{1}{x^3} \text{ and higher powers of } \frac{1}{x}.$$

Also $\left(1+\frac{b}{x}\right)^n - \left(1-\frac{b}{x}\right)^n = \frac{2nb}{x} + \text{terms in } \frac{1}{x^3}$ and higher powers of $\frac{1}{x}$.

Proceeding to the limit, we get

$$\lim_{x \to \infty} \left[x^{n-m}.\frac{(x+b)^m - (x-b)^m}{(x+b)^n - (x-b)^n}\right] = \frac{2mb}{2nb} = \frac{m}{n}.$$

Alternately we may write $x = 1/y$ and consider

$$\lim. \; y^{m-n}\frac{(y^{-1}+b)^m - (y^{-1}-b)^m}{(y^{-1}+b)^n - (y^{-1}-b)^n}.$$

as $y \to 0$ through positive values.

3·8. To find the number of homogeneous products of r dimensions that can be formed out of n letters, $a_1, a_2, a_3, \ldots$ and their powers

Let $a_1, a_2, a_3, \ldots, a_n$ denote n numbers such that

$$|a_r| \leqslant 1, \; r = 1, 2, 3, \ldots, n.$$

Then if $|x| < 1$, $|a_r x| < 1$. Hence by the binomial theorem

$$(1 - a_r x)^{-1} = \sum_{s=0}^{\infty} a_r^s x^s = 1 + a_r x + a_r^2 x^2 + \ldots$$

Thus the product $\prod_{r=1}^{n} (1 - a_r x)^{-1} = 1 + S_1 x + S_2 x^2 + \ldots,$

where $S_1 = a_1 + a_2 + \ldots + a_n$, $S_2 = a_1^2 + a_1 a_2 + \ldots$
$$+ a_2^2 + a_2 a_3 + \ldots$$
$$S_3 = a_1^3 + a_1^2 a_2 + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Then $S_1, S_2, S_3, \ldots$ are the sums of the homogeneous products of *one, two, three* $\ldots$, dimensions which can be formed out of the letters, $a_1, a_2, a_3, \ldots$ and their powers.

It will be seen that in the product $\prod_{r=1}^{n} (1 - a_r x)^{-1}$ we are considering the product of a *finite number* of *absolutely convergent* series, the subsequent product being arranged in ascending powers of x. That this process is legitimate follows from the theorem on the product of *two* absolutely convergent series. (Chap. I., § 1·6) By repeated application of the theorem, the result clearly applies to the product of a *finite number* of absolutely convergent series.

To find the number of terms in S_1, S_2, S_3, ... put $a_r = 1$ and denote the corresponding values by ${}_nH_1$, ${}_nH_2$, ${}_nH_3$, ..., i.e. ${}_nH_r$ is the number of homogeneous products of r dimensions. Hence

$$\{(1 - x)^{-1}\}^n = 1 + \sum_{r=1}^{\infty} {}_nH_r x^r.$$

Now $$(1 - x)^{-n} = 1 + \sum_{r=1}^{\infty} \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r$$

and from the theorem on identical equality of power series (Chap. I., § 1·8), it follows that we can equate corresponding coefficients. Thus

$${}_nH_r = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}.$$

Note.—It will be observed that ${}_nH_r = {}_{n+r-1}C_r$ so that we can use the notation of Combinations in calculating the value in any particular case.

3·9. To find the number of terms in the expansion of a positive integral power of any polynomial

Let the polynomial be

$$a_0 + a_1 x + a_2 x^2 + \dots + a_{r-1} x^{r-1}$$

so that it contains r terms: it is assumed that the coefficients are all different from zero. We require the number of *distinct* terms in

$$(a_0 + a_1 x + a_2 x^2 + \dots + a_{r-1} x^{r-1})^n$$

where n is a positive integer, *i.e.* terms involving the *same* power of x are counted as distinct if the *literal coefficients are different.*

The expansion contains all possible combinations of the r letters a_0, a_1, a_2, ..., a_{r-1} and of their powers. Hence the total number of terms in the expansion is the same as the number of homogeneous

products of n dimensions which can be formed out of r letters and their powers,

$$\text{i.e.}\quad {}_rH_n = \frac{r(r+1)(r+2)\ldots(r+n-1)}{n\,!}.$$

Note.—If all the terms involving like powers of x are grouped together and treated as a single term the number of terms will in general be $nr - n + 1$ since the highest power of x is $x^{n(r-1)}$. This statement assumes that none of the coefficients in the series so obtained, is equal to zero. If some of the coefficients vanish this would imply the existence of relations between the original coefficients $a_0, a_1, a_2, \ldots a_{r-1}$.

Example.—*Find the number of terms in* $(a + b + c)^n$, *n being a positive integer.*

The required number is

$${}_3H_n = {}_{3+n-1}C_n = {}_{n+2}C_n = {}_{n+2}C_2 = \tfrac{1}{2}(n+1)(n+2).$$

EXERCISES III

28. (i) Find the coefficient of x^9 in $(1 + x^2)^3(1 - x^3)^{-2}$. (ii) Use the binomial theorem to calculate the value of $(8{\cdot}4)^{-2/3}$ correct to three places of decimals.

29. In the expansion of $(1 - x)^{-n}$, find the conditions that the sum of coefficients of the first three terms may be equal to the coefficient of the fourth term. Find also the condition that the sum of coefficients of the first r terms may be equal to the coefficient of the $(r + 1)$th term.

30. Find the coefficient of x^{10} in $(1 + x)/(1 - x)^3$.

31. Find the coefficient of x^r in $(1 - x)^2/(1 - 2x)^3$.

32. Find the coefficient of x^n in $(1 + 2x + 3x^2)/(1 - x)^3$.

33. Show that the coefficient of x^n in $(1 + 2x)/(1 - 2x)^2$ is $2^n(2n + 1)$.

34. Find the coefficient of x^r in $(1 + ax + a^2x^2 + \ldots \text{to infinity})^n$.

35. Show that the coefficient of x^2 in $(1 - 2x + 3x^2 - \ldots \text{to infinity})^{-\frac{3}{2}}$ is 3.

36. Show that the coefficient of x^{n+1} in the expansion of $(1 - x)\sqrt{1 + x}$ is

$$(-1)^n \cdot \frac{1}{2^{n+1}} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{(n+1\,!)} \cdot (4n+1).$$

37. Find the coefficient of x^r in the expansion of $\{(1 + x)/(1 - x)\}^{\frac{1}{2}}$.

38. Show that the coefficient of x^{2n} in $\dfrac{1}{(1 - x)^2(1 + x)^4}$ is

$$\tfrac{1}{6}(n+1)(n+2)(n+3).$$

39. Show that the coefficient of x^n in $(1 - 2x)^n/(1 - 3x)$ is 1 and that of x^{n+r} is 3^r.

40. Prove that the product of $\sum_{r=0}^{\infty} x^r$ and $\sum_{r=0}^{\infty} (r+1)x^r$ is

$$\frac{1}{2} \sum_{r=0}^{\infty} (r+1)(r+2)x^r.$$

41. Show that $x^n = 1 + n\left(1 - \frac{1}{x}\right) + \frac{n(n+1)}{1.2}\left(1 - \frac{1}{x}\right)^2 + \dots$

42. Show that

$$(a-b)^n = a^n\left\{1 - n.\frac{b}{a-b} + \frac{n(n+1)}{1.2}\left(\frac{b}{a-b}\right)^2 - \dots\right\}.$$

43. Show that $\left(\frac{1+x}{1-x}\right)^n = 1 + n.\left(\frac{2x}{1+x}\right) + \frac{n(n+1)}{1.2}.\left(\frac{2x}{1+x}\right)^2 + \dots$

44. Prove that $1 + n\frac{2n}{1+n} + \frac{n(n+1)}{1.2}.\left(\frac{2n}{1+n}\right)^2 + \dots$

$$= 1 + n\frac{2n}{1-n} + \frac{n(n-1)}{1.2}\left(\frac{2n}{1-n}\right)^2 + \dots$$

45. Find the first two terms in the expansion of $\dfrac{2 + 3x + (1-3x)^{\frac{1}{3}}}{1 - \frac{1}{2}x + (4-x)^{\frac{1}{2}}}$ in ascending powers of x.

46. (a) Find the coefficient of x^4 and of x^7 in $(1 + x + x^2 + x^3 + x^4)^3$.
(b) Show that the coefficient of x^{4m} in $(1 - x + x^2 - x^3)^{-1}$ is 1.

47. A student is examined in 3 papers with a maximum of n marks for each paper. In how many ways can he get a total of n marks?

[In each paper he may get 0, 1, 2, ... or n marks.

Hence the total number of ways in which he can get a total of n marks is the coefficient of x^n in
$(1 + x + x^2 + \dots + x^n)(1 + x + x^2 + \dots + x^n)(1 + x + x^2 + \dots + x^n)$, since every way of forming the coefficient of x^n, say $(x^2 \times x^2 \times x^{n-4})$, corresponds to a way of getting a total of n marks (2 marks in the first paper, 2 marks in the second paper and $n - 4$ marks in the third).]

48. A man sits for an examination in which there are four papers with a maximum of m marks for each paper. Show that the number of ways of getting half marks on the whole is

$$\frac{1}{3}(m+1)(2m^2 + 4m + 3).$$

Prove the following results:

49. $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \dots$ to $\infty = \sqrt{2}$.

50. $\sqrt[3]{\frac{3}{2}} = 1 + \frac{1}{3^2} + \frac{1.4}{1.2}.\frac{1}{3^4} + \frac{1.4.7}{1.2.3}.\frac{1}{3^6} + \dots$ to infinity.

51. $2 = 1 + \frac{1}{2.2} + \frac{1.3}{2.3.2^2} + \frac{1.3.5}{2.3.4.2^3} + \dots$ to infinity.

52. Find the value of $\sum\limits_{r=1}^{\infty} r\left(\frac{x}{1+x}\right)^{r-1}$ and state for what values of x the series converges to this sum.

53. Find, by multiplication or otherwise, the cube of

$$1 + \tfrac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \ldots \text{ to infinity,}$$

as far as x^3, in its simplest form.

54. Find, to five places of decimals, the value of $\sqrt[3]{1003} - \sqrt[3]{997}$.

55. Find the value of $(998)^{\frac{1}{3}}$ correct to five decimal places by means of the binomial theorem.

56. Show that $\sqrt[3]{3} = \frac{1000}{343} \times \left(\frac{1029}{1000}\right)^{\frac{1}{3}} = \frac{10}{7}(1 + \cdot 029)^{\frac{1}{3}}$, and hence find the cube root of 3 to four places of decimals.

57. Prove that $\sqrt[3]{2} = \frac{5}{4}(1 + \cdot 024)^{\frac{1}{3}}$, and hence find the cube root of 2 to four places of decimals.

58. Find the cube root of 29 to four places of decimals.

If x be so small that x^2 and higher powers of x may be neglected, show that

59. $(1 + 2x)^{\frac{1}{2}}(1 - 4x)^{-2\frac{1}{2}} = 1 + 11x.$

60. $\dfrac{(9 + 2x)^{\frac{1}{2}}(3 + 4x)}{\sqrt[5]{1 - x}} = 9 + 14\tfrac{4}{5}x.$

61. $\dfrac{\sqrt{1 + x} + \sqrt[3]{1 + 2x}}{\sqrt[4]{1 + 3x} + \sqrt[5]{1 + 4x}} = 1 - \tfrac{23}{120}x.$

62. $\dfrac{2 + 3x + (1 - 3x)^{\frac{1}{2}}}{1 - \frac{1}{4}x + (4 - x)^{\frac{1}{2}}} = 1 + \tfrac{2}{3}x.$

63. If $y = x/\{(x + 1)(x + 4)\}$, prove that y has a maximum value when $x = 2$. Show also that approximately, when $x = 2 + h$, where h is small, $y = \frac{1}{9}\left(1 - \frac{h^2}{18}\ldots\right)$.

64. If c be a quantity so small that c^3 may be neglected in comparison with l^3, show that

$$\sqrt{\frac{l}{l + c}} + \sqrt{\frac{l}{l - c}}$$

is very nearly equal to $2 + 3c^2/4l^2$.

65. Prove that $\lim\limits_{x \to 1} \{(1 - x)(ax^b - bx^a)/(x^b - x^a)\} = 1.$

66. Prove that $\lim\limits_{x \to a} \dfrac{(5a^2x - x^3)^{\frac{1}{2}} - 2a^{\frac{1}{2}}x}{a(3ax - 2x^2)^{\frac{1}{4}} - a^{\frac{3}{2}}} = 6.$

67. If $u = 1 + \sqrt{1 + x^2}$, $v = 1 - \sqrt{(1 + x^2)}$ show that as $x \to \infty$

$$u^{2m} - v^{2m} \sim 2m\,(u^{2m+1} - v^{2m+1})/x^2$$

where m denotes a positive integer.

68. Show that if t_r denote the middle term of $(1 + x)^{2r}$, then will

$$t_0 + t_1 + t_2 + \ldots = (1 - 4x)^{-\frac{1}{2}}.$$

69. If N and n be nearly equal, then $\sqrt{\dfrac{N}{n}} = \dfrac{N}{N + n} + \tfrac{1}{4}\cdot\dfrac{N + n}{n}$ approx.

70. If $\sqrt{N} = a + x$ where x is very small, then $\sqrt{N} = a \cdot \dfrac{3N + a^2}{N + 3a^2}$ approx.

71. If $\sqrt[3]{N} = a - x$ where x is very small, then

$$\sqrt[3]{N} = a \cdot \frac{2N + a^3}{N + 2a^3} \text{ approx.}$$

72. Prove that if M differ from N^2 by a small quantity, the square root of M is approximately equal to

$$\tfrac{3}{2}N - \frac{(3N^2 - M)^2}{8N^3}.$$

73. Show that if $p - q$ be small compared with p or q, then

$$\sqrt[n]{\frac{p}{q}} = \frac{(n + 1)\,p + (n - 1)\,q}{(n - 1)\,p + (n + 1)\,q} \text{ approx.}$$

$$\left[\sqrt[n]{\frac{p}{q}} = \frac{\{(p + q) + (p - q)\}^{\frac{1}{n}}}{\{(p + q) - (p - q)\}^{\frac{1}{n}}} = \frac{\{1 + (p - q)/(p + q)\}^{\frac{1}{n}}}{\{1 - (p - q)/(p + q)\}^{\frac{1}{n}}} = \ldots\right]$$

74. Apply the foregoing rule to find to three places of decimals (*a*) the square root of $\frac{19}{20}$; (*b*) the cube root of $\frac{24}{25}$; (*c*) $\sqrt[7]{\frac{119}{121}}$.

75. If n is a positive integer, prove that the coefficient of x^{3n} in the expansion of $(1 + x)/(1 + x + x^2)^3$ in a series of ascending powers of x is

$$\tfrac{1}{2}(n + 1)(3n + 2).$$ [*Camb. Sch.*]

76. If x, $1 - x$, and $(n - 1)$ are all positive, prove that the remainder after r terms in the expansion of $(1 - x)^{-n}$ is less than

$$n(n + 1)\ldots(n + r - 1)\,x^r(1 - x)^{-n}/r\,!$$ [*Lond., B.Sc.*]

77. Write down the expansion of $(1 + x)^n$, $|\,x\,| < 1$ and n not an integer, in ascending powers of x. Prove that

$$1 + \tfrac{3}{8} + \tfrac{3}{8}\cdot\tfrac{5}{16} + \tfrac{3}{8}\cdot\tfrac{5}{16}\cdot\tfrac{7}{24} + \ldots = 8\sqrt{3}/9.$$

[*London, B.Sc.*]

CHAPTER IV

INFINITE SERIES WHOSE TERMS ARE FUNCTIONS OF A VARIABLE

In Chapter I. some of the properties of power series of the form

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

have been considered. In particular if $a_n = \binom{n}{r}$ the power series becomes the binomial expansion

$$(1 + x)^n = \sum^{\infty} \binom{n}{r} x^r.$$

The terms of such series depend on a variable x and we have proved in the case of the binomial series that if x is any fixed value in the range $-1 < x < 1$ the series converges absolutely.

4·1. Meaning attached to the Sum of a Series

Any series whose terms depend upon a variable x may be expressed in the form

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots = \sum_{n=1}^{\infty} u_n(x).$$

Let us now consider precisely what is meant by the sum of such a series.

Denote by $S(x)$ the sum of the series, $S_n(x)$ the sum of the first n terms, $R_n(x)$ the remainder after n terms,

$${}_pR_n(x) = S_{n+p}(x) - S_n(x)$$

a partial remainder, where p is any positive integer. Then when we assert that the series converges for values of x lying in a certain range, the following properties are implied.

(i) We *fix the value of* x in the range, for which the sum is required and *insert* this value in *each term* of the series.

(ii) The value of $S_n(x)$ is then found.

(iii) *Keeping x fixed,* $S(x)$ is the limit as $n \to \infty$ of $S_n(x)$, i.e. *corresponding to the arbitrary positive number ϵ there exists a number n_0 such that for all* $n \geqslant n_0$

$$| S(x) - S_n(x) | < \epsilon.$$

The conditions for convergence may also be expressed in the form

$$|\,{}_pR_n\,| < \epsilon,\ n \geqslant n_0,\ p \text{ a positive integer};$$

or $$|\,R_n\,| < \epsilon,\ n \geqslant n_0.$$

We have emphasised in Chapter I., the property that the number n_0 *depends in general on* ϵ and this has been indicated by the notation $n_0\,(\epsilon)$.

In order to fix precisely the number n_0 let us agree that it is to be the *smallest positive integer* with the property such that

$$|\,R_n(x)\,| < \epsilon, \text{ for all } n \geqslant n_0.$$

Now the terms of the series $\Sigma\, u_n(x)$ *depend on* x and consequently we would expect n_0 to depend also on the particular value of x used in determining this sum. This is equivalent to the statement that n_0 is a function of x, *i.e.* if we alter the value of x, the value of n_0 is, in general, altered. Thus n_0 is a function of *both* x *and* ϵ. This is conveniently indicated by the notation $n_0\,(x,\,\epsilon)$.

Examples.—(1) Let the sum of a given series to n terms be $1/(x + n)$ where $x > 0$. Thus $S_n(x) = \dfrac{1}{x+n}$.

Clearly if $x > 0$, $\lim\limits_{n \to \infty} S_n(x) = 0$. The condition for convergence is

$$\frac{1}{x+n} < \epsilon, \text{ i.e. } x + n > \frac{1}{\epsilon}, \text{ i.e. } n > \frac{1}{\epsilon} - x.$$

Since x is a *fixed* positive number we can choose ϵ satisfying $x < \dfrac{1}{\epsilon}$ so that $\dfrac{1}{\epsilon} - x$ is positive.

It follows that $n_0 = \left[\dfrac{1}{\epsilon} - x\right] + 1$. This equation* shows that n_0 depends on x.

(2) Consider the series $\sum\limits_{n=1}^{\infty} u_n(x)$ where

$$u_n(x) = x/\{(n-1)\,x + 1\}\,\{nx + 1\},$$

for $x > 0$. The series is

$$\frac{x}{x+1} + \frac{x}{(x+1)\,(2x+1)} + \frac{x}{(2x+1)\,(3x+1)} + \ldots$$

To find the sum to n terms of this series we observe that

$$u_n(x) = \frac{1}{(n-1)\,x+1} - \frac{1}{nx+1};$$

$$\therefore\ S_n(x) = \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{2x+1}\right) + \left(\frac{1}{2x+1} - \frac{1}{3x+1}\right) + \ldots$$
$$+ \left(\frac{1}{(n-1)\,x+1} - \frac{1}{nx+1}\right) = 1 - \frac{1}{nx+1}.$$

* Throughout this chapter the notation $[y]$ will be taken to denote *the integral part of* y.

It is necessary to consider two cases.

If $x > 0$, $nx \to \infty$ as $n \to \infty$. Thus

$$\lim_{n \to \infty} S_n(x) = 1 - \lim_{n \to \infty} 1/(nx + 1) = 1.$$

If $x = 0$, $nx = 0$ and $S_n(0) = 0$. Hence

$$\lim_{n \to \infty} S_n(0) = 0,$$

and it is obvious in this latter case that the sum of the series is 0, for each term is zero.

If $x > 0$ the condition for convergence is

$$\left|\frac{1}{nx + 1}\right| < \epsilon, \text{ i.e. } nx + 1 > 1/\epsilon, \text{ i.e. } n > \left\{\frac{1}{\epsilon} - 1\right\}\Big/x.$$

Thus $n_0 = \left[\left(\frac{1}{\epsilon} - 1\right)\Big/x\right] + 1$, using the same notation as in Ex. 1.

From this equation it follows that *the smaller x is the larger will be n_0.* In fact

$$\lim_{x \to 0} \left[\left(\frac{1}{\epsilon} - 1\right)\Big/x\right] = +\infty.$$

On the other hand $|S(0) - S_n(0)| < \epsilon$ for *every* positive integer n, since $S(0) = S_n(0) = 0$. In this case $n_0 = 1$.

This example shows that ***although all the terms of the series are continuous functions of x the sum function may be discontinuous.*** For in order that the sum function $S(x)$ be continuous at $x = 0$ it is necessary that

$$\lim_{x \to 0} S(x) = S(0).$$

The left-hand side is unity, while the right-hand side is zero.

Let us next consider the approximation curves as in Vol. I., Chap. XIII.

$$\text{Write } y_n = S_n(x) = 1 - \frac{1}{nx + 1},$$

$$\text{i.e. } (y_n - 1)\left(x + \frac{1}{n}\right) = -\frac{1}{n}.$$

We give the graphs for the cases $n = 1, 5, 10, 30$. The student will recognise that the graph of $y = y_n$ is a rectangular hyperbola whose asymptotes are the lines $x = -\frac{1}{n}$, $y = 1$. As $n \to \infty$, $-\frac{1}{n} \to 0$ so that the line $x = -\frac{1}{n}$ approaches the line $x = 0$, *i.e.* the axis of y. As we now are only concerned with positive values of x only a part of one branch of the hyperbola is shown in each case. (See Fig. 5.)

$x =$	0	0·2	0·5	1	1·5	2
$y_1 =$	0	0·17	0·33	0·50	0·60	0·67
$y_5 =$	0	0·50	0·71	0·83	0·88	0·91
$y_{10} =$	0	0·67	0·83	0·91	0·95	0·95
$y_{30} =$	0	0·86	0·94	0·97	0·978	0·983

As $n \to \infty$, $x \geqslant 0$ the graph $y = y_n$ approach nearer and nearer to the y-axis between $y = 0$ and $y = 1$, and to the line $y = 1$.

If we argued from the approximating curves it would appear that $S(x) = 1$ when $x > 0$, while when $x = 0$, $S(x)$ might have any value between 0 and 1. Of course $S(0)$ can only have *one value* since each term of the series is *single-valued*.

The actual graph of the sum function $y = S(x)$ is the line $y = 1$ for $x > 0$, and the origin corresponding to $x = 0$.

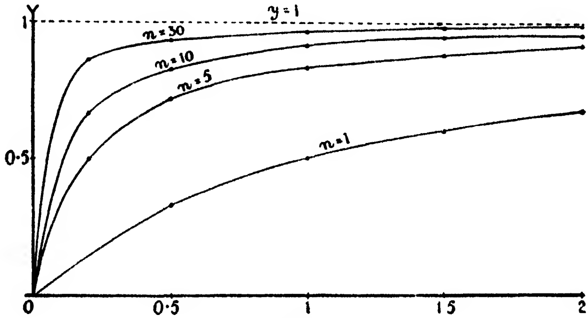


FIG. 5.

The example just discussed shows the dangers of arguing *intuitively*. Since $S_n(x) \to S(x)$ as $n \to \infty$ for $x \geqslant 0$, we would *expect* that if the curves $y = S_n(x)$, $y = S(x)$ can be represented graphically *the curves* $y = S_n(x)$ *would approach more and more closely to the curve* $y = S(x)$ *as* n *becomes larger and* larger. Actually this is *false* as the example shows.

The only sense in which we can assert that the curves do approach coincidence is that if we choose *a particular value of* x, then corresponding to ϵ there exists an integer n_0 such that for *this particular value of* x, the absolute value of the difference of the ordinates of $y = S_n(x)$ $y = S(x)$ will be less than ϵ for all $n \geqslant n_0$. Thus, *e.g.* it is clear if we take $x = 0{\cdot}2$ and $\epsilon = 0{\cdot}15$ that

$$| S(x) \quad S_n(x) | < \epsilon, \text{ for all } n \geqslant 30.$$

$$\text{For } | S(x) \quad\quad = 1 - 0{\cdot}86 = 0{\cdot}14 < 0{\cdot}15.$$

Also for $n > 30$ the corresponding approximation curve lies above $y = y_{30}$ and below $y = 1$.

4·2. Repeated Limits

The difficulties which arise in Ex. 2 of § 4·1 are due to what are called **repeated limits**. Let $\Sigma\, u_n(x)$ be a series which converges to $S(x)$ for each value of x in an interval (α, β) and let a be a value

of x which lies inside the interval. In accordance with this notation $S(a)$ will denote the sum, $u_n(a)$ the nth term, of the series corresponding to $x = a$.

If $S_n(x) = \sum_{r=1}^{n} u_r(x)$, then by definition

$$S(a) = \lim_{n \to \infty} S_n(a).$$

This is a particular case of $S(x) = \lim_{n \to \infty} S_n(x)$. In this last equation suppose that $x \to a$. Then

$$\lim_{x \to a} S(x) = \lim_{x \to a} [\lim_{n \to \infty} S_n(x)].$$

If it is legitimate to *invert the order of the limits*, i.e. let $x \to a$ *first* and $n \to \infty$ afterwards, then we would obtain

$$\lim_{n \to \infty} [\lim_{x \to a} S_n(x)].$$

Now $S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$ and as $u_n(x)$ has been assumed to be a continuous function of x it follows that $S_n(x)$ is a continuous function, being the sum of a finite number of continuous functions. Hence

$$\lim_{x \to a} S_n(x) = S_n(a).$$

It follows that

$$\lim_{n \to \infty} [\lim_{x \to a} S_n(x)] = \lim_{n \to \infty} S_n(a) = S(a).$$

Hence in order that we may assert that $\lim_{x \to a} S(x) = S(a)$ we require that

$$\lim_{x \to a} [\lim_{n \to \infty} S_n(x)] = \lim_{n \to \infty} [\lim_{x \to a} S_n(x)],$$

that is, *we must be able to reverse the order of the limiting processes without affecting its value.* Ex. 2 of § 4·1 shows that this reversal is not always legitimate.

4·3. Uniform Convergence

The discussion of the previous section brings us to the notion of *uniformity* as applied to an infinite series.

Consider the series $\Sigma \dfrac{x}{\{(n-1)x+1\}\{nx+1\}}$. Then it has

been shown in § 4·1 that corresponding to ϵ we can take

$$n_0 = \left[\left(\frac{1}{\epsilon} - 1\right)\Big/ x\right] + 1$$

such that $| R_n(x) | < \epsilon$, all $n \geqslant n_0$, where $R_n(x)$ denotes the remainder after n terms. Since

$$\left[\left(\frac{1}{\epsilon} - 1\right)\Big/ x\right]$$

becomes infinite as $x \to 0$ it follows that there does not exist a fixed number n_0 such that *for all values of* x in the range $x \geqslant 0$,

$$| R_n(x) | < \epsilon, \text{ for all } n \geqslant n_0,$$

i.e. there is no number n_0 which will serve for *all values* of x in the the interval.

On the other hand if we exclude $x = 0$ and suppose that $x \geqslant \lambda > 0$ we can find a number n_0 which will suffice. For in this case the *greatest value* of $\left(\frac{1}{\epsilon} - 1\right)\Big/ x$ is

$$\left(\frac{1}{\epsilon} - 1\right)\Big/ \lambda \text{ and the value of } n_0 \text{ is } \left[\left(\frac{1}{\epsilon} - 1\right)\Big/ \lambda\right] + 1.$$

We say then that the series converges *uniformly* for all values of x in the range $x \geqslant \lambda > 0$. Thus we are led to the following definition. *A series $\Sigma u_n(x)$ is said to converge* **uniformly** *in an interval $\alpha \leqslant x \leqslant \beta$ if corresponding to the arbitrary positive number ϵ there exists a number n_0 such that*

$$| R_n(x) | < \epsilon, \text{ all } n \geqslant n_0,$$

the same n_0 serving for all values of x in the interval.

The student should note carefully the *extra* condition which implies uniformity. Ordinary convergence asserts that *corresponding to each value* of x we can find a number n_0, while uniform convergence asserts the existence of a number n_0 which is the same *for all values* of x in the range, *i.e.* a number which is *independent* of x. Clearly uniform convergence implies convergence in the ordinary sense, but the converse is not true.

Examples.—(1) *Show that the series whose sum to n terms is x^n is not uniformly convergent in the interval* $0 \leqslant x \leqslant 1$.

Write $S_n(x) = x^n$, $0 \leqslant x \leqslant 1$.

If $0 \leqslant x < 1$, $\lim\limits_{n \to \infty} x^n = 0$; if $x = 1$, $S_n(x) = 1$ and $\lim\limits_{n \to \infty} S_n(x) = 1$.

Thus the sum function is discontinuous at $x = 1$.

Consider $0 < x < 1$. The condition for uniform convergence is that corresponding to ϵ we can find n_0 such that

$$|x^n| < \epsilon, \text{ all } n > n_0,$$

the *same* n_0 holding for all values of x in the interval.

If $x = 0$ the condition is satisfied for all values of n so that we may take $n_0 = 1$.

If $x > 0$, $x^n > 0$ and $|x^n| = x^n$. Taking logarithms to base 10, the condition $x^n < \epsilon$ becomes

$$n \log x < \log \epsilon.$$

Since $0 < x < 1$, $\log x < 0$. Dividing both sides of the inequality by the *negative* number $\log x$ we obtain

$$n > \log \epsilon/\log x.$$

It will be observed that when ϵ is small, $\log \epsilon < 0$ so that the ratio $\log \epsilon/\log x$ is always positive.

As $x \to 1$, $\log x \to 0$ and $\log \epsilon/\log x \to \infty$.

For any *fixed* value of x between 0 and 1 we can take

$$n_0 = [\log \epsilon/\log x] + 1.$$

Since $n_0 \to \infty$ as $x \to 1$ it follows that no n_0 exists which will be sufficient for all values of x in the range $0 \leqslant x \leqslant 1$.

On the other hand when $x = 1$ we may take $n_0 = 1$; for in this case $R_n(x) = 0$.

(2) *Prove that the series* $\sum_{n=1}^{\infty} x^{n-1}(1 - x)^2$ *converges uniformly to* $1 - x$ in $0 \leqslant x \leqslant 1$.

$$\begin{aligned} \text{Now } S_n(x) &= (1 - x)^2 + x(1 - x)^2 + \ldots + x^{n-1}(1 - x)^2 \\ &= (1 - x)^2 \{1 + x + \ldots + x^{n-1}\} \\ &= (1 - x)^2 (1 - x^n)/(1 - x) = (1 - x)(1 - x^n). \end{aligned}$$

If $0 \leqslant x < 1$, $\lim\limits_{n \to \infty} x^n = 0$, so that

$$S(x) = 1 - x \text{ for } 0 \leqslant x < 1.$$

When $x = 1$, $\lim\limits_{n \to \infty} x^n = 1$ and $S(x) = 0$.

Since $1 - x = 0$ when $x = 1$ it follows that

$$S(x) = 1 - x, \quad 0 \leqslant x \leqslant 1.$$

This proves that the series converges to $(1 - x)$ for $0 \leqslant x \leqslant 1$.

To prove that the convergence is uniform in $0 \leqslant x \leqslant 1$ it is necessary to find a number n_0 such that for all $n > n_0$

$$|(1 - x) - (1 - x)(1 - x^n)| < \epsilon$$

the *same* n_0 serving for all x in $0 \leqslant x \leqslant 1$.

Now $(1 - x) - (1 - x)(1 - x^n) = x^n(1 - x) \geqslant 0, 0 \leqslant x \leqslant 1$.

Let ϵ be an arbitrary positive number. Divide the interval $(0, 1)$ into two parts,

$$0 \leqslant x \leqslant 1 - \epsilon, \quad 1 - \epsilon < x \leqslant 1.$$

Suppose that x lies in the first interval. Then

$$(1 - x)\, x^n < x^n < \epsilon \text{ provided } n > \log \epsilon/\log x.$$

Now the numerically least value of $\log x$ is $\log (1 - \epsilon)$. Thus the greatest value of $\log \epsilon/\log x$ is $\log\epsilon/\log (1 - \epsilon)$.

Hence we may take $n_0 = [\log \epsilon/\log (1 - \epsilon)] + 1$. This value of n_0 will suffice for all values of x in the range $0 \leqslant x \leqslant 1 - \epsilon$.

Next consider the second interval $1 - \epsilon < x \leqslant 1$. Since $x^n \leqslant 1$ it follows that

$$(1 - x)\, x^n < \epsilon \text{ provided } (1 - x) < \epsilon, \text{ i.e. } 1 - \epsilon < x.$$

This condition is satisfied for all values of n so that we may take $n_0 = 1$.

Since the former value of n_0 is in general greater than unity it follows that for all values of $n > n_0$ where

$$n_0 = [\log \epsilon/\log (1 - \epsilon)] + 1,$$

$$|\,(1 - x) - (1 - x)\,(1 - x^n)\,| < \epsilon.$$

As we have found a value of n_0 *independent of* x the property of *uniform* convergence is proved.

4·31. A Necessary and Sufficient Condition for Uniform Convergence

Corresponding to the general principle of convergence for series (Chap. I., § 1·3) we have the following theorem: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be an infinite series whose terms are defined in the interval* (α, β). *Then a necessary and sufficient condition that the series should converge uniformly in* (α, β) *is that corresponding to the arbitrary positive number* ϵ *there exists a* positive *number* n_0 *independent of* x *such that*

$$|\, S_{n+p}(x) - S_n(x)\,| < \epsilon$$

for $n > n_0$ *and every positive integer* p.

(i) *The condition is necessary.* Suppose the series converges uniformly to $S(x)$. Then there exists a number n_0 *independent of* x such that

$$|\, S(x) - S_n(x)\,| < \tfrac{1}{2}\epsilon, \text{ all } n \geqslant n_0.$$

Let n, $n + p$ be two such values of n. Then

$$\begin{aligned} |\, S_{n+p}(x) - S_n(x)\,| &= |\, S_{n+p}(x) - S(x) + S(x) - S_n(x)\,| \\ &\leqslant |\, S_{n+p}(x) - S(x)\,| + |\, S(x) - S_n(x)\,| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon, \quad n \geqslant n_0. \end{aligned}$$

Hence the condition is necessary.

(ii) *The condition is sufficient.* First we observe that the given condition implies that the series converges for each value of x to some function $S(x)$. The given condition is

$$| S_{n+p}(x) - S_n(x) | < \epsilon$$

for $n \geqslant n_0$, and every positive integral value of p, n_0 being *independent* of x. Thus

$$S_n(x) - \epsilon < S_{n+p}(x) < S_n(x) + \epsilon.$$

Now keep n fixed and let $p \to \infty$. Since $S_{n+p}(x) \to S(x)$ as $p \to \infty$ it follows that

$$S_n(x) - \epsilon \leqslant S(x) \leqslant S_n(x) + \epsilon,$$

$$\text{i.e. } | S(x) - S_n(x) | < \epsilon.$$

Now n is any integer which is greater than or equal to n_0, n_0 being *independent of* x. Hence $S_n(x)$ converges uniformly to $S(x)$.

4·32. Uniform Convergence in a Closed Interval

We now prove that if the functions involved are *continuous*, an interval of uniform convergence is always closed. The theorem may be stated precisely as follows. *Let $S_n(x)$ be the sum to n terms of a series whose terms are all continuous in the* closed *interval $\alpha \leqslant x \leqslant \beta$, so that $S_n(x)$ is continuous in this interval. Suppose also that $S_n(x)$ converges uniformly to $S(x)$ in the* open *interval $\alpha < x < \beta$. Then $S_n(x)$ converges uniformly to $S(x)$ in the* closed *interval $\alpha \leqslant x \leqslant \beta$.*

Let ϵ denote an arbitrary positive number. Since $S_n(x)$ converges uniformly to $S(x)$ in $\alpha < x < \beta$ there exists a number n_0 *independent* of x such that

$$| S_{n+p}(x) - S_n(x) | < \tfrac{1}{3}\epsilon$$

for all $n \geqslant n_0$ and for every positive integer p.

Since $S_n(x)$, $S_{n+p}(x)$ are continuous at $x = \alpha$, there exist numbers of η_1, η_2 corresponding to ϵ such that

$$| S_n(x) - S_n(\alpha) | < \tfrac{1}{3}\epsilon,$$

for all x in the interval $0 \leqslant x - \alpha \leqslant \eta_1$,

$$| S_{n+p}(x) - S_{n+p}(\alpha) | < \tfrac{1}{3}\epsilon,$$

for all x in the interval $0 \leqslant x - \alpha \leqslant \eta_2$.

Let η be the smaller of η_1 and η_2 and let $0 \leqslant x - \alpha \leqslant \eta$.

Then

$$\begin{aligned}|S_{n+p}(a) - S_n(a)| &= |S_{n+p}(a) - S_{n+p}(x) + S_{n+p}(x) \\ &\qquad - S_n(x) + S_n(x) - S_n(a)| \\ &< |S_{n+p}(a) - S_{n+p}(x)| + |S_{n+p}(x) \\ &\qquad - S_n(x)| + |S_n(x) - S_n(a)| \\ &< \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon;\end{aligned}$$

i.e. $|S_{n+p}(a) - S_n(a)| < \epsilon$, for $n \geqslant n_0$, p a positive integer.

From the way n_0 has been defined it is clear that it is independent of x. Hence $S_n(x)$ converges at $x = a$, and further since n_0 is independent of x the convergence is uniform. Similarly for $x = \beta$.

4·33. Tests for Uniform Convergence

As in the case of ordinary convergence, when we are dealing with particular series it is convenient to develop some simple tests which may be used in determining uniform convergence.

The simplest test, and by far the most important, is known as Weierstrass' M-test. *Let* $\{M_n\}$ *denote a series of* positive constants independent of x *such that* ΣM_n *converges. Then if the series* $\Sigma u_n(x)$ *has the property that*

$$|u_n(x)| \leqslant M_n$$

in some interval $a \leqslant x \leqslant \beta$, *for all values of* n, *then the series* $\Sigma u_n(x)$ *converges* absolutely *and* uniformly *in the interval* (a, β).

The property of absolute convergence follows immediately from the comparison test of Chap. I., § 1·35.

To prove the uniform property we may proceed as follows.

$$\begin{aligned}\left|\sum_{r=n+1}^{n+p} u_r(x)\right| &\leqslant \sum_{r=n+1}^{n+p} |u_r(x)| \\ &\leqslant \sum_{r=n+1}^{n+p} M_r.\end{aligned}$$

Since ΣM_r converges, there exists a number n_0 depending upon an arbitrary positive ϵ such that

$$\sum_{r=n+1}^{n+p} M_r < \epsilon, \text{ all } n > n_0.$$

Since M_r is independent of x it follows that n_0 is independent of x. Thus

$$\left|\sum_{r=n+1}^{n+p} u_r(x)\right| < \epsilon, \text{ for } n > n_0,$$

the same n_0 serving for all values of x in the interval (α, β). Hence the series converges uniformly in (α, β).

Examples.—(1) *The series* $\Sigma r^n \cos n\theta$, $\Sigma r^n \sin n\theta$ *converge uniformly for all real values* θ if $0 < r < 1$.

It should be observed that uniform convergence is *with respect to the variable* θ.

For $|r^n \cos n\theta| \leqslant r^n$, since $|\cos n\theta| \leqslant 1$ for all values of θ.

Since the series Σr^n, $0 < r < 1$, is a series of positive constants which converges, the uniform convergence of the series $\Sigma r^n \cos n\theta$ follows immediately from the M-test. Similarly for the second series.

(2) *Prove that the series whose nth term is* $x/n(1 + nx^2)$ *converges uniformly for all real values of* x.

Write $\dfrac{x}{n(1 + nx^2)} = y$. Then $yn^2x^2 - x + ny = 0$.

From the Theory of Quadratic Equations [Vol. I., § 17·4] x is real if $1 > 4y^2n^3$.

$$\text{Hence } y^2 \leqslant 1/4n^3, \text{ i.e. } |y| \leqslant 1/2n^{\frac{3}{2}}.$$

$$\text{Thus } \left|\frac{x}{n(1 + nx^2)}\right| \leqslant \frac{1}{2n^{\frac{3}{2}}}, \text{ for all real values of } x.$$

Since $\Sigma n^{-\frac{3}{2}}$ converges, the series $\sum\limits_{r=1}^{\infty} \dfrac{x}{n(1 + nx^2)}$ converges absolutely and uniformly for all real values of x.

4·4. Continuity of the Sum Function

So far we have not given any reason for the introduction of the idea of uniformity into convergence. The first reason is given in this section. It is that the sum of a uniformly convergent series is continuous. There are, however, many other important reasons which cannot be given here but which will become clear during the study of more advanced work.

(i) **Uniform convergence implies continuity in the sum.** *Let* $\Sigma u_n(x)$ *denote a series of which each term is continuous in an interval* $\alpha \leqslant x \leqslant \beta$. *Suppose further that* $\Sigma u_n(x)$ *converges uniformly to* $S(x)$ *in* $\alpha \leqslant x \leqslant \beta$. *Then* $S(x)$ *is continuous in* $\alpha \leqslant x \leqslant \beta$.

Let ϵ be an arbitrary positive number. Then there exists a number n_0 *independent of* x such that

$$|S(x) - S_n(x)| < \tfrac{1}{3}\epsilon, \text{ all } n \geqslant n_0.$$

Now $S_n(x)$ is a continuous function of x, being the sum of a finite number of continuous functions. Let a be a value of x in $\alpha \leqslant x \leqslant \beta$ so that

$$|S(a) - S_n(a)| < \tfrac{1}{3}\epsilon, \text{ all } n \geqslant n_0.$$

There exists a number of η such that

$$| S_n (x) - S_n (a) | < \tfrac{1}{3}\epsilon.$$

for all x in the interval $| x - a | \leqslant \eta$.

To prove continuity at $x = a$ it is sufficient to show that corresponding to ϵ there exists a number η such that

$$| S (x) - S (a) | < \epsilon$$

for all x in the interval $| x - a | \leqslant \eta$. Now

$$\begin{aligned} | S (x) - S (a) | &= | S (x) - S_n(x) + S_n(x) - S_n(a) + S_n(a) - S (a) | \\ &\leqslant | S (x) - S_n(x) | + | S_n(x) - S_n(a) | \\ &\qquad\qquad + | S_n(a) - S (a) | \\ &< \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon \\ &= \epsilon, \text{ for } | x - a | \leqslant \eta. \end{aligned}$$

Hence $S (x)$ is continuous at $x = a$.

Since a is any value of x in $\alpha \leqslant x \leqslant \beta$ it follows that $S (x)$ is continuous in (α, β).

In order to illustrate how the argument fails when $\Sigma u_n(x)$ is not uniformly convergent, consider the series

$$\sum_{1}^{\infty} u_n(x) \quad \text{where} \quad \sum_{r=1}^{n} u_r (x) = - x^n, \quad 0 \leqslant x \leqslant 1.$$

We know that the sequence $\{x^n\}$ converges uniformly in any closed interval contained in $0 \leqslant x \leqslant 1$ which excludes $x = 1$. We treat then $x = 1$ as our testing point.

$$u_n(x) = \sum_{r=1}^{n} u_r(x) - \sum_{r=1}^{n-1} u_r(x) = x^{n-1} - x^n.$$

Thus the given series is

$$(1 - x) + (x - x^2) + (x^2 - x^3) + \ldots + (x^{n-1} - x^n) + \ldots$$

We need to consider $| S (x) - S_n(x) |$, $| S_n(x) - S_n(1) |$, $| S_n (1) - S (1) |$. In the general proof each of these terms is less than $\tfrac{1}{3}\epsilon$.

$$\begin{aligned} \text{Now} \qquad S (x) - S_n(x) &= (x^n - x^{n+1}) + (x^{n+1} - x^{n+2}) + \ldots \\ &= x^n, \ 0 < x < 1 \\ S_n(x) - S_n(1) &= (1 - x^n) - 0 = 1 - x^n. \\ S_n (1) - S (1) &= 0. \end{aligned}$$

Clearly $| S_n(1) - S(1) | < \frac{1}{3}\epsilon$, so that we need only consider $| S(x) - S_n(x) | < \frac{1}{3}\epsilon$ and $| S_n(x) - S_n(1) | < \frac{1}{3}\epsilon$. The former requires $x^n < \frac{1}{3}\epsilon$ and the latter $1 - x^n < \frac{1}{3}\epsilon$, or $x^n > 1 - \frac{1}{3}\epsilon$. The two latter inequalities are contradictory and so the two corresponding steps are incompatible. The fact is that $x^n < \frac{1}{3}\epsilon$ does not lead to the determination of an n_0 *independent* of x as $x \to 1$. Uniform convergence ensures that the two inequalities $| S(x) - S(a) |$ and $| S_n(x) - S(a) |$ do not contradict each other.

(ii) **Conversely** *if the terms of the series are* **continuous** *and the sum function is* **discontinuous** *then the series cannot converge uniformly in any interval which contains the discontinuity.*

For if the series converged uniformly the sum function would be continuous.

(iii) *Uniform convergence is a* **sufficient** *condition for continuity of the sum of an infinite series whose terms are continuous, but the condition is* **not necessary.**

This property may be shown by an example. Consider the series whose sum to n terms is $n^2x/(1 + n^3x^2)$. Thus

$$S_n(x) = n^2x/(1 + n^3x^2).$$

If $x = 0$, $S_n(0) = 0$ and hence $S(0) = 0$.

If $x \neq 0$, $S_n(x) = x\Big/\left(\dfrac{1}{n^2} + nx\right) \to 0$ as $n \to \infty$.

Hence $S(x) = 0$ for all finite values of x. Thus the sum function is continuous.

We now show that the series does not converge uniformly at $x = 0$. The condition $| S(x) - S_n(x) | < \epsilon$ becomes

$$\frac{n^2 \, | x |}{1 + n^3x^2} < \epsilon.$$

Suppose there exists a number n_0 such that this condition is satisfied for all $n \geqslant n_0$, the same n_0 serving for all values of x in the range $x \geqslant 0$. Write $x = 1/n_0^{\frac{3}{2}}$. Then

$$\frac{n^2 \, | x |}{1 + n^3x^2} = \frac{n^2/(n_0)^{\frac{3}{2}}}{1 + n^3/n_0^{3}}.$$

Taking $n = n_0$, this gives $\frac{1}{2}\, n_0^{\frac{1}{2}}$, the least value of which is $\frac{1}{2}$.

Thus if $\epsilon < \frac{1}{2}$ the condition $\dfrac{n^2 \, | x |}{1 + n^3x^2} < \epsilon$ is not satisfied.

Since $1/n_0^{\frac{3}{2}} \to 0$ as $n_0 \to \infty$ it follows that the series cannot converge uniformly at $x = 0$.

Example.—*Express* $u_n(x) = x/\{nx + 1\}\{(n+1)x + 1\}\{(n+2)x + 1\}$ *as the sum of three partial fractions,* $x \geqslant 0$. *Hence find the sum of the series* $\sum_{n=1}^{\infty} u_n(x)$ *and show that it is not uniformly convergent at* $x = 0$.

$$\frac{x}{\{nx+1\}\{(n+1)x+1\}\{(n+2)x+1\}}$$
$$\equiv \frac{A}{nx+1} + \frac{B}{(n+1)x+1} + \frac{C}{(n+2)x+1}$$

i.e. $x \equiv A\{(n+1)x+1\}\{(n+2)x+1\}$
$$+ B\{nx+1\}\{(n+2)x+1\} + C\{nx+1\}\{(n+1)x+1\}.$$

The expression on the right is a quadratic equation in x so that if we equate corresponding coefficients on both sides of the identity we obtain three equations to determine the quantities A, B, C.

It is simpler to write $x = -1/n,\ -1/(n+1),\ -1/(n+2)$ in succession. Thus

$$-\frac{1}{n} = A\left(-\frac{1}{n}\right)\left(-\frac{2}{n}\right),\ A = -\tfrac{1}{2}n.$$

$$-\frac{1}{n+1} = B\left(\frac{1}{n+1}\right)\left(-\frac{1}{n+1}\right),\ B = (n+1).$$

$$-\frac{1}{n+2} = C\left(\frac{2}{n+2}\right)\left(\frac{1}{n+2}\right),\ C = -\tfrac{1}{2}(n+2);$$

$$\therefore\ u_n(x) = \frac{-\frac{1}{2}n}{nx+1} + \frac{n+1}{(n+1)x+1} - \frac{\frac{1}{2}(n+2);}{(n+2)x+1}.$$

$$S_n(x) = \sum_{r=1}^{n} u_r(x) = \frac{-\frac{1}{2}}{x+1} + \frac{2}{2x+1} + \frac{-\frac{1}{2}\cdot 3}{3x+1}$$
$$+ \frac{-\frac{1}{2}\cdot 2}{2x+1} + \frac{3}{3x+1} + \frac{-\frac{1}{2}\cdot 4}{4x+1}$$
$$+ \frac{-\frac{1}{2}\cdot 3}{3x+1} + \frac{4}{4x+1} + \frac{-\frac{1}{2}\cdot 5}{5x+1}.$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$+ \frac{-\frac{1}{2}(n-2)}{(n-2)x+1} + \frac{n-1}{(n-1)x+1} + \frac{-\frac{1}{2}n}{nx+1}$$
$$+ \frac{-\frac{1}{2}(n-1)}{(n-1)x+1} + \frac{n}{nx+1} + \frac{-\frac{1}{2}(n+1)}{(n+1)x+1}$$
$$+ \frac{-\frac{1}{2}n}{nx+1} + \frac{n+1}{(n+1)x+1} + \frac{-\frac{1}{2}(n+2)}{(n+2)x+1}.$$

$$\therefore\; S_n(x) = \frac{-\frac{1}{2}}{x+1} + \frac{1}{2x+1} + \frac{\frac{1}{2}(n+1)}{(n+1)x+1} - \frac{\frac{1}{2}(n+2)}{(n+2)x+1}$$

$$= \tfrac{1}{2}\cdot\frac{1}{(x+1)(2x+1)} - \tfrac{1}{2}\cdot\frac{1}{\{(n+1)x+1\}\{(n+2)x+1\}}.$$

Hence provided $x > 0$

$$\lim_{n\to\infty} S_n(x) = \tfrac{1}{2}\cdot\frac{1}{(x+1)(2x+1)} = S(x).$$

When $x = 0$ each member of the given series is zero so that $S(0) = 0$.

$$\text{Also}\quad \lim_{x\to 0}\ \tfrac{1}{2}\cdot\frac{1}{(x+1)(2x+1)} = \tfrac{1}{2}.$$

Thus $S(x)$ is discontinuous at $x = 0$.

Since $u_n(x)$ is continuous for $x \geqslant 0$ it follows that the series cannot converge uniformly at $x = 0$.

4·5. Tannery's Theorem

The student will observe that this useful theorem is similar to Weierstrass' M-test: *Let $f(n)$ denote the sum of the series:*

$$u_1(n) + u_2(n) + \ldots + u_p(n),$$

where $u_r(n)$, $r = 1, 2, 3, \ldots p$, *denotes a function of n, and p is an integer which tends to infinity* as $n \to \infty$. Then if *for fixed r,*

$$\lim_{n\to\infty} u_r(n) = v_r,$$

then $\quad \lim\limits_{n\to\infty} f(n) = v_1 + v_2 + v_3 + \ldots$ *to infinity,* provided

$|u_r(n)| \leqslant M_r$ where M_r is independent of n and $\sum\limits_{r=1}^{\infty} M_r$ converges.

Since $\sum\limits_{r=1}^{\infty} M_r$ converges there exists a number ν_0 such that

$$\sum_{r=\nu}^{\infty} M_r < \epsilon,\ \text{all}\ \nu \geqslant \nu_0.$$

Let ν denote a *fixed* value which satisfies $\nu \geqslant \nu_0$. Since $p \to \infty$ as $n \to \infty$ we can assume that $p > \nu$. Thus

$$|f(n) - u_1(n) - u_2(n) - \ldots - u_{\nu-1}(n)|$$
$$= |u_\nu(n) + u_{\nu+1}(n) + \ldots + u_p(n)|$$
$$\leqslant |u_\nu(n)| + |u_{\nu+1}(n)| + \ldots + |u_p(n)|$$
$$\leqslant M_\nu + M_{\nu+1} + \ldots M_p$$
$$< \epsilon,\ p > \nu \geqslant \nu_0.$$

Again, since $v_r = \lim\limits_{n \to \infty} u_r(n)$, r fixed, and $|u_r(n)| \leqslant M_r$ where M_r is independent of n, it follows that $|v_r| \leqslant M_r$. Hence

$$|v_\nu + v_{\nu+1} + v_{\nu+2} + \ldots| \leqslant M_\nu + M_{\nu+1} + M_{\nu+2} + \ldots < \epsilon.$$

Hence $\sum\limits_{\nu - 1}^{\infty} v_r$ converges. Let its sum be denoted by V. Then

$$\begin{aligned} |f(n) - V| &= \left| \sum_{r=1}^{\nu-1} u_r + \sum_{r=\nu}^{p} u_r - \sum_{r=1}^{\nu-1} v_r - \sum_{r=\nu}^{\infty} v_r \right| \\ &\leqslant \left| \sum_{r=1}^{\nu-1} u_r - \sum_{r=1}^{\nu-1} v_r \right| + \left| \sum_{r=\nu}^{p} u_r \right| + \left| \sum_{r=\nu}^{\infty} v_r \right| \\ &< \left| \sum_{r=1}^{\nu-1} u_r - \sum_{r=1}^{\nu-1} v_r \right| + 2\epsilon. \end{aligned}$$

Now the only restriction which has so far been introduced on n is that $p > \nu$. Since ν is a number which *does not depend on* n we may let n tend to infinity in this last equality. Now

$$\sum_{r=1}^{\nu-1} u_r - \sum_{r=1}^{\nu-1} v_r = \sum_{r=1}^{\nu-1} (u_r - v_r).$$

Since $u_r \to v_r$ as $n \to \infty$ and there are only a finite number of terms,

$$\sum_{r=1}^{\nu-1} (u_r - v_r) \to 0 \text{ as } n \to \infty.$$

Hence there exists a number n_1 such that

$$\left| \sum_{r=1}^{\nu-1} (u_r - v_r) \right| < \epsilon, \text{ all } n \geqslant n_1.$$

Hence $|f(n) - V| < 3\epsilon$, all $n \geqslant n_1$,

Thus $f(n) \to V$ as $n \to \infty$.

4·6. An Important Application

In Chapter II., § 2·21, it has been proved that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right.$$

exists and lies between 2 and 3. We now consider a proof of the existence of the limit, by means of Tannery's theorem.

Expanding by the binomial theorem, n a positive integer

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \ldots$$
$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right).$$

This expansion contains $(n + 1)$ terms so that in the notation of Tannery's theorem $p = n + 1$,

$$u_r(n) = \frac{1}{(r-1)!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-2}{n}\right).$$

Keeping r *fixed*, $\lim\limits_{n\to\infty} u_r(n) = \dfrac{1}{(r-1)!}$

Again $|u_r(n)| < \dfrac{1}{(r-1)!}$ since the other factors of $u_r(n)$ are all positive and less than unity.

Since the series $\sum\limits_{r=1}^{\infty} \dfrac{1}{(r-1)!}$ converges, the conditions of Tannery's theorem are satisfied. Hence

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = \sum_{r=0}^{\infty}\frac{1}{r!}.$$

We can generalise the result by considering the case in which n tends to infinity *in any way*, not necessarily through integral values only. Consider then

$$\lim_{\nu\to\infty}(1+\xi)^\nu$$

where ν tends to infinity in any way and

$$\lim_{\nu\to\infty}(\nu\xi) = 1.$$

We first consider the case in which ν takes integral values n.

Write $n\xi = \lambda$ where n is a positive integer where $n\xi \to 1$ as $n \to \infty$. Then proceeding as before

$$(1+\xi)^n = 1 + \lambda + \frac{\lambda^2}{2!}\left(1-\frac{1}{n}\right) + \ldots + \frac{\lambda^n}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$\ldots\left(1-\frac{n-1}{n}\right)\cdot$$

$$| u_r(n) | = \left| \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{r-2}{n}\right) \frac{\lambda^{r-1}}{(r-1)\,!} \right|$$
$$< \frac{| \lambda^{r-1} |}{(r-1)\,!} < \frac{\rho^{r-1}}{(r-1)!}$$

where ρ is the greatest value of $| \lambda |$ for all values of n. The number ρ can obviously be chosen independent of n.

Since ρ is a constant the series $\Sigma\, \rho^{r-1}/(r-1)\,!$ converges and the conditions of Tannery's theorem are satisfied.

$$\text{Also} \quad \lim_{n \to \infty} u_r(n) = \frac{1}{(r-1)\,!} \lim_{n \to \infty} \lambda^{r-1} = \frac{1}{(r-1)\,!}.$$

$$\text{Hence} \quad \lim_{n \to \infty} (1 + \xi)^n = 1 + 1 + \frac{1}{2\,!} + \dots + \frac{1}{(r-1)\,!} + \dots$$

where $n\,\xi \to 1$ as $n \to \infty$, n being a positive integer.

If $\nu \to \infty$ through values other than integral values we can find at any point in the process an integer n such that $n \leqslant \nu < n + 1$. Then with the usual interpretation of the index $(1 + \xi)^\nu$ will lie between $(1 + \xi)^n$ and $(1 + \xi)^{n+1}$.

$$\text{Also } n\xi \leqslant \nu\xi < (n+1)\,\xi.$$

$$\text{Thus} \quad \lim_{n \to \infty} (n\xi) = \lim_{n \to \infty} (n+1)\xi = 1.$$

From the argument given above

$$\lim_{n \to \infty} (1 + \xi)^n = \sum_{r=0} 1/r\,! = \lim_{n \to \infty} (1 + \xi)^{n+1}.$$

As $(1 + \xi)^\nu$ lies between $(1 + \xi)^n$ and $(1 + \xi)^{n+1}$ and both these expressions tend to the *same* limit it follows that $(1 + \xi)^\nu$ tends to this limit. Hence

$$\lim_{\nu \to \infty} (1 + \xi)^\nu = \sum_{r=0}^{\infty} 1/r\,!$$

where ν tends to infinity in any way and $\lim_{\nu \to \infty} (\nu\xi) = 1$.

4·7. Properties of Power Series

We now consider some properties of the power series

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots = \sum^{\infty} a_nx^n.$$

I. *If* $\lim\limits_{n \to \infty} |a_n|^{\frac{1}{n}} = 1/\rho$ *the power series converges absolutely if* $|x| < \rho$, *and cannot converge for* $|x| > \rho$. It is assumed that ρ is finite and positive.

First we observe that the series Σa_n converges absolutely if

$$\lim_{n \to \infty} |a_n|^{\frac{1}{n}} < 1 \text{ . and diverges if } \lim_{n \to \infty} |a_n|^{\frac{1}{n}} > 1 \text{ .}$$

For let the limit be $k < 1$. Choose ϵ so that $k + \epsilon < 1$, e.g. take $\epsilon = \frac{1}{2}(1 - k)$. Then there exists an integer n_0 such that for all $n \geqslant n_0$

$$|a_n|^{\frac{1}{n}} < k + \epsilon \text{ ; i.e. } |a_n| < (k + \epsilon)^n.$$

Since $\Sigma (k + \epsilon)^n$ converges, $0 < k + \epsilon < 1$, it follows that Σa_n is absolutely convergent.

Now consider the power series and write $\lim\limits_{n \to \infty} |a_n|^{\frac{1}{n}} = \frac{1}{\rho}$. Applying the previous result, the series will be absolutely convergent if

$$\lim_{n \to \infty} |a_n x^n|^{\frac{1}{n}} < 1,$$

$$\text{i.e. } \lim_{n \to \infty} |a_n|^{\frac{1}{n}} |x| < 1, \text{ i.e. } |x| < \rho.$$

If $\lim\limits_{n \to \infty} |a_n x^n| > 1$, the limit as n tends to ∞ of the nth term of the series does not tend to zero. Hence the series cannot converge. [Chap. I., § 1·32.]

The interval $(-\rho, \rho)$ may be called **the interval of convergence** of the power series. *Inside* the interval the series *converges absolutely*, *outside* the interval convergence is *impossible*, at the end points the power series may or may not converge.

In dealing with power series it is sometimes convenient to replace the interval $(-\rho, \rho)$ by the interval $(-1, 1)$. This may be done by substituting x/ρ for x in the given series. For the condition of convergence $|x| < \rho$ is equivalent to $\left|\frac{x}{\rho}\right| < 1$.

The behaviour of a power series at the end points of the interval of convergence is illustrated by the following examples.

Examples.—(1) $\sum_{n=0}^{\infty} (n + 1)x^n = 1 + 2x + 3x^2 + \ldots$

The series converges absolutely for $|x| < 1$, diverges for $|x| > 1$ and the interval of convergence is $(-1, 1)$. When $x = \pm 1$ the series becomes

$$1 + 2 + 3 + \ldots$$

$$1 - 2 + 3 - \ldots$$

neither of which is convergent.

(2) $$\sum_{n=1}^{\infty} x^n/n = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

In this case also the interval of convergence is $(-1, 1)$. When $x = +1$ the series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots$$

which is divergent. When $x = -1$ the series becomes

$$-1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \ldots$$

which is convergent, but not absolutely. Thus the power series converges at one end of the interval and diverges at the other.

(3) $$\sum_{n=1}^{\infty} x^n/n^2 = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots$$

The interval of convergence is $(-1, 1)$. When $x = \pm 1$ the series becomes

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots$$

both of which are absolutely convergent.

II. *If the power series* $\Sigma a_n x^n$ *converges for* $x = \lambda$ *it is absolutely convergent for* $|x| < \lambda$.

For $|a_n x^n| = \left| a_n \lambda^n . \left(\frac{x}{\lambda}\right)^n \right| < k \left(\frac{x}{\lambda}\right)^n$, k a constant, since $a_n \lambda^n$ is finite for all values of n, and tends to zero as $n \to \infty$.

Since $\left|\frac{x}{\lambda}\right| < 1$ the series $\Sigma \left|\frac{x}{\lambda}\right|^n$ is convergent. Thus $\Sigma |a_n x^n|$ is convergent, giving the required result.

COROLLARY. *If the series does not converge for* $x = \lambda$ *it does not converge for any value of* x *such that* $|x| > \lambda$.

III. *If* $(-\rho, \rho)$ *is the interval of convergence of the power series* $\Sigma a_n x^n$ *then the series converges* **uniformly in any closed interval entirely inside** $(-\rho, \rho)$.

The result follows immediately from the M-test. Let δ be a number as small as we please. Then in the interval

$$-\rho + \delta \leqslant x \leqslant \rho - \delta$$

the series is uniformly convergent.

For $|a_n x^n| < |a_n(\rho - \delta)^n|$ which is the nth term of an absolutely convergent series, as $\rho - \delta$ lies inside the interval of convergence.

From this theorem we deduce the following result.

IV. *A power series $\Sigma\, a_n x^n$ is a continuous function of x in any closed interval lying entirely inside the interval of convergence.*

V. ABEL'S THEOREM. *If the series $\Sigma a_n x^n$ converges for $|x| < \rho$, and the series converges at $x = \rho$ or $x = -\rho$ then the interval of uniform convergence extends up to and includes that point, and continuity of the sum function $f(x) = \Sigma a_n x^n$ extends up to and includes that point.*

The proof depends on Abel's test for uniform convergence (§ 7·81).

Thus suppose the interval of convergence of $\Sigma a_n x^n$ is $(-1, 1)$ and Σa_n converges. Then from Abel's theorem the series is uniformly convergent in the closed interval $-1 + \delta \leqslant x \leqslant 1$, where δ is an arbitrary positive number. Thus if $f(x) = \Sigma a_n x^n$, then lim. $f(x) = \Sigma a_n$.

4·8. Application to the Binomial Series

It has been proved in Chapter III. that if $|x| < 1$

$$(1 + x)^n \qquad \sum_{r=0}^{\infty} \binom{n}{r} x^r.$$

The series converges absolutely in the *open* interval, $-1 < x < 1$, and uniformly in any *closed* interval lying inside $(-1, 1)$.

Now in Chapter I., § 1·52, it is shown that when $x = 1$ and $n + 1 > 0$ the series $\sum_{x=0}^{\infty} \binom{n}{r} x^r$ converges.

Thus if $n + 1 > 0$ the value $x = 1$ is a point of uniform convergence and consequently $(1 + x)^n$ is continuous at $x = 1$. Hence if $n + 1 > 0$

$$2^n = \sum_{r=0}^{\infty} \binom{n}{r}.$$

Again, it has been shown in Chapter I., § 1·52, that the binomial series converges at $x = -1$ if $n > 0$. Thus $(1+x)^n$ is continuous at $x = -1$ and hence by Abel's theorem

$$0 = \sum_{r=0}^{\infty} (-1)^r \binom{n}{r}, \quad n > 0.$$

EXERCISES IV

1. If the sum to n terms of a series is $nx/(1+n^2x^2)$ for all finite values of x, find the sum of the series to infinity and prove that it is continuous for all finite values of x. Prove also that the series does not converge uniformly to its sum.

2. Prove that the geometric progression $1 + x + x^2 + \ldots$ converges uniformly to $1/(1-x)$ in the range $0 \leqslant x \leqslant \lambda < 1$.

3. Prove from first principles that series $\sum_{n=1}^{\infty} (-1)^{n-1}/(n+x^2)$ converges uniformly for $x \geqslant 0$.

4. Prove that the series

$$\cos\theta + \frac{\cos 2\theta}{2^p} + \frac{\cos 3\theta}{3^p} + \frac{\cos 4\theta}{4^p} + \ldots$$

converges absolutely and uniformly for all real values of θ, if $p > 1$.

5. Prove that the series whose nth term is $1/(n^2 + n^4x^2)$ converges uniformly for all values of x.

6. Show that the series $\Sigma \dfrac{\sin(x+nx)}{n(n+1)}$ is uniformly convergent for all real values of x.

7. Prove that the series whose nth term is $x^2/(1+x^2)^{n-1}$ converges for all real values of x, but is not uniformly convergent in any interval which contains the value $x = 0$.

8. Find the sum of the series $\sum_{n=1}^{\infty} x^n(1-x^n)$ for $0 \leqslant x \leqslant 1$. Show that it is not uniformly convergent in this range.

9. Show that the series whose nth term is $x/(1+n^2x^2)$ converges uniformly if x is finite and $x \geqslant \lambda > 0$. Does the series converge uniformly at $x = 0$?

10. Show that the exponential function defined by the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ is continuous for all finite values of x.

11. Show that the cosine series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \ldots,$$

converges absolutely and uniformly for all finite values of x.

12. Assuming that the series

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \ldots$$

converges at $x = 1$, prove that the expansion converges absolutely and uniformly in the closed interval $0 \leqslant x \leqslant 1$ and hence show that

$$\tfrac{1}{2}\pi = 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \ldots$$

It is assumed that the given expansion holds for *principal* values of $\sin^{-1} x$.

CHAPTER V

THE EXPONENTIAL AND LOGARITHMIC SERIES

It has been proved in Chap. IV., § 4·6, that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{r\,!} + \ldots$$

where n may tend to infinity through any set of values. The value of the limit has been denoted by e and it has been shown in § 2·21 that

$$2 < e \leqslant 3.$$

5·1. The Exponential Series as a Limit

We now show that ***if x is any real number such that***

$$\lim_{\nu \to \infty} (\nu y) = x$$

then

$$\lim_{\nu \to \infty} (1 + y)^\nu = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

This is a generalisation of the result just stated.

First suppose that ν is a positive integer n. Then by the binomial theorem

$$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2\,!}y^2 + \ldots$$
$$+ \frac{n(n-1)\ldots(n-r+1)}{r\,!}y^r + \ldots + y^n.$$

In the notation of § 4·5,

$$u_{r+1}(n) = \frac{n(n-1)\ldots(n-r+1)}{r\,!}y^r$$
$$= \left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{r-1}{n}\right)\frac{n^r y^r}{r\,!}$$
$$= \left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{r-1}{n}\right)\frac{z^r}{r\,!},$$

where $z = ny$ so that $z \to x$ as $n \to \infty$.

It follows that z is finite for all values of n and is bounded as $n \to \infty$. Hence we can find a positive number ρ such that $|z| < \rho$ for all values of n, ρ being independent of n.

Hence $|u_{r+1}(n)| < \rho^r/r\,!$.

Now it has been proved in Chap. I., § 1·53, that the series $\Sigma\, \rho^r/r\,!$ converges for all finite values of ρ. Again, for *fixed* r,

$$\lim_{n\to\infty} u_{r+1}(n) = \left[\lim_{n\to\infty} \frac{(ny)^r}{r\,!}\right] . \lim_{n\to\infty} \left[\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\right]$$
$$= x^r/r\,!.$$

Applying Tannery's theorem it follows that

$$\lim_{n\to\infty} (1+y)^n = 1 + x + \frac{x^2}{2\,!} + \ldots + \frac{x^r}{r\,!} + \ldots$$

Now suppose that the index ν is not necessarily an integer and that $\nu \to \infty$ in any way. Then at any point in the process we can find an integer n such that $n \leqslant \nu < n + 1$.

Thus $(1+y)^\nu$ will lie between $(1+y)^n$ and $(1+y)^{n+1}$. This is on the assumption that the positive value of $(1+y)^\nu$ is taken. The result is clearly true if ν is rational, while if ν is irrational the result is also true since we may approximate to an irrational number as closely as we please by rational numbers.

Again $ny \leqslant \nu y \leqslant (n+1)\,y$.

Since $\nu y \to x$ as $\nu \to \infty$, $y \to 0$ as $\nu \to \infty$.

$$\text{Hence} \quad \lim_{n\to\infty} ny = \lim_{n\to\infty} (n+1)\,y.$$

Since $\lim\limits_{n\to\infty} \nu y = x$, lies between these values it follows that

$$\lim_{n\to\infty} ny = \lim_{n\to\infty} (n+1)\,y = x.$$

But it has already been proved that

$$\lim_{n\to\infty} (1+y)^n = 1 + x + \frac{x^2}{2\,!} + \ldots$$
$$+ \frac{x^r}{r\,!} + \ldots = \lim_{n\to\infty} (1+y)^{n+1}.$$

As $(1+y)^\nu$ lies between $(1+y)^n$ and $(1+y)^{n+1}$ it follows that $(1+y)^\nu$ tends to the *same* limit. Thus

$$\lim_{\nu\to\infty} (1+y)^\nu = \sum_{r=0}^{\infty} x^r/r\,!, \text{ where } \lim_{\nu\to\infty} (\nu y) = x.$$

In particular, if $\nu = nx$, $y = 1/n$ it follows that

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{nx} = \sum_{r=0}^{\infty} x^r/r\,!$$

We now proceed to prove that if $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ then

$$e^x = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{nx}.$$

It has been proved in Chap. II., § 2·5, that if $f(n) \to l$, $l > 0$ as $n \to \infty$ then $\{f(n)\}^k \to l^k$ where k is any rational number. Let x denote any rational number and write

$$f(n) = \left(1 + \frac{1}{n}\right)^n.$$

$$\text{Then } f(n) \to e \text{ as } n \to \infty.$$

$$\text{Hence } \{f(n)\}^x \to e^x \text{ as } n \to \infty.$$

Since $$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{nx} = \sum_{r=0}^{\infty} x^r/r!$$

it follows that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots,$$

the expansion being valid for all finite values of x.*

5·11. The Remainder in the Exponential Series

Let R_n denote the remainder after n terms. Then

$$R_n = \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \ldots + \frac{x^{n+r}}{(n+r)!} + \ldots$$

$$= \frac{x^n}{n!}\left\{1 + \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \ldots + \frac{x^r}{(n+1)\ldots(n+r)} + \ldots\right\}$$

$$\therefore \ |R_n| < \frac{|x|^n}{n!}\left\{1 + \frac{|x|}{n+1} + \frac{|x|^2}{(n+1)(n+2)} + \ldots + \frac{|x|^r}{(n+1)\ldots(n+r)} + \ldots\right\}$$

$$\frac{|x|^n}{n!}\left\{1 + \frac{|x|}{n} + \left(\frac{|x|}{n}\right)^2 + \ldots + \left(\frac{|x|}{n}\right)^r + \ldots\right\}$$

$$\frac{|x|^n}{n!}\,\frac{1}{1 - |x|/n}, \text{ provided } |x|/n < 1, \text{ i.e. } |x| < n.$$

* In the last part of the argument a theorem proved only for rational values of the variable is used. The theorem is, however, true for all real values and enables the exponential theorem to be deduced *for all real values* of x.

Since x is fixed and $n \to \infty$ we can always find a value of n such that $|x| < n$.

Again, if $x > 0$ all the terms in the series are positive. Hence $R_n > x^n/n!$.

If $x < 0$ the terms are alternatively positive and negative and *steadily decrease* in absolute value. Put $x = -y$ so that $y > 0$. Thus

$$R_n = (-1)^n \frac{y^n}{n!}\left\{1 - \frac{y}{n+1} + \frac{y^2}{(n+1)(n+2)} + \dots\right\}$$

$$= (-1)^n \frac{y^n}{n!}\left\{1 - \frac{y}{n+1}\left(1 - \frac{y}{n+2}\right) - \frac{y^3}{(n+1)(n+2)(n+3)}\left(1 - \frac{y}{n+4}\right) - \dots\right\}$$

Since it has been assumed that $|x| < n$, i.e. $y < n$ it follows that the factor of $(-1)^n y^n/n!$ is less than unity. Hence

$$|R_n| < y^n/n!, \text{ i.e. } |R_n| < |x|^n/n!.$$

Combining the inequalities, we have for $|x| < n$,

$$\frac{x^n}{n!} < |R_n| < \frac{}{(n-1)!\,(n-x)}, \quad x > 0; \quad |R_n| < \frac{|x|^n}{n!} \quad x < 0.$$

Thus, e.g. if $x = 1$, $n = 5$ the remainder lies between $\frac{1}{120}$ and $\frac{1}{96}$.

5·12. The Number e is not Rational

For suppose $e = p/q$ where p and q are positive integers. Then

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!} + R_{q+1}$$

$$\text{where} \quad \frac{1}{(q+1)!} < R_{q+1} < \frac{1}{q\,(q!)};$$

$$\therefore\ p\,(q-1)! = q! + q! + \frac{q!}{2!} + \dots + 1 + q!\,R_{q+1}.$$

Now all the terms in the equation except the last are clearly integers. Hence $q!\,R_{q+1}$ is an integer. But $q!\,R_{q+1} < 1/q$, which is a fraction. Hence we have a contradiction and e cannot be of the form p/q where p and q are integers.

5·13. The Exponential Theorem for Real Values of x

We have proved that when x is *rational*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Now take the expansion and suppose the function e^x to be defined by it *for irrational* x. The series has been proved to be *absolutely convergent* for all finite values of x. Further the M-test shows that it *converges uniformly* for all finite values of x. Hence the sum function is a continuous function for all finite x, since each term of the series is continuous. (Chap. IV., § 4·4.)

Thus the expansion *defines a continuous function* for all values of x.

Again, $f(x)$ is a *strictly monotonic increasing* function of x. For suppose $x_2 > x_1 > 0$. Then

$$e^{x_1} = \sum_{r=0}^{\infty} x_1^r/r\,!, \quad e^{x_2} = \sum_{r=0}^{\infty} x_2^r/r\,!$$

Since $x_2^r > x_1^r$ it follows that $e^{x_2} > e^{x_1}$.

Again, if $x_1 < x_2 < 0$ write $y_1 = -x_1$, $y_2 = -x_2$ so that $y_1 > y_2 > 0$. Thus $e^{y_1} > e^{y_2}$ from the first case, and

$$e^{-y_1} < e^{-y_2}, \text{ i.e. } e^{x_1} < e^{x_2}.$$

Thus the function steadily increases with x, for all values of x.

Hence to any given x there corresponds *one and only one* value of the function e^x.

Thus adopting the exponential series as a definition of the function e^x we have shown e^x is a *unique continuous monotonic increasing* function for all values of x.

Again, if $x = 0$, $e^x = 1$.

If $x > 0$, $e^x > 1$ and it is clear from the expansion that by sufficiently increasing x we can make e^x as large as we like. In fact it is clear that

$$\lim_{x \to \infty} e^x = +\infty.$$

Next consider $x < 0$ and write $x = -y$ where $y > 0$. Then as y increases from 0 to $+\infty$, e^y steadily increases from 1 to ∞. Hence $e^{-y} = e^x$ steadily *decreases from* 1 *to* 0,

$$\text{i.e.} \quad \lim_{x \to -\infty} e^x = 0.$$

Thus we have shown that the function e^x can take any value between 0 and $+\infty$.

Hence corresponding to any value $y > 0$ there will exist a number x such that

$$y = e^x, \text{ i.e. } x = \log_e y.$$

It follows that corresponding to any real positive number there exists a *unique number* which is its *logarithm*.

Now let a be any positive number. Then

$$a^x = 1 + x \log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots + \frac{x^r}{r!}(\log_e a)^r + \ldots$$

for $a^x = e^{\log_e a^x}$, from the definition of a logarithm. Thus $a^x = e^{x \log_e a}$.

Substituting in the exponential series we obtain the expansion for a^x. Thus

$$a^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}(\log_e a)^r.$$

The series being uniformly convergent for all finite values of x, it follows that a^x is continuous for all finite values of x, $a > 0$.

Arguing as before it follows that there exists a *unique logarithm* of any *positive number* to any *positive base*.

Example.—*From the expansion of e^x in ascending powers of x, deduce the expansion of a^x, where a denotes any positive number, and hence calculate the hundredth root of* 100 *to four decimal places, given* $\log_e 10 = 2{\cdot}3026$.

$$a^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}(\log_e a)^n.$$

Now, writing $a = 100$, $x = 1/100 = 10^{-2}$,

$$100^{\frac{1}{100}} = \sum_{n=0}^{\infty} (\log_e 100)^n/(n!\, 10^{2n}).$$

The second term is $(2 \log_e 10)/100 = {\cdot}046052$.

The third term is $4 (\log_e 10)^2/(2.10^4) = {\cdot}001060$, to 6 decimal places.

The fourth term is $8 (\log_e 10)^3/(6.10^6) = {\cdot}000016$, to 6 decimal places.

Clearly higher terms will not affect the fourth decimal place. Thus

$$\begin{aligned} 100^{\frac{1}{100}} &= 1 + {\cdot}046052 + {\cdot}001060 + {\cdot}000016 + \ldots \\ &= 1{\cdot}0471 \text{ correct to four decimal places.} \end{aligned}$$

5·14. Illustrative Examples

(1) *Find the coefficient of x^r in the expansion of* $(2 + 3x)\, e^{-2x}$.

$$(2 + 3x)e^{-2x} = (2 + 3x)\left\{1 - 2x + \frac{2^2x^2}{2!} - \ldots + \frac{(-1)^{r-1}\,2^{r-1}x^{r-1}}{(r-1)!} + \frac{(-1)^r\,2^r x^r}{r!} + \ldots\right\};$$

$$\therefore \text{ the coefficient of } x^r = 3 . \frac{(-1)^{r-1}\,2^{r-1}}{(r-1)!} + 2 . \frac{(-1)^r\,2^r}{r!}$$

$$= \frac{(-1)^r}{r!}\{2^{r+1} - 3r.2^{r-1}\} = (-1)^r.2^{r-1}(4 - 3r)/r!.$$

(2) *If x is a quantity so small that powers higher than its square may be neglected, show how to find A, B, C, so that $e^{a+x} = Ae^a + Be^{a+rx} + Ce^{a+sx}$, to this degree of approximation, a, r, s being supposed known.*

In particular, find A, B, C so that in the circumstances stated above, $e^{a+x} = Ae^a + Be^{a+\frac{1}{2}x} + Ce^{a+\frac{1}{3}x}$.

$e^{a+x} = Ae^a + Be^{a+rx} + Ce^{a+sx}$ is equivalent to $e^x = A + Be^{rx} + Ce^{sx}$.

Expanding each term and neglecting powers of x higher than the second,

$$1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} = A + B\left(1 + \frac{rx}{1!} + \frac{r^2x^2}{2!}\right) + C\left(1 + \frac{sx}{1\,!} + \frac{s^2x^2}{2\,!}\right).$$

Equating coefficients of corresponding powers of x,

$$1 = A + B + C,$$
$$1 = rB + sC,$$
$$1 = r^2B + s^2C.$$

The solution of these three linear equations determines A, B, C in terms of r and s.

In particular if $r = \frac{1}{2}$, $s = \frac{1}{3}$, the equations become

$$A + B + C = 1 \quad \text{.........................(i)}$$
$$3B + 2C = 6 \quad \text{.........................(ii)}$$
$$9B + 4C = 36 \quad \text{.........................(iii)}$$

From (ii) and (iii) $B = 8$, $C = -9$. Substituting in (i), $A = 2$.

(3) *Show that the relations* $y = 1 + 2{\cdot}5/(10 - x)$ and $y = {\cdot}94 + {\cdot}31e^{{\cdot}08x}$ *are approximately equivalent to the linear relation* $y = 1{\cdot}25 + {\cdot}025x$ *for small values of x.*

Obtain the expansions of the two expressions $a + b/(10 - x)$ and $p + qe^{rx}$ in ascending powers of x up to and including the terms in x^2. Show that p, q, r, can be so chosen that the two expressions agree for the terms obtained, and that the latter then takes the form $a + b\,(1 + e^{\frac{1}{5}x})/20$.

Now $y = 1 + 2{\cdot}5/(10 - x) = 1 + {\cdot}25\,(1 - \frac{1}{10}x)^{-1} = 1 + {\cdot}25\,(1 + \frac{1}{10}x)$, neglecting x^2 and higher powers. Thus $y = 1{\cdot}25 + {\cdot}025x$.

Again if $y = {\cdot}94 + {\cdot}31e^{{\cdot}08x}$, $y = {\cdot}94 + {\cdot}31\,(1 + {\cdot}08x)$, by the exponential theorem, if x^2 and higher powers are neglected.

Thus $y = 1{\cdot}25 + {\cdot}0248x = 1{\cdot}25 + {\cdot}025x$ approximately.

Expanding $a + b/(10 - x)$ and $p + qe^{rx}$ as far as terms in x^2 we obtain

$$a + b/(10 - x) = a + \frac{1}{10}b\left(1 - \frac{x}{10}\right)^{-1} = a + \frac{1}{10}b + \frac{1}{100}bx + \frac{1}{1000}bx^2,$$

$$p + qe^{rx} = p + q\,(1 + rx + \tfrac{1}{2}r^2x^2) = p + q + qrx + \tfrac{1}{2}qr^2x^2.$$

If these expressions agree then

$$p + q = a + \tfrac{1}{10}b,\quad qr = \tfrac{1}{100}b,\quad \tfrac{1}{2}qr^2 = \tfrac{1}{1000}b.$$

This gives three equations to determine p, q, r. Hence p, q, r can be chosen so that the two expressions agree. Solving the three equations, $r = \frac{1}{5}$, $q = 20b$, $p = a + \frac{1}{20}b$. Thus $p + qe^{rx}$ becomes

$$a + \tfrac{1}{20}b + \tfrac{1}{20}be^{\frac{1}{5}x} = a + b\,(1 + e^{\frac{1}{5}x})/20.$$

(4) *Show that the expansion of* $(1 + y + \frac{1}{3}y^2)/(1 - y + \frac{1}{3}y^2)$ *in ascending powers of* y *agrees with that of* e^{2y} *up to the term involving* y^4. *Hence find approximately the value of* x *which satisfies the equation* $e^x = 1{\cdot}1$ *and check your result by using tables of logarithms, assuming* $e = 2{\cdot}718$.

$$\begin{aligned}(1 + y + \tfrac{1}{3}y^2)/(1 - y + \tfrac{1}{3}y^2) &= (1 + y + \tfrac{1}{3}y^2)\{1 - y(1 - \tfrac{1}{3}y)\}^{-1}\\ &= (1 + y + \tfrac{1}{3}y^2)\{1 + y(1 - \tfrac{1}{3}y) + y^2(1 - \tfrac{1}{3}y)^2\\ &\qquad + y^3(1 - \tfrac{1}{3}y)^3 + y^4(1 - \tfrac{1}{3}y)^4 + \ldots\}\\ &= (1 + y + \tfrac{1}{3}y^2)(1 + y + \tfrac{2}{3}y^2 + \tfrac{1}{3}y^3 + \tfrac{1}{9}y^4 + \ldots),\end{aligned}$$

i.e. $(1 + y + \frac{1}{3}y^2)/(1 - y + \frac{1}{3}y^2) = 1 + 2y + 2y^2 + \frac{4}{3}y^3 + \frac{2}{3}y^4 + \ldots$

Again $$e^{2y} = 1 + 2y + \frac{(2y)^2}{2!} + \frac{(2y)^3}{3!} + \frac{(2y)^4}{4!} + \ldots$$
$$= 1 + 2y + 2y^2 + \tfrac{4}{3}y^3 + \tfrac{2}{3}y^4 + \ldots$$

Thus the expansions agree as far as the terms in y^4.

Next consider the equation $e^x = 1{\cdot}1$. Write $x = 2y$. Then approximately

$$1 + y + \tfrac{1}{3}y^2 = 1{\cdot}1\,(1 - y + \tfrac{1}{3}y^2),$$
$$\text{i.e. } y^2 - 63y + 3 = 0.$$
$$\text{Hence } y = \{63 \pm \sqrt{63^2 - 12}\}/2.$$

Since x is small the negative square root must be taken. Thus

$$x = 63\,\{1 - (1 - 12/63^2)^{\frac{1}{2}}\}.$$

By the binomial theorem,

$$(1 - 12/63^2)^{\frac{1}{2}} = 1 - 6/63^2 \text{ approximately.}$$
$$\text{Hence } x = 63 \times 6/63^2 = 2/21 = {\cdot}0952.$$

To check the result, we have

$$x \log_{10} e = \log 1{\cdot}1, \quad \text{i.e. } x = {\cdot}04139/{\cdot}43423 = {\cdot}0953.$$

(5) *Show that the coefficient of* x^r *in the expansion of* e^{e^x} *is*

$$\frac{1}{r!}\left(\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!} + \ldots\right).$$

$$\begin{aligned}e^{e^x} &= 1 + e^x + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \ldots + \frac{e^{nx}}{n!} + \ldots\\ &= 1 + \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots\right) + \frac{1}{2!}\left(1 + 2x + \frac{4x^2}{2!} + \ldots\right.\\ &\qquad \left. + \frac{2^r x^r}{r!} + \ldots\right) + \frac{1}{3!}\left(1 + 3x + \frac{9x^2}{2!} + \ldots + \frac{3^r x^r}{r!} + \ldots\right) + \ldots\end{aligned}$$

Hence coefficient of x^r in the expansion is $\dfrac{1}{r!}\left(\dfrac{1^r}{1!} + \dfrac{2^r}{2!} + \dfrac{3^r}{3!} + \ldots\right)$.

(6) *Show that the coefficient of* x^3 *in the expansion of* e^{e^x} *is* $5e/3!$.

$$e^{e^x} = e^{1 + x + x^2/2! + \ldots} = e.e^{x + x^2/2! + x^3/3! + \ldots}$$

$$= e\left\{1 + \left(x + \frac{x^2}{2!} + \ldots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2!} + \ldots\right)^2 + \frac{1}{3!}\left(x + \frac{x^2}{2!} + \ldots\right)^3 + \ldots\right\};$$

$$\therefore \text{ coefficient of } x^3 = e\left(\frac{1}{3!} + \frac{1}{2!} + \frac{1}{3!}\right) = \frac{5e}{3!}.$$

Note.—From Ex. 5,

$$e^{e^x} = \Sigma \frac{1}{r!}\left(\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!} + \ldots\right)x^r.$$

Ex. 6 gives an alternative way of obtaining the power series which is the expansion of e^{e^x}.

Since the power series converges to the same function it follows that they are identical and that coefficients of corresponding powers of x are equal. (Chap. I., § 1·8.) In particular, if we compare the coefficients of x^3,

$$5e = \sum_{n=1} \frac{n^3}{n!}.$$

(7) *Write down the first five terms in the expansion of* $pe^{qx} + qe^{px}$ *in ascending powers of* x.

If p, q *are the roots of the quadratic equation*

$$\lambda^2 - a\lambda + b = 0$$

show that the first four terms of the expansion are

$$a + 2bx + \tfrac{1}{2}abx^2 + \tfrac{1}{6}b(a^2 - 2b)x^3,$$

and express the coefficient of x^4 *in terms of* a *and* b. [*Lond. Inter. Econ.*]

$$pe^{qx} + qe^{px} = p\left(1 + qx + \frac{q^2x^2}{2!} + \frac{q^3x^3}{3!} + \frac{q^4x^4}{4!} + \ldots\right.$$

$$+ q\left(1 + px + \frac{p^2x^2}{2!} + \frac{p^3x^3}{3!} + \frac{p^4x^4}{4!} + \ldots\right.$$

$$= p + q + 2pqx + pq(p + q)\frac{x^2}{2!} + pq(p^2 + q^2)\frac{x^3}{3!}$$

$$+ pq(p^3 + q^3) \qquad + \ldots$$

Now if p, q are the roots of the equation $\lambda^2 - a\lambda + b = 0$, then

$$p + q = a, \quad pq = b.$$

Also $p^2 + q^2 = (p + q)^2 - 2pq = a^2 - 2b$, and

$$p^3 + q^3 = (p + q)(p^2 - pq + q^2) = (p + q)\{(p + q)^2 - 3pq\} = a(a^2 - 3b).$$

Thus the first four terms become

$$a + 2bx + \tfrac{1}{2}abx^2 + \tfrac{1}{6}b(a^2 - 2b)x^3$$

and the coefficient of x^4 is $\frac{1}{24}ab(a^2 - 3b)$.

(8) *Sum to infinity the series*

$$\frac{1}{1!} + \frac{1 + 3}{2!} + \frac{1 + 3 + 3^2}{3!} + \frac{1 + 3 + 3^2 + 3^3}{4!} + \ldots$$

The nth term is

$$\frac{1 + 3 + 3^2 + \ldots + 3^{n-1}}{n!} = \frac{1}{2}\cdot\frac{3^n - 1}{n!} = \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right).$$

Hence the required sum

$$= \tfrac{1}{2}\left\{\left(\frac{3}{1\,!} + \frac{3^2}{2\,!} + \frac{3^3}{3\,!} + \ldots\right) - \left(\frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \ldots\right)\right\}$$

(Chap. I., § 1·36.)

$$= \tfrac{1}{2}\left\{\left(1 + 3 + \frac{3^2}{2\,!} + \frac{3^3}{3\,!} + \ldots\right) - \left(1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \ldots\right)\right\}$$

$$= \tfrac{1}{2}\,(e^3 - e) = \tfrac{1}{2}\,e\,(e^2 - 1).$$

(9) *Show that* $n + \frac{1}{n} = 2\left\{1 + \frac{(\log_e n)^2}{2\,!} + \frac{(\log_e n)^4}{4\,!} + \ldots\right\}$.

In $a^x = 1 + (\log_e a)\,x + \frac{(\log_e a)^2 x^2}{2\,!} + \frac{(\log_e a)^3 x^3}{3\,!} + \ldots$ first put $a = n$, and $x = 1$; and then put $a = n$, and $x = -1$.

$$\text{Thus } n = 1 + \log_e n + \frac{(\log_e n)^2}{2\,!} + \frac{(\log_e n)^3}{3\,!} + \ldots,$$

$$\text{and } n^{-1} = 1 - \log_e n + \frac{(\log_e n)^2}{2\,!} - \frac{(\log_e n)^3}{3\,!} + \ldots$$

By addition, we get the required result.

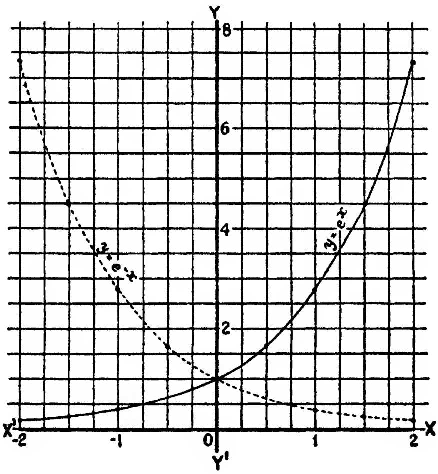


FIG. 6.

5·2. The Graphs of e^x and e^{-x}

Using the series for e we may approximate to e as closely as we please. Thus correct to ten decimal places $e = 2·7182818285\ldots$ Using a suitable approximation for e we can construct a table of values and plot the curve $y = e^x$.

It should first be observed that a number of properties of the curve are already known. Thus e^x steadily increases from 0 to

$+\infty$ as x increases from $-\infty$ to ∞. Also e^x is never negative for real values of x. Thus the graph lies always above the X-axis and steadily rises as x increases. Further, since $e^0 = 1$, the point $(0, 1)$ lies on the graph.

These facts in themselves give the approximate form of the curve. In constructing a table of values we observe that $e^{-x} = 1/e^x$ so that values of the function for negative values of x are readily obtained when the values for the corresponding positive values of x are known. The table given below is obtained by taking $e = 2{\cdot}718$.

$x =$	0	·5	1	1·5	2
$e^x =$	1	1·65	2·72	4·48	7·39
$e^{-x} =$	1	0·61	0·37	0·22	0·14

In Fig. 6 the continuous line shows the curve $y = e^x$, the dotted line $y = e^{-x}$. If we plot the graphs of $y = a^{\pm x}$, $a > 0$ we obtain similar curves.

Example.—*Show on a rough sketch the general form of the graph of the expression* $(e^x - 1)/(e^x + 1)$.

Draw carefully, on a suitable large scale, the graph of

$$3\,(10^x - 1)/(10^x + 1)$$

between $x = 0$ and $x = 1$. *Use this to solve the equation*

$$3\,(10^x - 1) = (3 - 2x)\,(10^x + 1).$$

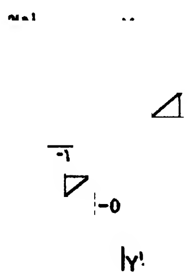


FIG. 7

$$y = (e^x - 1)/(e^x + 1) = 1 - 2/(e^x + 1).$$

As x ranges from $-\infty$ to $+\infty$, e^x ranges from 0 to ∞. Hence y lies between ± 1 and tends to these values as x tends to $\pm\infty$ respectively. When $x = 0$, $y = 0$; when $x > 0$, $y > 0$ and when $x < 0$, $y < 0$. Also if we change x into $-x$, y changes into $-y$. This enables the curve to be sketched. The lines $y = \pm 1$ are asymptotes. (See Fig. 7.)

$$y = 3\,(10^x - 1)/(10^x + 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

The following table of values is first calculated.

x	0	0·25	0·5	0·75	1
y	0	0·84	1·56	2·08	2·44

$3\,(10^x - 1) = (3 - 2x)\,(10^x + 1)$ is equivalent to

$$3 - 2x = 3\,(10^x - 1)/(10^x + 1).$$

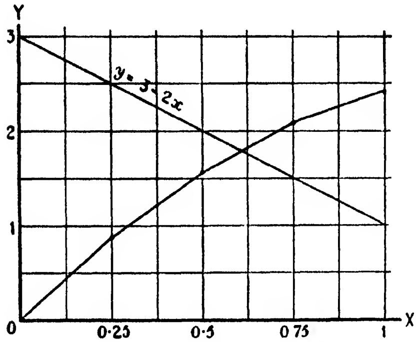


FIG. 8.

The required solution is given by the point of intersection of the graph of (i) and the straight line $y = 3 - 2x$. (See Fig. 8.) The solution is $x = 0{\cdot}61$ correct to two decimal places.

5·3. The Application of the Exponential Expansion to Limits

We now prove an important property of the exponential function true for *real* values of x: *As $x \to \infty$, e^x tends to infinity faster than a positive integral power of x. In symbols, if n denote any positive integer,*

$$\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} x^n e^{-x} = 0.$$

Now $e^x = 1 + x + \frac{x^2}{2\,!} + \ldots + \frac{x^n}{n\,!} + R_{n+1}(x)$

$$= x^n \left\{ \frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2\,!\,x^{n-2}} + \ldots + \frac{1}{n\,!} \right. + R_{n+1}(x),$$

where $R_{n+1}(x)$ denotes the remainder after $(n + 1)$ terms.

No matter how large x is, the expression

$$\frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2\,!\,x^{n-2}} + \ldots + \frac{1}{n\,!} = f(x)$$

is always finite. In fact, as $x \to \infty$ this expression approaches nearer and nearer to $1/n\,!$, n being a *fixed positive integer*.

Now $R_{n+1}(x) = \frac{x^{n+1}}{(n + 1)\,!} + \frac{x^{n+2}}{(n + 2)\,!} + \ldots > \frac{x^{n+1}}{(n + 1)\,!}$.

Hence $e^x > x^n f(x) + \frac{x^{n+1}}{(n + 1)\,!}$, where $f(x)$ is always finite and positive. Then

$$\frac{\ }{e^x} < \frac{\ }{f(x) + x/(n + 1)\,!} < \frac{(n + 1)\,!}{x}$$

Since $1/x \to 0$ as $x \to \infty$ and all the quantities are positive, it follows that

$$\lim_{\infty} \{x^n/e^x\} = 0.$$

It follows immediately from this result that if ν is *any positive number,*

$$\lim_{x \to \infty} \{x^\nu/e^x\} = 0,$$

for there exists an integer n such that $n \leqslant \nu < n + 1$, and since

$$\lim_{x \to \infty} \{x^{n+1}/e^x\} = \lim_{x \to \infty} \{x^n/e^x\} = 0$$

the result follows immediately.

The limit proved above may be expressed in different ways by means of substitutions. Thus, *e.g.* if we write $y = e^x$, $\log_e y = x$ the limit takes the form

$$\lim_{y \to \infty} (\log_e y)^n/y = 0 \quad \text{or} \quad \lim_{y \to \infty} (\log_e y)/y^{\frac{1}{n}} = 0,$$

where n is any positive integer.

Substituting $y = 1/x$ the limit takes the form

$$\lim_{x \to +0} \{x\ (\log_e x)^n\} = 0.$$

In particular, if $n = 1$, $\lim\limits_{x \to +0} x \log x = 0$

Another limit which is of importance is $\lim \dfrac{a^x - 1}{x}$, $a > 0$.

Now $\quad a^x = 1 + x \log_e a + \dfrac{x^2}{2\,!} (\log_e a)^2 + \dfrac{x^3}{3\,!}(\log_e a)^3 + \dots$

Thus $\quad \dfrac{a^x - 1}{x} = \log_e a + x\left\{\dfrac{1}{2\,!} (\log_e a)^2 + \dfrac{x}{3\,!} (\log_e a)^3 + \dots\right\}.$

The series inside the bracket converges for all finite values of x and hence is finite if x be finite. Hence

$$\frac{a^x - 1}{x} \quad \log_e a \quad < Ax, \text{ where } A \text{ is a constant.}$$

Since $x \to 0$, so also does Ax. Hence if $a > 0$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a.$$

Example.—*If* $y = x\,(1 + e^{-x})/(1 - e^{-x})$, *find the limit to which* y *tends as* x *tends to zero.*

If y *is expanded in a series of ascending powers of* x, *show that the coefficients of all the odd powers of* x *are zero and determine the expansion as far as the term involving* x^4. [*Lond., B.Sc.*]

$$y = x\,(1 + e^{-x})/(1 - e^{-x}) = x\,(e^x + 1)/(e^x - 1).$$

Now $\lim\limits_{x \to 0} (e^x + 1) = 2$, since e^x is continuous and $e^0 = 1$.

Again $\lim\limits_{x \to 0} \dfrac{x}{e^x - 1} = \lim\limits_{x \to 0} \dfrac{x}{1 + x + \dfrac{x^2}{2!} + \ldots - 1}$

$$= \lim_{x \to 0} \frac{1}{1 + \dfrac{1}{2!}x + \ldots} = 1.$$

Hence $\lim\limits_{x \to 0} y = 2.$

Now $1 - e^{-x} = 1 - 1 + x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{x^4}{4!} + \dfrac{x^5}{5!} - \ldots$

$$= x\left(1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \ldots\right);$$

$$\therefore\ y = \left\{2 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right\} \Big/ \left\{1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} \ldots\right\}$$

Now it is easily verified by the ratio test that the series

$$\frac{1}{2!} - \frac{x}{3!} + \frac{x^2}{4!} - \frac{x^3}{5!} + \ldots$$

converges absolutely for all finite values of x, and hence is finite. Thus the expression

$$-z = -x\left\{\frac{1}{2!} - \frac{x}{3!} + \frac{x^2}{4!} - \frac{x^3}{5!} + \ldots\right\} = -\frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \ldots$$

can be made as small as we like by taking x sufficiently small; in particular there will be a range of values of x such that

$$-\frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \qquad < 1.$$

Let $y = a_0 + a_1x + a_2x^2 + \ldots a_{2n-1}x^{2n-1} + a_{2n}x^{2n} + \ldots$

Changing x into $-x$, y becomes

$$-x(1 + e^x)/(1 - e^x) = x(e^x + 1)/(e^x - 1)$$

which is y itself. Thus y is unaltered if x is changed into $-x$, *i.e.* y is an *even function* of x. Changing x into $-x$ in the expansion we have

$$y = a_0 - a_1x + a_2x^2 - \ldots a_{2n-1}x^{2n-1} + a_{2n}x^{2n} - \ldots$$

Since both the expansions are power series and converge to the same function the expansions must be identical (Chap. I., § 1·8). Hence $a_{2n-1} = -a_{2n-1}$, i.e. $a_{2n-1} = 0$, for all positive integral values of n. Thus the coefficients of odd powers of x are zero.

Now $y = \left\{2 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \ldots\right\} \{1 - z\}^{-1}$, where $|z| < 1$.

Expanding by the binomial theorem

$$y \quad \left\{2 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right\} \{1 + z + z^2 + z^3 + z^4 + \ldots\}$$

Now $z = \dfrac{x}{?} - \dfrac{x^2}{6} + \dfrac{x^3}{24} - \dfrac{x^4}{120} +$

$$z^2 = \frac{x^2}{4} - \frac{x^3}{6} + \tfrac{5}{72}x^4 - \dots$$

$$z^3 = \frac{x^3}{8} - \frac{x^4}{8} + \dots$$

$$z^4 = \frac{x^4}{16} - \dots$$

Thus $1 + z + z^2 + z^3 + z^4 + \dots = 1 + \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 - \dots$

$$\begin{array}{rllllll}
\text{Hence } y = & 2 & + x & + \tfrac{1}{6}x^2 & + \ 0 & - \tfrac{1}{360}x^4 & \dots \\
& & - x & - \tfrac{1}{2}x^2 & - \tfrac{1}{12}x^3 & + \ 0 & \dots \\
& & & + \tfrac{1}{2}x^2 & + \tfrac{1}{4}x^3 & + \tfrac{1}{24}x^4 & \dots \\
& & & & - \tfrac{1}{6}x^3 & - \tfrac{1}{12}x^4 & \dots \\
& & & & & + \tfrac{1}{24}x^4 & \dots
\end{array}$$

$$\therefore\ y = 2 + \tfrac{1}{6}x^2 - \tfrac{1}{360}x^4 \dots$$

5·4. Series Related to the Exponential Series

There are two important series which are closely related to the exponential series. They are the expansions of the trigonometrical functions $\sin x$, $\cos x$. Thus

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r-1}}{(2r-1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r-2}}{(2r-2)!},$$

x denoting *circular measure.*

We first observe that each of the series is both ***absolutely*** and ***uniformly*** convergent for all finite values of x. For consider the first series and let u_r denote the rth term. Then

$$\frac{u_{r+1}}{u_r} = -\frac{x^{2r+1}}{(2r+1)!} \cdot \frac{(2r-1)!}{x^{2r-1}} = -\frac{x^2}{2r(2r+1)}$$

$$\text{Thus} \quad \lim_{r \to \infty} \frac{u_{r+1}}{u_r} = 0$$

and the series converges absolutely for any finite value of x. Uniform convergence follows immediately from Weierstrass' M-test. (Chap. IV., § 4·33.)

The second series may be considered in a similar way. It follows that the sum function in each case is continuous for all finite values of x.

One method of discussing these trigometrical functions would be to *define* $\sin x$ and $\cos x$ by the corresponding series and show by algebraic methods that the functions so defined satisfy the

fundamental properties of sin x and cos x as defined by elementary trigonometrical methods. Thus, *e.g.* write

$$S(x) = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r-1}}{(2r-1)!}, \quad C(x) = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r-2}}{(2r-2)!}.$$

First we observe that $S(0) = 0$, $C(0) = 1$.

Then we can show that $S(x)$ is an *odd* function, *i.e.*

$$S(-x) = -S(x),$$

while $C(x)$ is an *even* function, i.e. $C(x) = C(-x)$: for

$$S(-x) = \sum_{r=1}^{\infty} (-1)^{r-1} (-1)^{2r-1} \frac{x^{2r-1}}{(2r-1)!}$$

$$= -\sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r-1}}{(2r-1)!},$$

$$\text{i.e. } S(-x) = -S(x).$$

As for $C(x)$ it will be observed that only *even* powers of x occur so that the substitution of $-x$ for x does not alter the series.

Next it may be proved by using the theorems on the product and rearrangement of absolutely convergent series (Chap. I., §§ 1·6, 1·34) that

$$\{S(x)\}^2 + \{C(x)\}^2 = 1.$$

Finally we consider the property, $\lim\limits_{x \to 0} \dfrac{S(x)}{x} = 1$. Write $f(x) = S(x)/x$. Then

$$f(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r+1)!}$$

The ratio test shows that the expansion of $f(x)$ is *absolutely* convergent for all finite values of x. From the properties of power series it follows that the series is *uniformly* convergent for all finite values of x. Hence $f(x)$ is continuous.

In particular $\lim\limits_{x \to 0} f(x) = f(0)$. Now $f(0) = 1$, since all terms of the series after the first are zero. Hence

$$\lim_{x \to 0} \frac{S(x)}{x} = 1.$$

The student will now observe that the functions $S(x)$, $C(x)$ defined by the series given at the beginning of the section satisfy the following fundamental properties of the sine and cosine.

(i) $\sin 0 = 0$; $\cos 0 = 1$.

(ii) $\sin(-x) = -\sin x;\ \cos(-x) = \cos x.$

(iii) $\sin^2 x + \cos^2 x = 1.$

(iv) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$

The results (i)-(iii) would be true for both degrees and radians if we adopt the ordinary elementary definitions, since the trigonometrical ratios sine and cosine are independent of the units in which x is measured. On the other hand, (iv) is true only if x is measured in radians. It follows that the number x in the given series must represent an angle measured in radians.

Example.—*If* $f(x) = \dfrac{1 - \cos x}{x^2}$, $x \neq 0$, and $f(0) = \frac{1}{2}$, *prove that* $f(x)$ *is continuous.*

Since $\cos x$ is continuous for all finite values of x, so also is $1 - \cos x$. Also $1/x^2$ is continuous provided $x \neq 0$.

Hence $(1 - \cos x)/x^2$ is continuous, except possibly for $x = 0$. To prove continuity at $x = 0$, it is sufficient to show that $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = f(0)$.

Now $\cos x = 1 - \dfrac{x^2}{2!} + O(x^4)$, where $O(x^4)$ implies a function $\phi(x)$ which is such that $\phi(x)/x^4$ is bounded as $x \to 0$.

Hence $$\frac{1 - \cos x}{x^2} = \frac{1}{2} + O(x^2),\quad x \neq 0.$$

Letting $x \to 0$, $1 - \cos x$ $\frac{1}{2}$ which is the defined value of $f(0)$.

5.41 The Hyperbolic Functions

The hyperbolic sine and cosine, sin*h* x, cos*h* x, are defined by the equations

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}),\quad \cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

$$\text{Now}\quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

$$\text{Thus}\quad \tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

The expansions are absolutely and uniformly convergent for all finite values of x. Thus the hyperbolic sine and cosine are defined and are continuous for all finite values of x.

Since the expansion for $\sinh x$ contains only odd powers of x, $\sinh x$ is an *odd* function, *i.e.* $\sinh(-x) = -\sinh x$. Similarly, since the expansion of $\cosh x$ contains only even powers of x, this function is *even*, i.e. $\cosh(-x) = \cosh x$. Further, since

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

it follows that $\cosh x$ is least when $x = 0$ and its value is then unity. As x increases from zero $\cosh x$ steadily increases, since each term of the expansion steadily increases. Similarly, as x decreases from zero, $\cosh x$ steadily increases, since the expansion contains only even powers of x. Thus $\cosh x$ is monotonic decreasing for $x < 0$ and monotonic increasing for $x > 0$. The graph of $y = \cosh x$ is shown in Fig. 9 by the continuous curve. The curve is symmetrical about the y-axis since the function is even.

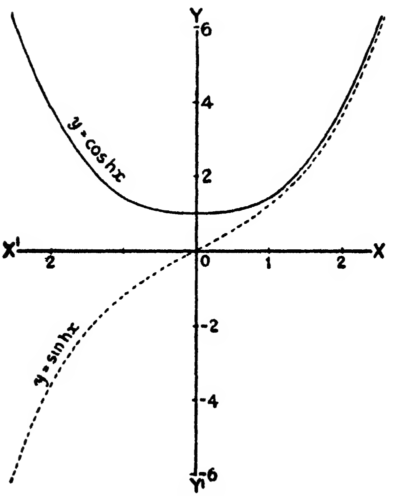


FIG. 9.

Next consider

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

When $x = 0$, $\sinh x = 0$ since each term of the expansion is zero.

When $x > 0$, all terms of the expansion are positive and steadily increase with x. Thus $\sinh x > 0$ for $x > 0$ and is a monotonic increasing function of x. When $x < 0$, all the terms of the expansion are negative since it contains only odd powers.

Further, as x decreases, $|x|$ increases and each term of the expansion increases in absolute value, so that $\sinh x$ steadily decreases as x decreases.

It follows that $\sinh x$ is a monotonic increasing function of x for all values of x.

The graph of $y = \sinh x$ is shown by the dotted curve in Fig. 9.

5·42. The Expansion of $x/(e^x - 1)$

Now $(e^x - 1)/x = 1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \ldots = 1 + y$, say.

Then $$\frac{x}{e^x - 1} = \frac{1}{1 + y} = 1 - y + y^2 - y^3 + \ldots$$

provided $|y| < 1$. Thus the expansion will be valid for $|x| < \lambda$ where

$$\frac{\lambda}{2!} + \frac{\lambda^2}{3!} + \frac{\lambda^3}{4!} + \ldots \leqslant 1.$$

Now $$\frac{\lambda}{2!} + \frac{\lambda^2}{3!} + \frac{\lambda^3}{4!} + \ldots < \tfrac{1}{2}\lambda\left\{1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \ldots\right\}$$
$$= \tfrac{1}{2}\lambda/(1 - \tfrac{1}{3}\lambda), \text{ provided } \lambda < 3.$$

The equation $\dfrac{\frac{1}{2}\lambda}{1 - \frac{1}{3}\lambda} = 1$ give $\lambda = 1{\cdot}2$. Hence the expansion is certainly valid for $|x| < 1{\cdot}2$. The important point here is not that we should know the best possible value of λ, but that there should be some definite interval of convergence.

It is clear that the coefficient of x is $-\frac{1}{2}$. Hence we may write

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$$

Consider the function $f(x) = \dfrac{x}{e^x - 1} + \frac{1}{2}x$. Then

$$f(-x) = \frac{-x}{e^{-x} - 1} - \tfrac{1}{2}x = \frac{xe^x}{e^x - 1} - \tfrac{1}{2}x$$
$$= \frac{x}{e^x - 1} + \frac{xe^x - x}{e^x - 1} - \tfrac{1}{2}x = \frac{x}{e^x - 1} + \tfrac{1}{2}x.$$

Hence $f(x) = f(-x)$ and $f(x)$ is an *even* function. Thus the expansion in ascending powers of x contains only *even* powers. (Chap. I., § 1·9.) Thus

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \sum_{n=1}^{\infty} a_{2n}x^{2n}.$$

Write $a_{2n} = B_{2n}/(2n)!$ so that

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \sum_{n=1}^{\infty} B_{2n}\frac{x^{2n}}{(2n)!}$$

Thus if $n \geqslant 1$, B_{2n} is the coefficient of $x^{2n}/(2n)!$ in the expansion of $x/(e^x - 1)$ in ascending powers of x and, for particular values of n, can be calculated from the identity

$$\left(1 - \frac{1}{2}x + B_2\frac{x^2}{2!} + B_4\frac{x^4}{4!} + B_6\frac{x^6}{6!} + \dots\right)$$
$$\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \dots\right) = 1.$$

The identity asserts that the coefficients of x^2, x^4, x^6, ..., x^{2n}, ... on the left-hand side are zero. Thus

$$\frac{1}{3!} - \frac{1}{2}\cdot\frac{1}{2!} + \frac{B_2}{2!} = 0$$

$$\frac{1}{5!} - \frac{1}{2}\cdot\frac{1}{4!} + \frac{B_2}{2!}\cdot\frac{1}{3!} + \frac{B_4}{4!} = 0$$

$$\frac{1}{7!} - \frac{1}{2}\cdot\frac{1}{6!} + \frac{B_2}{2!}\cdot\frac{1}{5!} + \frac{B_4}{4!}\cdot\frac{1}{3!} + \frac{B_6}{6!} = 0$$

....

$$\frac{1}{(2n+1)!} - \frac{1}{2}\cdot\frac{1}{(2n)!} + \frac{B_2}{2!}\cdot\frac{1}{(2n-1)!} + \frac{B_4}{4!}\cdot\frac{1}{(2n-3)!} + \dots$$
$$+ \frac{B_{2n-2}}{(2n-2)!}\cdot\frac{1}{3!} + \frac{B_{2n}}{(2n)!} = 0.$$

....

Considering these equations in succession we find

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730},$$
$$B_{14} = \frac{7}{6}.$$

The coefficients B_2, B_4, B_6, ... B_{2n}, ... are called *Bernouillian numbers*. They have important applications in the expansions of trigonometrical expressions.

Example.—*Prove that* $\tanh x = 2\coth 2x - \coth x$, *and find by means of the expansion* $\dfrac{x}{e^x - 1} = 1 - \dfrac{x}{2} + \sum_{n=1}^{\infty} \dfrac{B_{2n}x^{2n}}{(2n)!}$, *where* B_{2n} *is the Bernouilli number of order* $2n$, *a power series for* $\tanh x$.

$$2 \coth 2x - \coth x = \frac{2\,(e^{2x} + e^{-2x})}{e^{2x} - e^{-2x}} - \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$= \frac{2\,(e^{4x} + 1)}{e^{4x} - 1} - \frac{e^{2x} + 1}{e^{2x} - 1}$$

$$= \frac{(e^{2x} - 1)^2}{e^{4x} - 1} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^{x} - e^{-x}}{e^{x} + e^{x}},$$

provided $e^{2x} - 1 \neq 0$. Hence the result is true for real x provided $x \neq 0$.

Now $\coth x = \dfrac{e^{2x} + 1}{e^{2x} - 1} = 1 + \dfrac{2}{e^{2x} - 1}$; $\coth 2x = 1 + \dfrac{2}{e^{4x} - 1}$.

Hence $\tanh x = 1 + \dfrac{4}{e^{4x} - 1} - \dfrac{2}{e^{2x} - 1}$, $x \neq 0$,

$$= 1 + \frac{1}{x}\cdot\frac{4x}{e^{4x} - 1} - \frac{1}{x}\cdot\frac{2x}{e^{2x} - 1}, \quad x \neq 0.$$

Using the expansion of $x/(e^{x} - 1)$ we have

$$\frac{4x}{e^{4x} - 1} = 1 - 2x + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} 4^{2n} x^{2n}$$

$$\frac{2x}{e^{2x} - 1} = 1 - x + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} 2^{2n} x^{2n}.$$

It has been shown above that there exists a number λ such that the expansion of $x/(e^{x} - 1)$ is absolutely convergent for $|x| < \lambda$. Hence the expansion of $4x/(e^{4x} - 1)$ is absolutely convergent for $|4x| < \lambda$, and that of $2x/(e^{2x} - 1)$ for $|2x| < \lambda$. Thus both expansions are absolutely convergent for $|x| < \lambda/4$ and it is legitimate to write

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} 4^{2n} x^{2n} - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} 2^{2n} x^{2n} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (4^{2n} - 2^{2n})\, x^{2n}, \quad |x| < \lambda/4.$$

Hence $\tanh x = 1 + \dfrac{1}{x}\left[1 - 2x + \displaystyle\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} 4^{2n} x^{2n} - 1 + x - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} 2^{2n} x^{2n}\right]$

$$= 1 + \frac{1}{x}\left[-x + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (4^{2n} - 2^{2n})\, x^{2n}\right]$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} (2^{2n} - 1)\, x^{2n-1}.$$

This is the required power series for $\tanh x$. Substituting the values of $B_2, B_4, B_6, B_8, \ldots$ found above we have

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots.$$

More than one notation has been used for Bernouillian numbers. In the above definition the Bernouilli number of order n, $n > 1$, has been taken to be the coefficient of $x^n/n!$ in the expansion of $x/(e^{x} - 1)$. This gives $B_{2n-1} = 0$, and B_{2n} are alternately positive and negative.

An alternative way of writing the expansion of $x/(e^x - 1)$ is as follows:—

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B'_1\frac{x^2}{2!} - B'_2\frac{x^4}{4!} + B'_3\frac{x^6}{6!} + \dots$$
$$+(-1)^{n-1} B'_n\frac{x^{2n}}{(2n)!} + \dots.$$

This form has also been used for a definition of Bernouillian numbers and in accordance with it, the number of order n, B'_n is the coefficient of $(-1)^{n-1}x^{2n}/(2n)!$ The relation between the two notations is

$$B'_1 = B_2,\ B'_2 = -B_4,\ \dots,\ B'_n = (-1)^{n-1}B_{2n},\ \dots.$$

Bernouillian numbers can be expressed as series by using the following expansion of $x/(e^x - 1)$. Thus*

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{2x^2}{x^2 + 4n^2\pi^2}$$

This series of rational functions is absolutely convergent for $|x| < 2\pi$.

Now for $n \geqslant 1$, and $|x| < 2\pi$ we can expand each term of the series by the Binomial theorem. Thus

$$\frac{1}{x^2 + 4n^2\pi^2} = \frac{1}{4n^2\pi^2}\left(1 + \frac{x^2}{4n^2\pi^2}\right)^{-1}$$

$$= \frac{1}{4n^2\pi^2}\left[1 - \frac{x^2}{4n^2\pi^2} + \frac{x^4}{(4n^2\pi^2)^2} - \dots + (-1)^{r-1}\frac{x^{2r-2}}{(4n^2\pi^2)^{r-1}} + \dots\right]$$

By expanding for $n = 1, 2, 3, \dots$ the series $\sum_{n=1}^{\infty} \frac{2x^2}{x^2 + 4n^2\pi^2}$ can now be represented as a double series in the following form.

$$\frac{2x^2}{4\pi^2} - \frac{2x^4}{(4\pi^2)^2} + \frac{2x^6}{(4\pi^2)^3} - \dots + (-1)^{r-1}\frac{2x^{2r}}{(4\pi^2)^r} + \dots$$

$$+\frac{2x^2}{4.2^2\pi^2} - \frac{2x^4}{(4.2^2\pi^2)^2} + \frac{2x^6}{(4.2^2\pi^2)^3} - \dots + (-1)^{r-1}\frac{2x^{2r}}{(4.2^2\pi^2)^r} + \dots$$

$$+\frac{2x^2}{4\cdot3^2\pi^2} - \frac{2x^4}{(4\cdot3^2\pi^2)^2} + \frac{2x^6}{(4\cdot3^2\pi^2)^3} - \dots + (-1)^{r-1}\frac{2x^{2r}}{(4\cdot3^2\pi^2)^r} + \dots$$

....

....

* A proof of the expansion is given in Bromwich, *Theory of Infinite Series*

$$+\frac{2x^2}{4n^2\pi^2}-\frac{2x^4}{(4n^2\pi^2)^2}+\frac{2x^6}{(4n^2\pi^2)^3}-\ldots+(-1)^{r-1}\frac{2x^{2r}}{(4n^2\pi^2)^r}+\ldots$$

....

....

The sum by columns is

$$a_1x^2-a_2x^4+a_3x^6-\ldots+a_r(-1)^{r-1}x^{2r}+\ldots$$

where $a_1=\dfrac{2}{(2\pi)^2}\cdot\sum_1^\infty\dfrac{1}{n^2}$, $a_2=\dfrac{2}{(2\pi)^4}\sum_1^\infty\dfrac{1}{n^4}$, $a_3=\dfrac{2}{(2\pi)^6}\sum_1^\infty\dfrac{1}{n^6}$, ...

$$a_r=\frac{2}{(2\pi)^{2r}}\sum_{n=1}^\infty\frac{1}{n^{2r}},\ \ldots.$$

Hence, if B'_r denote the coefficient of $(-1)^{r-1}x^{2r}/(2r)!$ in the expansion of $x/e^x-1)$ in ascending powers of x then

$$B'_r=\frac{2\,(2r)!}{(2\pi)^{2r}}\sum_{n=1}^\infty\frac{1}{n^{2r}}.$$

This provides a general formula for the Bernouillian numbers, in the form of an infinite series.

Now write

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{n=1}^{r}(-1)^{n-1}B'_n\frac{x^{2n}}{(2n)!}+(-1)^rF_r(x)$$

and $$F_r(x)=B'_{r+1}\frac{x^{2r+2}}{(2r+2)!}-B'_{r+2}\frac{x^{2r+1}}{(2r+4)!}+\ldots$$

$$=\sum_{s=1}^\infty(-1)^{s-1}B'_{r+s}\frac{x^{2r+2s}}{(2r+2s)!}$$

The function $(-1)^rF_r(x)$ represents the remainder after $(r+2)$ terms of the expansion of $x/(e^x-1)$. We now prove that $F_r(x)$ is a positive function whose magnitude is less than its first term, i.e. $B'_{r+1}x^{2r+2}/(2r+2)!$.

Put $\dfrac{2x^2}{x^2+4n^2\pi^2}=\dfrac{x^2}{2n^2\pi^2}\left(1+\dfrac{x^2}{4n^2\pi^2}\right)^{-1}$ and substitute $y=\dfrac{x^2}{4n^2\pi^2}$ in the identity

$$1/(1+y)=1-y+y^2-\ldots+(-1)^{r-1}y^{r-1}+(-1)^ry^r/(1+y).$$

Then

$$\frac{2x^2}{x^2+4n^2\pi^2} = \frac{x^2}{2n^2\pi^2}\left[1 - \frac{x^2}{4n^2\pi^2} + \ldots + (-1)^{r-1}\left(\frac{x^2}{4n^2\pi^2}\right)^{r-1} + (-1)^r\left(\frac{x^2}{4n^2\pi^2}\right)^{r-1}\frac{x^2}{x^2+4n^2\pi^2}\right]$$

The series $\sum\limits_{n=1}^{\infty}\dfrac{2x^2}{x^2+4n^2\pi^2}$ can then be represented by the following array.

$$\frac{2x^2}{4\pi^2} - 2\left(\frac{x^2}{4\pi^2}\right)^2 + \ldots + (-1)^{r-1}2\left(\frac{x^2}{4\pi^2}\right)^r + (-1)^r\left(\frac{x^2}{4\pi^2}\right)^r\frac{2x^2}{x^2+4\pi^2}$$

$$+ 2\frac{x^2}{4.2^2\pi^2} - 2\left(\frac{x^2}{4.2^2\pi^2}\right)^2 + \ldots + (-1)^{r-1}2\left(\frac{x^2}{4.2^2\pi^2}\right)^r + (-1)^r\left(\frac{x^2}{4.2^2\pi^2}\right)^r\frac{2x^2}{x^2+4.2^2\pi^2}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ 2\frac{x^2}{4n^2\pi^2} - 2\left(\frac{x^2}{4n^2\pi^2}\right)^2 + \ldots + (-1)^{r-1}2\left(\frac{x^2}{4n^2\pi^2}\right)^r + (-1)^r\left(\frac{x^2}{4n^2\pi^2}\right)^r\frac{2x^2}{x^2+4n^2\pi^2}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Adding by columns we have

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + B'_1\frac{x^2}{2!} - B'_2\frac{x^4}{4!} + \ldots + (-1)^{r-1}B'_r\frac{x^{2r}}{(2r)!} + (-1)^r F_r(x)$$

where $$F_r(x) = \left(\frac{x^2}{4\pi^2}\right)^r\frac{2x^2}{x^2+4\pi^2} + \left(\frac{x^2}{4.2^2\pi^2}\right)^r\frac{2x^2}{x^2+4.2^2\pi^2} + \ldots + \left(\frac{x^2}{4n^2\pi^2}\right)^r\frac{2x^2}{x^2+4n^2\pi^2} + \ldots.$$

All the terms of this expansion are positive, since x is *real*. Hence $F_r(x)$ is a positive function for real values of x.

Since $$\frac{x^2}{x^2 + 4n^2\pi^2} < \frac{x^2}{4n^2\pi^2},$$

x real and n a positive integer.

$$F_r(x) < \left(\frac{x^2}{4\pi^2}\right)^r \frac{2x^2}{4\pi^2} + \left(\frac{x^2}{4.2^2\pi^2}\right)^r \frac{2x^2}{4.2^2\pi^2} + \ldots$$

$$+ \left(\frac{x^2}{4n^2\pi^2}\right)^r \frac{2x^2}{4n^2\pi^2} + \ldots$$

$$= \sum_{n=1}^{\infty} 2\left(\frac{x^2}{4n^2\pi^2}\right)^{r+1} = \frac{2x^{2r+2}}{(2\pi)^{2r+2}} \sum_{n=1}^{\infty} \frac{1}{n^{2r+2}} = B'_{r+1}\frac{x^{2r+2}}{(2r+2)!}$$

Thus $$\frac{x}{e^x - 1} - \left[1 - \tfrac{1}{2}x + \sum_{n=1}^{r} (-1)^{n-1}B'_n\frac{x^{2n}}{(2n)!}\right] = (-1)^r F_r(x),$$

where $0 < F_r(x) < B'_{r+1}\dfrac{x^{2r+2}}{(2r+2)!}$

If r is even so that $(-1)^r = +1$, it follows that

$$\frac{x}{e^x - 1} - \left[1 - \tfrac{1}{2}x + \sum_{n=1}^{r} (-1)^{n-1}B'_n\frac{x^{2n}}{(2n)!}\right] < B'_{r+1}\frac{x^{2r+2}}{(2r+2)!},$$

while if n is odd so that $(-1)^r = -1$,

$$\frac{x}{e^x - 1} - \left[1 - \tfrac{1}{2}x + \sum_{n=1}^{r} (-1)^{n-1}B'_n\frac{x^{2n}}{(2n)!}\right] > B'_{r+1}\frac{x^{2r+2}}{(2r+2)!}$$

The expression in square brackets is the expansion in powers of x of $x/(e^x - 1)$ to $(r + 2)$ terms. Hence if we stop the expansion at this point we see that *the magnitude of the error term is less than the numerical value of the next item in the expansion.* Thus, for example, writing $B'_1 = \frac{1}{6}$, $B'_2 = \frac{1}{30}$, $B'_3 = \frac{1}{42}$ we have

$$\frac{x}{e^x - 1} - \left[1 - \frac{x}{2}\right] < \frac{x^2}{12}$$

$$\frac{x}{e^x - 1} - \left[1 - \frac{x}{2} + \frac{x^2}{12}\right] > -\frac{x^4}{720}$$

$$\frac{x}{e^x - 1} - \left[1 - \frac{x^2}{2} + \frac{x^2}{12} - \frac{x^4}{180}\right] < \frac{x^6}{30240},$$

and so on.

5·43. Bernouillian Polynomials

Consider the expansion, in ascending powers of t, of

$$F(x, t) = t(e^{xt} - 1)/(e^t - 1)$$

and denote by $\phi_n(x)$ the coefficient of $t^n/n!$ in this expansion. Then $\phi_n(x)$ which is a polynomial of degree n in x is called a Bernouillian polynomial.

Writing $$e^{xt} - 1 = \sum_{n=1}^{\infty} x^n t^n/n!,$$

$$F(x, t) = \sum_{n=1}^{\infty} \phi_n(x)\frac{t^n}{n!} = \left[1 - \tfrac{1}{2}t + B'_1\frac{t^2}{2!} - B'_2\frac{t^4}{4!} + \dots \right.$$

$$\left. + (-1)^{r-1}B'_r\frac{t^{2r}}{(2r)!} + \dots\right]\left[\sum_{n=1}^{\infty}\frac{x^n t^n}{n!}\right]$$

Equating coefficients of $t^n/n!$,

$$\phi_n(x) = x^n - \frac{n}{2}x^{n-1} + \frac{n(n-1)}{2!}B'_1 x^{n-2}$$

$$- \frac{n(n-1)(n-2)n-3)}{4!}B'_2 x^{n-4} + \dots,$$

the last term in $\phi_n(x)$ involving either x^2 or x, depending on the value of n. If $n > 1$ this polynomial has $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+3)$ terms according as n is even or odd. The first four polynomials are

$$\phi_1(x) = x,\ \phi_2(x) = x^2 - x,\ \phi_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$
$$\phi_4(x) = x^4 - 2x^3 + x^2,$$

When $x = 1$, $F(x, t) = t$ and hence $\phi_n(1) = 0$, $n > 1$.

Now $$F(x+1, t) - F(x, t) = t\frac{e^{(x+1)t} - 1}{e^t - 1} - t\frac{e^{xt} - 1}{e^t - 1} = te^{xt}$$

Hence $$\sum_{n=1}^{\infty}\{\phi_n(x+1) - \phi_n(x)\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n t^{n+1}}{n!}.$$

Equating coefficient of t^n in the two power series,

$$\phi_n(x+1) - \phi_n(x) = nx^{n-1} \quad \dots\dots\dots\dots\dots\dots \quad (1)$$

Using the known values for $\phi_n(x)$, $n = 2, 3, 4$, we have

$$\phi_2(x+1) = \phi_2(x) + 2x = x^2 + x$$
$$\phi_3(x+1) = \phi_3(x) + 3x^2 = x^3 + \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$
$$\phi_4(x+1) = \phi_4(x) + 4x^3 = x^4 + 2x^3 + x^2.$$

Changing n into $(n + 1)$ in the difference equation (1), we have

$$\phi_{n+1}(x + 1) - \phi_{n+1}(x) = (n + 1)\, x^n \quad \ldots\ldots\ldots \quad (2).$$

Now take x to be a positive integer and write $x = 1, 2, 3, \ldots$ in succession in (2). Then

$$\phi_{n+1}(2) - \phi_{n+1}(1) = (n + 1).1^n$$
$$\phi_{n+1}(3) - \phi_{n+1}(2) = (n + 1).2^n$$

$$\phi_{n+1}(x + 1) - \phi_{n+1}(x) = (n + 1).x^n$$

Adding, and observing that $\phi_{n+1}(1) = 0$,

$$1^n + 2^n + 3^n + \ldots + x^n = \frac{1}{n + 1}\phi_{n+1}(x + 1) = \frac{1}{n + 1}\phi_{n+1}(x) + x^n.$$

This gives a formula for the sum of the nth powers of the positive integers. The function $\phi_{n+1}(x)$ is the coefficient of $t^{n+1}/(n + 1)!$ in the expansion of $F(x, t)$. Hence the last equation can be written in the form

$$\sum_{r=1}^{x} r^n = \frac{1}{n + 1}\left[x^{n+1} \quad \frac{n + 1}{2}x^n \quad \frac{(n + 1)\, n}{2!}B'_1 x^{n-1} \quad \frac{(n + 1)\, n\, (n - 1)\, (n - 2)}{4!}B'_2 x^{n-3} + \ldots\right] + x^n$$

$$\frac{x^{n+1}}{n + 1} + \tfrac{1}{2}x^n + \frac{n}{2!}B'_1 x^{n-1} \quad \frac{n\,(n - 1)\,(n - 2)}{4!}B'_2 x^{n-3} + \ldots$$

$$= \int x^n dx + \tfrac{1}{2}x^n + \frac{B'_1}{2!}\frac{d}{dx}(x^n) - \frac{B'_2}{4!}\frac{d^3}{dx^3}(x^n) + \ldots$$

Writing $f(x) = x^n$, this formula becomes

$$f(1) + f(2) + \ldots + f(x) = \int f(x)\, dx + \tfrac{1}{2}f(x) + \frac{B'_1}{2!}f'(x) - \frac{B'_2}{4!}f'''(x) + \ldots,$$

where $\int f(x)\, dx$ denote the indefinite integral of $f(x)$, and $f'(x)$, $f'''(x)$, $\ldots$ denote the differential coefficients of $f(x)$. Every term on the right-hand side is ultimately divisible by x. This

result extends immediately to any polynomial in x since multiplication of x^n by a constant a does not affect the result and a polynomial is made up of terms of the form ax^n. The result, proved for a polynomial, is known as the *Euler-Maclaurin summation formula.*

In practice the majority of the most interesting applications are to transcendental and algebraic functions which on expansion give an infinite series and not a polynomial. In these cases questions of convergence arise and it is necessary to consider the magnitude of the remainder term which expresses the error committed when an infinite series is stopped at a particular term.

Example.—*If* S_r *denotes* $\sum_{x=1}^{n} x^r$, *prove that* $S_5 + S_7 = 2S_3^2$.

Applying the Euler-Maclaurin formula for $f(x) = x^3, x^5$, and x^7 in succession, we have

$$S_3 = \int x^3dx + \tfrac{1}{2}x^3 + \frac{1}{2!}B'_1(3x^2), \quad x = n,$$

$$S_5 = \int x^5dx + \tfrac{1}{2}x^5 + \frac{1}{2!}B'_1(5x^4) - \frac{1}{4!}B'_2(5\cdot4\cdot3x^2), \quad x = n,$$

$$S_7 = \int x^7dx + \tfrac{1}{2}x^7 + \frac{1}{2!}B'_1(7x^6) - \frac{1}{4!}B'_2(7.6.5x^4)$$
$$+ \frac{1}{6!}B'_3(7.6.5\cdot4\cdot3x^2), \; x = n.$$

Substituting $B'_1 = \frac{1}{6}$, $B'_2 = \frac{1}{30}$, $B'_3 = \frac{1}{42}$, these expressions reduce to

$$S_3 = \frac{1}{4}n^2(n+1)^2, \quad S_5 = \frac{1}{12}n^2(2n^4 + 6n^3 + 5n^2 - 1),$$

$$S_7 = \frac{1}{24}n^2(3n^6 + 12n^5 + 14n^4 - 7n^2 + 2).$$

$$S_5 + S_7 = \frac{n^2}{24}(3n^6 + 12n^5 + 18n^4 + 12n^3 + 3n^2) = \frac{n^4}{8}(n+1)^4 = 2S_3^2.$$

5·5 The Logarithmic Series

We now prove that if $|x| < 1$,

$$\log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$
$$= \sum^{\infty} (-1)^{r-1}\frac{x^r}{}$$

$$(1+x)^y = e^{y\log(1+x)*}$$

* In more advanced mathematics the base of a logarithm is *always* taken to be e unless otherwise indicated. Thus in what follows, when the base is omitted the student should understand that the base is e.

i.e. $$(1+x)^y = 1 + y\log(1+x) + \frac{y^2}{2!}\{\log(1+x)\}^2 + \frac{y^3}{3!}\{\log(1+x)\}^3 + \ldots \ldots\ldots \quad \text{(i)}$$

Expanding $(1+x)^y$ by the binomial theorem we have

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{2!}x^2 + \frac{y(y-1)(y-2)}{3!}x^3 + \ldots, \quad |x| < 1. \qquad \text{(ii)}$$

Assuming that it is legitimate to rearrange this series in ascending powers of y it follows from Chapter I., § 1·8 that coefficients of corresponding powers of y in (i) and (ii) are equal. In particular if we equate coefficients of y, we have

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots$$

We may justify the rearrangement of the terms of the series by comparing term by term the binomial expansion of $(1+x)^y$ with that of $(1-\xi)^{-\eta}$ where $\xi = |x|$, $\eta = |y|$, the expansions being valid for $|x| < 1$. The $(r+1)$th term is $\eta(\eta+1)\ldots(\eta+r-1)\,\xi^r/r!$ which is the product of a series of positive factors and when written in powers of η contains $(r+1)$ *positive* terms. This series can accordingly be rearranged in powers of η and by comparison the result follows for the series in y.

Note.—It will be observed that we can obtain series for $\{\log(1+x)\}^2$ and for higher powers by equating coefficients of y^2 and higher powers of y. Thus from the coefficients of y^2,

$$\tfrac{1}{2}\{\log(1+x)\}^2 = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3(1+\tfrac{1}{2}) + \tfrac{1}{4}x^4(1+\tfrac{1}{2}+\tfrac{1}{3}) - \ldots,$$

and so on.

VALIDITY OF THE LOGARITHMIC SERIES.—The series converges absolutely and uniformly for $|x| \leqslant \rho < 1$. Hence $\log(1+x)$ is a continuous function of x for $|x| < 1$.

When $x = 1$ the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ which is known to be convergent. Hence by Abel's theorem (Chapter IV., § 4·7) the interval of uniform convergence extends up to and includes $x = 1$ and

$$\lim_{x \to 1} \log(1+x) = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots,$$

$$\text{i.e.} \quad \log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots$$

When $x = -1$ the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ which is divergent.

Thus the series is valid in the range $-1 < x \leqslant 1$ and in this range $\log(1+x)$ is continuous.

It is convenient to remember the form of the series in the case in which x is negative. Thus

$$\log(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \ldots$$

$$i.e. \quad -\log(1-x) = \sum_{=1}^{\infty} \frac{x^r}{r}.$$

5·51. Inequalities for $\log(1+x)$

Now $x - \log(1+x) = \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{x^4}{4} - \dfrac{x^5}{5} + \ldots;$

$$= x^2(\tfrac{1}{2} - \tfrac{1}{3}x) + x^4(\tfrac{1}{4} \quad \tfrac{1}{5}x) + \ldots > 0,$$

for $-1 < x \leqslant 1$. Again if $0 < x \leqslant 1$,

$$x - \log(1+x) - \tfrac{1}{2}x^2 = -x^3(\tfrac{1}{3} - \tfrac{1}{4}x) - x^5(\tfrac{1}{} \quad \tfrac{1}{6}x) - \ldots < 0.$$

Hence $0 < x - \log(1+x) < \tfrac{1}{2}x^2, \quad 0 < x \leqslant 1.$

Now suppose that $-1 < x < 0$ and write $y = -x$, so that $0 < y < 1$. Then

$\log(1+x) = \log(1-y) = -y - \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 - \ldots$ Hence

$$\begin{aligned} -y - \log(1-y) &= \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + \tfrac{1}{4}y^4 + \ldots \\ &< \tfrac{1}{2}y^2(1 + y + y^2 + \ldots) \\ &= \tfrac{1}{2}y^2/(1-y), \quad |y| < 1. \end{aligned}$$

Thus $0 < x - \log(1+x) < \tfrac{1}{2}x^2/(1+x), \quad -1 < x < 0.$

Example.—*Prove that if n is a positive integer*

$$0 < \frac{1}{n} - \log\frac{n+1}{n} < \frac{1}{2n^2}$$

and $0 < 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \dfrac{1}{n-1} - \log n < 1$ [*M.T.*]

The first inequality follows immediately by writing $x = 1/n$ in the first inequality proved above.

Now put $n = 1, 2, 3, \ldots (n-1)$ in succession. Then

$$0 < 1 - \log\tfrac{2}{1} < \frac{}{2.1^2},$$

$$0 < \tfrac{1}{2} \quad \log\tfrac{3}{2} < \frac{1}{2.2^2},$$

$$0 < \tfrac{1}{3} - \log\tfrac{4}{3} < {}_{2.3^2}$$

$0 <$

Adding, $0 < 1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n-1} - \log n < \frac{1}{2} \sum_{r=1}^{n-1} \frac{1}{r^2}$.

Now $$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{1}{1^2} + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \ldots$$

$$< 1 + 2 \cdot \frac{1}{2^2} + 4 \cdot \frac{1}{4^2} + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots = 2$$

Now $\sum_{r=1}^{n-1} \frac{1}{r^2} < \sum_{r=1}^{\infty} \frac{1}{r^2}$, *i.e.* < 2 and result follows.

5·6. Modification of the Logarithmic Series

For purposes of numerical calculation the logarithmic series is inconvenient for two reasons.

Firstly the series converges very slowly, *i.e.* it would require many terms to obtain a good approximation.

Secondly the series is not directly available for the cases in which $x > 1$.

To obviate the first difficulty the logarithmic series is replaced by others which converge more rapidly.

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad |x| < 1,$$

$$\log (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \quad |x| < 1.$$

Thus $$\log (1 + x) - \log (1 - x) = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right\}$$

Hence $$\log \frac{1 + x}{1 - x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right\} \ldots\ldots\ldots\ldots (1)$$

the expansion being valid for $|x| < 1$.

Example.—*Show that the logarithms of all integers from 1 to 10 can be calculated by putting* $x = \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ *and* $\frac{1}{8}$ *in turn in the expansion*

$$\log_e \frac{1 + x}{1 - x} = 2\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right).$$

Write $f(x) = \log_e \frac{1 + x}{1 - x}$. Then

$f(\frac{1}{5}) = \log \frac{3}{2} = \log 3 - \log 2$ ....(i)
$f(\frac{1}{6}) = \log \frac{7}{5} = \log 7 - \log 5$ ....(ii)
$f(\frac{1}{7}) = \log \frac{4}{3} = 2 \log 2 - \log 3$ ....(iii)
$f(\frac{1}{8}) = \log \frac{9}{7} = 2 \log 3 - \log 7$ ....(iv)

All logarithms are to base e. The left-hand side of each equation can be calculated as accurately as is desired by considering the corresponding series.

From (i) and (iii) $\log 2$ and $\log 3$ are found: then $\log 4 = 2 \log 2$ is known. Eliminating $\log 7$ between (ii) and (iv), $\log 5$ is found: then

$$\log 6 = \log 3 + \log 2$$

is known. Log 7 may be found from (ii) since $\log 5$ is known. Also $\log 8 = 3 \log 2$, $\log 9 = 2 \log 3$ and $\log 10 = \log 5 + \log 2$ are determined.

From (1) we may deduce series which are useful in numerical computation, and which converge when the variable involved is greater than unity.

Write $x = 1/(2n + 1)$ so that $(1 + x)/(1 - x) = (n + 1)/n$. Then

$$\log \frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\} \quad (2)$$

the expansion being valid provided $|1/(2n + 1)| < 1$, i.e. $n > 0$ or $n < -1$.

Example.—*By means of the expansion of* $\log \{(n + 1)/n\}$ *in ascending powers of* $1/(2n + 1)$, *evaluate* $\log_e \frac{4}{3}$ correct to four decimal places.

To find $\log \frac{4}{3}$ write $n = 3$. Then

$$\log \tfrac{4}{3} = 2\left\{\frac{1}{7} + \frac{1}{3 \cdot 7^3} + \frac{1}{5 \cdot 7^5} + \frac{1}{7 \cdot 7^7} + \dots\right\}$$

The work of calculation may be set out as follows:

	$1/7 = \cdot 142857$.
$1/7^2 = \cdot 020408$,	
$1/7^3 = \cdot 002915$,	$1/3 \cdot 7^3 = \cdot 000972$.
$1/7^4 = \cdot 000416$,	
$1/7^5 = \cdot 000059$,	$1/5 \cdot 7^5 = \cdot 000012$.

Clearly the next term of the series will not affect the fifth decimal place. Adding $\log \frac{4}{3} = 2\,(\cdot 143841) = \cdot 287682 = \cdot 2877$ correct to 4 decimal places.

We now deduce another formula suitable for numerical computation. In (1) write $(1 + x)/(1 - x) = m/n$ so that $x = (m - n)/(m + n)$. The condition for the validity of the expansion is $|(m - n)/(m + n)| < 1$, i.e.

$$\left(\frac{m-n}{m+n}\right)^2 - 1 < 0. \text{ or } \left(\frac{m-n}{m+n} - 1\right)\left(\frac{m-n}{m+n} + 1\right) < 0.$$

Multiplying throughout by the factor $(m + n)^2$ which is positive for real m and n, this condition reduces to $mn > 0$, i.e. m, n either both positive or both negative. It will be observed that this is

the condition that the logarithm be *real*. For $mn > 0$ implies $m/n > 0$, when m and n are real. The expansion becomes

$$\log\frac{m}{n} = 2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + 5\left(\frac{m-n}{m+n}\right)^5 + \ldots\right] \quad \ldots(3)$$

the expansion being valid for $mn > 0$.

Example.—*Show that* $\log_e 1{\cdot}01$ *and* $\log_e 1{\cdot}02$ *can be calculated correct to five decimal places by using only the first term of the above series, and that* $\log_e 1{\cdot}1$ *can be calculated to the same degree of accuracy by using the first two terms.*

Write $m = 101$, $n = 100$. Then

$$m - n = 1,\ m + n = 201,\ (m-n)/(m+n) = 1/201$$

Hence $\left(\dfrac{m-n}{m+n}\right)^3 < \left(\frac{1}{100}\right)^3 = 0{\cdot}000001$. Thus $\frac{2}{3}\left(\dfrac{m-n}{m+n}\right)^3 < 0{\cdot}000001$. Clearly this and higher terms cannot affect the fifth decimal place in the determination of $\log 1{\cdot}01$.

For $\log 1{\cdot}02$ write $m = 102$, $n = 100$. Then

$$\frac{2}{3}\left(\frac{m-n}{m+n}\right)^3 = \frac{2}{3}\cdot\left(\frac{2}{202}\right)^3 = \frac{2}{3}\left(\frac{1}{101}\right)^3 < 0{\cdot}000001$$

Thus the second term will not affect the fifth decimal place.*

For $\log 1{\cdot}1$ write $m = 11$, $n = 10$. Then

$$\frac{2}{5}\left(\frac{m-n}{m+n}\right)^5 = \frac{2}{5}\left(\frac{1}{21}\right)^5 < \frac{2}{5}\cdot\left(\frac{1}{10}\right)^5 = \frac{2}{5} \times {\cdot}00001 < {\cdot}000005.$$

Hence this and higher terms will not affect the fifth decimal place.

The series (3) may be represented in a slightly different form by writing $m/n = x$ or we may deduce the result by proceeding directly from the expansion

$$\log\frac{1+y}{1-y} = 2\{y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \ldots\}$$

by writing $(1+y)/(1-y) = x$, so that $y = (x-1)/(x+1)$. In either case we obtain the expansion

$$\log x = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right\}. \quad (4)$$

the expansion being valid for all positive values of x.

The condition $x > 0$ may be deduced from the validity condition of either (1) or (3).

* A discussion on the conditions under which we may approximate to the expansion by the first term of the series is given in § 5·9.

Example.—*Show that in the series*

$$\log_e x = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots\right\},$$

when $x = 2$, *the remainder after* n *terms is less than* $1/\{4(2n+1)\,3^{2n-1}\}$.
[*Lond. B.Sc.*]

When $x = 2$, $\log_e x = 2\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \dots\right]$

$$i.e.\ \log 2 = 2 \sum_{r=1} \frac{}{2r+1}$$

The remainder after n terms is

$$\sum_{r=n}^{\infty} \frac{3^{-(2r+1)}}{2r+1} < \frac{2}{2n+1} \sum_{r=n}^{\infty} 3^{-(2r+1)}.$$

Summing the series as a G.P. we obtain required result.

We can deduce from (1) another form which is sometimes useful. In the expansion

$$\log\frac{1+y}{1-y} = 2\{y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \dots\},\ |y| \quad 1,$$

write $y = 1/(2x-1)$ so that $(1+y)/(1-y) = x/(x-1)$.

The condition $|y| < 1$ becomes $|1/(2x-1)| < 1$, i.e. $(2x-1)^2 > 1$, i.e. $x(x-1) > 0$. Thus the expansion will be valid provided $x < 0$ or $x > 1$. Hence

$$\log\frac{x}{x-1} = 2\left\{\frac{1}{2x-1} + \frac{1}{3}\frac{1}{(2x-1)^3} + \frac{1}{5}\frac{1}{(2x-1)^5} + \dots\right\}. \quad (5)$$

provided $x < 0$ or $x > 1$.

5·7. The expansion of $\log_e(a + x)$

If $a > 0$ and $|x| < a$ we can obtain from the logarithmic series an expansion of $\log(a + x)$ in ascending powers of x. Thus

$$\log(a + x) = \log\{a(1 + x/a)\}$$
$$= \log a + \log(1 + x/a),\ \text{if } a > 0.$$

If $|x| < a$, $\log(1 + x/a)$ may be expanded as follows:

$$\log\left(1 + \frac{x}{a}\right) = \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 - \frac{1}{4}\left(\frac{x}{a}\right)^4 + \dots$$

Thus $\log(a + x) = \log a + \sum_{r=1}^{\infty} (-1)^{r-1}\frac{1}{r}\left(\frac{x}{a}\right)^r,\quad |x| < a.$

If $a < 0$ but $a + x > 0$ and $|a| < x$ we can obtain in a similar way an expansion in ascending powers of $1/x$.

Example.—*From the expansion* $\log_e(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ *deduce the expansion of* $\log_{10}(a + x)$ *in ascending powers of* x. *For what values of* x *is it valid?*

Given $\log_{10}2 = \cdot 30103$, *and* $\log_{10}e = \cdot 43429$ *calculate* $\log_{10}2\cdot 1$ *and* $\log_{10}1\cdot 8$, *without using tables.*

$$\log_{10}(a + x) = \log_e(a + x)/\log_e 10 = \log_{10}e \,.\, \log_e(a + x)$$

$$= \mu \log_e\left\{ a\left(1 + \frac{x}{a}\right)\right\}, \text{ where } \mu = \log_{10}e,$$

$$= \mu \log_e a + \mu \log_e\left(1 + \frac{x}{a}\right)$$

$$= \mu \log_e a + \mu \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r}\left(\frac{x}{a}\right)^r, \quad |x| < |a|;$$

$$\text{thus } \log_{10}(a + x) = \log_{10}a + \mu \sum_{n=1}^{\infty} \frac{(-1)^{r-1}}{r}\left(\frac{x}{a}\right)^r.$$

Write $a = 2$, $x = 0\cdot 1$, $x/a \doteq 0\cdot 05$. Then

$$\log_{10}2\cdot 1 = \log_{10}2 + \mu\{\cdot 05 - \tfrac{1}{2}(\cdot 05)^2 + \tfrac{1}{3}(\cdot 05)^3 - \tfrac{1}{4}(\cdot 05)^4 + \ldots\}$$
$$= \cdot 30103 + \cdot 43429\{\cdot 05 - \cdot 00125 + \cdot 000042 - \ldots\}.$$

Clearly higher powers of $\cdot 05$ will not affect the fourth decimal place. Thus

$$\log_{10}2\cdot 1 = \cdot 30103 + \cdot 43429 \times \cdot 048792$$
$$= \cdot 3222 \text{ correct to four decimal places.}$$

To obtain $\log_{10}1\cdot 8$ write $a = 2$, $x = -0\cdot 2$, $x/a = -0\cdot 1$. Then

$$\log_{10}1\cdot 8 = \log_{10}2 + \cdot 43429\{-\cdot 1 - \tfrac{1}{2}(\cdot 1)^2 - \tfrac{1}{3}(\cdot 1)^3 - \tfrac{1}{4}(\cdot 1)^4 - \ldots\}$$
$$= \cdot 30103 - \cdot 43429\{\cdot 1 + \cdot 005 + \cdot 000333 + \cdot 000025 + \ldots\}$$
$$= \cdot 30103 - \cdot 43429 \times \cdot 10536$$
$$= \cdot 2553 \text{ correct to four decimal places.}$$

5·8. The Application of the Logarithmic Series to Limits

When x is small $\log(1 + x) = x$, provided x^2 and higher powers may be neglected. If this approximation is not sufficient for the purpose the next term must be added. Thus as a second approximation

$$\log(1 + x) = x - \tfrac{1}{2}x^2.$$

The application is shown in the following examples.

Examples.—(1) *Find the limit as* $x \to 1$ *of* $(\log x)/(x^2 - 3x + 2)$.

Put $x = 1 + h$. Then

$$\lim_{x \to 1} \frac{\log x}{x^2 - 3x + 2} = \lim_{h \to 0} \frac{\log(1 + h)}{(1 + h)^2 - 3(1 + h) + 2}$$

$$\lim \frac{\log(1 + h)}{\quad} = \lim_{h \to 0} \frac{h - \frac{1}{2}h^2 + \ldots}{h(-1 + h)}$$

$$\lim \frac{1 - \frac{1}{2}h +}{-1 + h} = -1.$$

(2) *Prove that* $\log_e(1 - xe^{-x}) = -x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \ldots$, *and deduce the limiting value of*

$$[\log_e \{(1 - xe^{-x})(1 + xe^{x})\}]/x^2$$

as x tends to zero.

Now
$$\begin{aligned}\log(1 - xe^{-x}) &= -xe^{-x} - \tfrac{1}{2}x^2e^{-2x} - \tfrac{1}{3}x^3e^{-3x} - \ldots, \quad |xe^{-x}| < 1\\ &= -x(1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \ldots)\\ &\quad - \tfrac{1}{2}x^2(1 - 2x + \tfrac{1}{2}.4x^2 - \ldots)\\ &\quad - \tfrac{1}{3}x^3(1 - 3x + \ldots)\\ &\quad - \tfrac{1}{4}x^4(1 - 4x + \ldots)\\ &\quad \ldots\ldots\ldots\ldots\\ &= -x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 \ldots\end{aligned}$$

Changing x into $-x$ and assuming that $|xe^{x}| < 1$,

$$\log(1 + xe^{x}) = x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 \ldots$$

$$\begin{aligned}\therefore \log\{(1 - xe^{-x})(1 + xe^{x})\} &= \log(1 - xe^{-x}) + \log(1 + xe^{x})\\ &= x^2 + \text{terms in } x^4 \text{ and higher powers.}\end{aligned}$$

Hence $\lim\limits_{x \to 0} [\log\{(1 - xe^{-x})(1 + xe^{x})\}]/x^2 = 1.$

(3) *Prove* that $\left(\dfrac{n+1}{n-1}\right)^n = e^2\left(1 + \dfrac{2}{3n^2}\right)$ *approximately, when n is large.*

Write $u = \{(n+1)/(n-1)\}^n$;

$$\therefore \log u = n \log\{(n+1)/(n-1)\} = n\log\left(1 + \frac{1}{n}\right) - n\log\left(1 - \frac{1}{n}\right)$$

$$= 2n\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \ldots\right\} = 2 + \frac{2}{3n^2},$$

where $1/n^3$ and higher powers of $1/n$ are neglected.

Now when n is large, $\log\left(1 + \dfrac{2}{3n^2}\right) \quad \dfrac{2}{3n^2}$;

$$\therefore \log u = 2 + \log\left(1 + \frac{2}{3n^2}\right) \text{ approx., i.e. } u = e^2\left(1 + \frac{2}{3n^2}\right).$$

(4) *Prove that* $\lim\limits_{x \to 0} \sqrt[x]{\dfrac{1+x}{1-x}} = e^2.$

Write $u = \sqrt[x]{\dfrac{1+x}{1-x}}.$

Then
$$\begin{aligned}\log u = \frac{1}{x}\log\{(1+x)/(1-x)\} &= \frac{2}{x}\{x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots\}\\ &= 2 + \tfrac{2}{3}x^2 + \tfrac{2}{5}x^4 + \ldots \qquad \text{(i)}\end{aligned}$$

Letting $x \to 0$, $\log u \to 2$. Hence $\lim\limits_{x \to 0} u = e^2$.

Alternatively we may proceed as follows. From (i)

$$u = e^{2 + \frac{2}{3}x^2 + \frac{2}{5}x^4 + \ldots} = e^2 \times e^{\frac{2}{3}x^2 + \frac{2}{5}x^4 + \ldots}$$

As $x \to 0$, $e^{\frac{2}{3}x^2 + \frac{2}{5}x^4 + \ldots} \to e^0 = 1.$

Hence $\lim u = e^2$ as before.

(5) *Find* $\lim\limits_{x \to 0} \{(e^x - e^{-x})/\log(1 + x)\}$.

$$e^x - e^{-x} = 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$\log(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots, \quad |x| < 1.$$

Thus $$\frac{e^x - e^{-x}}{\log(1+x)} = \frac{2\left(1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \ldots\right)}{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \ldots}$$

Thus $$\lim_{x \to 0} \frac{e^x - e^{-x}}{\log(1+x)} = 2.$$

The student should note that in a case like this we are assuming that series in the numerator and denominator represent continuous functions so that the limit as $x \to 0$ is the actual value of the functions at $x = 0$.

(6) *Defining* $\log_e a = \lim\limits_{x \to 0} \{(a^x - 1)/x\}$, *prove that*

$$\log_e ab = \log_e a + \log_e b.$$

Now $\log_e ab = \lim\limits_{x \to 0} (a^x b^x - 1)/x$;

$$\begin{aligned}
\therefore \log_e a + \log_e b &= \lim_{x \to 0} \{(a^x - 1)/x\} + \lim_{x \to 0} \{(b^x - 1)/x\} \\
&= \lim_{x \to 0} \{(a^x + b^x - 2)/x\} \text{ (Chap. II., § 2·63)} \\
&= \lim_{x \to 0} \{(a^x b^x - 1 + a^x + b^x - a^x b^x - 1)/x\} \\
&= \lim_{x \to 0} \{(a^x b^x - 1)/x\} \\
&\qquad + \left[\lim_{x \to 0} (1 - b^x)\right]\left[\lim_{x \to 0} \{(a^x - 1)/x\}\right] \\
&= \log_e ab,
\end{aligned}$$

since $\lim\limits_{x \to 0} (1 - b^x) = 0$ and $\lim\limits_{x \to 0} \{(a^x - 1)/x\} = \log_e a$, which is finite.

5·81. Miscellaneous Examples

(1) *Find the coefficient of* x^n *in the expansion of* $\log_e (1 + x + x^2 + x^3)$ *in ascending powers of* x, (i) *when* n *is a multiple of* 4, (ii) *when* n *is not a multiple of* 4.

Now $1 + x + x^2 + x^3 = (1 - x^4)/(1 - x)$. Thus

$$\begin{aligned}
\log_e(1 + x + x^2 + x^4) &= \log(1 - x^4) - \log(1 - x) \\
&= -\sum_{r=1}^{\infty} \frac{x^{4r}}{r} + \sum_{r=1}^{\infty} \frac{x^r}{r}.
\end{aligned}$$

Hence if n is a multiple of 4 there is a term from each expansion. That from the first is $-4x^n/n$ while that from the second is x^n/n. Thus the coefficient of x^n is $-3/n$. If n is not a multiple of 4, then there is no corresponding term in the first expansion, and the coefficient is $1/n$.

(2) *Find the ranges of values of x for which $\log_e(1 - 3x + x^2)$ can be expanded in a series of ascending powers of x and give the expansion as far as the term involving x^4.* [*Lond. B.Sc.*]

$$\log(1 - 3x + x^2) = \log\{1 - x(3 - x)\} = -\sum_{r=1}^{\infty} x^r(3 - x)^r/r,$$

provided $|3x - x^2| < 1$, i.e. $(3x - x^2)^2 - 1 < 0$.

Now $(3x - x^2)^2 - 1 = (3x - x^2 + 1)(3x - x^2 - 1)$

$$= (x^2 - 3x - 1)(x^2 - 3x + 1).$$

The roots of $x^2 - 3x + 1 = 0$ are $x = \frac{1}{2}(3 \pm \sqrt{5})$, while the roots of $x^2 - 3x - 1 = 0$ are $x = \frac{1}{2}(3 \pm \sqrt{13})$. Arranged in ascending order, let the four roots of $(x^2 - 3x - 1)(x^2 - 3x + 1) = 0$ be $\alpha, \beta, \gamma, \delta$ and write

$$E = (x^2 - 3x - 1)(x^2 - 3x + 1).$$

For $x < \alpha$, $E > 0$; for $\alpha < x < \beta$, $E < 0$; for $\beta < x < \gamma$, $E > 0$; for $\gamma < x < \delta$, $E < 0$, while for $x > \delta$, $E > 0$. Hence

$$E < 0 \text{ provided } \alpha < x < \beta \text{ or } \gamma < x < \delta,$$

$$\textit{i.e.}\ \tfrac{1}{2}(3 - \sqrt{13}) < x < \tfrac{1}{2}(3 - \sqrt{5})$$

$$\text{or } \tfrac{1}{2}(3 + \sqrt{5}) < x < \tfrac{1}{2}(3 + \sqrt{13}).$$

For these values of x

$$\begin{aligned}
\log(1 - 3x + x^2) &= -\{x(3 - x) + \tfrac{1}{2}x^2(3 - x)^2 + \tfrac{1}{3}x^3(3 - x)^3 \\
&\qquad + \tfrac{1}{4}x^4(3 - x^4) + \dots\} \\
&= -3x + x^2 \\
&\qquad - \tfrac{9}{2}x^2 + 3x^3 - \tfrac{1}{2}x^4 \\
&\qquad\qquad - 9x^3 + 9x^4 + \dots \\
&\qquad\qquad\qquad - \tfrac{81}{4}x^4 + \dots
\end{aligned}$$

$$\therefore \log(1 - 3x + x^2) = -3x - \tfrac{7}{2}x^2 - 6x^3 - \tfrac{47}{4}x^4 \dots$$

(3) *Expanding $\log(1 - x^3)$ in power series in two different ways, prove that for any positive integer k,*

$$\sum_{\frac{1}{2}k<n<k} \frac{(-1)^{n-1}(n-1)!}{(k-n)!\,(2n-k)!} = \frac{1}{k} \text{ if } k = 3p \pm 1$$

$$= -\frac{2}{k} \text{ if } k = 3p,$$

where p is an integer. [*M. T.*]

$$\log(1 - x^3) = -\sum_{t=1}^{\infty} \frac{x^{3t}}{t}, \ |x| < 1.$$

Again $\log(1 - x^3) = \log(1 - x) + \log(1 + x + x^2)$.

$$\log(1 - x) = -\sum_{r=1}^{\infty} \frac{x^r}{r}.$$

$$\log(1 + x + x^2) = \log\{1 + x(1 + x)\}$$

$$= \sum_{s=1}^{\infty} (-1)^s \frac{x^s}{s}(1 + x)^s, \ |x(1 + x)| < 1.$$

Thus $\sum\limits_{s=1}^{\infty} (-1)^s \frac{x^s}{s}(1+x)^s = \sum\limits_{r=1}^{\infty} \frac{x^r}{r} - \sum\limits_{t=1}^{\infty} \frac{x^{3t}}{t}$.

$$\text{Now } (1+x)^s = \sum_{u=0}^{s} {}_sC_u x^u.$$

$$\text{Thus } \sum_{s=1}^{\infty} (-1)^s \frac{x^s}{s}(1+x)^s = \sum_{s=1}^{\infty} \frac{(-1)^s}{s} \sum_{u=0}^{s} {}_sC_u x^{u+s}.$$

Consider the coefficient of x^k is this expansion. Then $u + s = k$. The coefficient is

$$\frac{(-1)^k}{k} {}_kC_0 + \frac{(-1)^{k-1}}{k-1} {}_{k-1}C_1 + \frac{(-1)^{k-2}}{k-2} {}_{k-2}C_2 + \ldots \frac{(-1)^n}{n} {}_nC_{k-n} + \ldots$$

The last term will correspond to $n = [\frac{1}{2}k]$ where $[\frac{1}{2}k]$ is the integral part of $\frac{1}{2}k$.

$$\text{Now } \frac{(-1)^n}{n} {}_nC_{k-n} = \frac{(-1)^n}{n} \frac{n\,!}{(k-n)\,!\,(2n-k)\,!} = \frac{(-1)^n (n-1)\,!}{(k-n)\,!\,(2n-k)\,!}$$

Thus the coefficient of x^k may be written in the form

$$\sum_{\frac{1}{2}k \leqslant n < k} (-1)^n \frac{(n-1)\,!}{(k-n)\,!\,(2n-k)\,!}$$

Now if $k = 3p \pm 1$ the coefficient of x^k in $\sum\limits_{r=1}^{\infty} \frac{x^r}{r} - \sum\limits_{t=1}^{\infty} \frac{x^{3t}}{t}$ is the coefficient of x^k in $\sum\limits_{r=1}^{\infty} \frac{x^r}{r}$, i.e. $\frac{1}{k}$.

If $k = 3p$ there will be a term from the second series and the coefficient is $\frac{1}{k} - \frac{1}{\frac{1}{3}k} = -\frac{2}{k}$.

(4) *Show that if $e^x = 1 + xe^{yx}$, and x^3 and higher powers of x may be neglected,* $y = \frac{1}{2\,!} + \frac{x}{4\,!}$.

Now $e^x - 1 = x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots$ Thus the given equation is equivalent to

$$x\left(1 + \frac{x}{2\,!} + \frac{x^2}{3\,!} + \frac{x^3}{4\,!} + \ldots\right) = xe^{yx}.$$

If $x = 0$, the given equation is true for all y. Assume then that $x \neq 0$.

$$\text{Then } 1 + \frac{x}{2\,!} + \frac{x^2}{3\,!} + \frac{x^3}{4\,!} + \ldots = e^{yx}.$$

Taking logarithms to base e

$$\begin{aligned} xy &= \log\left\{1 + \frac{x}{2}\left(1 + \frac{x}{3} + \frac{x^2}{12} + \ldots\right)\right\} \\ &= \frac{x}{2}\left(1 + \frac{x}{3} + \frac{x^2}{12} + \ldots\right) - \frac{1}{2}\frac{x^2}{4}\left(1 + \frac{x}{3} + \frac{x^2}{12} + \ldots\right)^2 \\ &\quad + \frac{1}{3}\frac{x^3}{8}\left(1 + \frac{x}{3} + \frac{x^2}{12} + \ldots\right)^3 - \ldots \end{aligned}$$

$$= \frac{x^2}{2} + \frac{x^3}{24} + \text{terms in } x^4 \text{ and higher powers.}$$

Hence $y = \frac{1}{2\,!} + \frac{x}{4\,!} +$ terms in x^3 and higher powers.

Observe that in the exponential expansion it was necessary to take terms as for x^4 because of the factor x which is divided twice.

(5) *By using the fact that* $\left(1 + \frac{x}{n}\right)^n = e^{n \log_e (1 + x/n)}$, *prove that*

$$\left(1 + \frac{x}{n}\right)^n + \left(1 - \frac{x}{n}\right)^{-n} = 2e^x\left\{1 + \frac{1}{n^2}\left(\frac{x^3}{3} + \frac{x^4}{8}\right)\right\},$$

if $1/n^4$ *and higher powers of* $1/n$ *are neglected.* [*Lond. B.A.*]

$$\left(1 + \frac{x}{n}\right)^n = e^{n \log_e (1 + x/n)}$$

$$= e^{n\left(\frac{x}{n} - \frac{1}{2}\frac{x^2}{n^2} + \frac{1}{3}\frac{x^3}{n^3} - \frac{1}{4}\frac{x^4}{n^4} + \dots\right)}$$

$$= e^{x - \frac{1}{2}\frac{x^2}{n} + \frac{1}{3}\frac{x^3}{n^2} - \frac{1}{4}\frac{x^4}{n^3}} \quad \dots\dots\dots\dots \text{(i)}$$

where $1/n^4$ and higher powers of $1/n$ are neglected.

Similarly $\left(1 - \frac{x}{n}\right)^{-n} = e^{-n \log (1 - x/n)}$

$$= e^{n\left(\frac{x}{n} + \frac{1}{2}\frac{x^2}{n^2} + \frac{1}{3}\frac{x^3}{n^3} + \frac{1}{4}\frac{x^4}{n^4} + \dots\right)}$$

$$= e^{x + \frac{1}{2}\frac{x^2}{n} + \frac{1}{3}\frac{x^3}{n^2} + \frac{1}{4}\frac{x^4}{n^3}}.$$

It will be observed that this result may be deduced from (i) by changing n into $-n$.

$$\text{Hence } \left(1 + \frac{x}{n}\right)^n + \left(1 - \frac{x}{n}\right)^{-n} = e^{x + \frac{1}{3}\frac{x^3}{n^2}} . (e^{-u} + e^{u}),$$

where $u = \frac{1}{2}\frac{x^2}{n}\left(1 + \frac{1}{2}\frac{x^2}{n^2}\right)$.

Expanding in accordance with the exponential theorem and neglecting $1/n^4$ and higher powers it follows that

$$\left(1 + \frac{x}{n}\right)^n + \left(1 - \frac{x}{n}\right)^{-n} = 2e^x\left(1 + \frac{1}{8}\frac{x^4}{n^2}\right)\left(1 + \frac{1}{3}\frac{x^3}{n^2}\right) = 2e^x\left\{1 + \frac{1}{n^2}\left(\frac{x^4}{8} + \frac{x^3}{3}\right)\right\}.$$

(6) *If* x *and* y *are small, show that*

$$(1 + y)^x/(1 + x)^y = 1 - xy\,(y - x)/2$$

provided the ratio x/y *is finite, and that terms of the fourth and higher orders are neglected.* [*Lond. B.Sc.*]

Now $(1 + y)^x = e^{x \log (1 + y)} = e^{x\,(y - \frac{1}{2}y^2)}$,

where y^3 and higher powers of y are neglected.

Again $(1 + x)^y = e^{y \log (1 + x)} = e^{y\,(x - \frac{1}{2}x^2)}$,

where x^3 and higher powers of x are omitted.

$$\therefore\ (1+y)^x/(1+x)^y = e^{xy - \frac{1}{2}xy^2 - xy + \frac{1}{2}x^2y} = e^{\frac{1}{2}xy\,(x-y)}$$
$$= 1 + \tfrac{1}{2}xy\,(x-y) + \text{higher powers of } x, y$$
$$= 1 - \tfrac{1}{2}xy\,(y-x) \quad \text{as a first approximation.}$$

(7) *In a population of N persons, where N is large, one person out of p dies each year, one person out of q emigrates each year, and one out of r is born each year. Show that the population after s years is*

$$N\left(1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{r}\right)^s.$$

Find the number of years which elapse before the population doubles itself if $p = 46$, $r = 33$ and there is no emigration.

Let X_t be the population at the end of t years. Then in the $(t+1)$th year, X_t/p die, X_t/q emigrate and X_t/r are born. Thus at the end of the $(t+1)$th year the population is

$$X_t\left\{1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{r}\right\}.$$

Hence at the end of s years the population is

$$N\left\{1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{r}\right\}^s.$$

If $p = 46$, $r = 33$, there is no emigration and the population doubles itself in s years then

$$N\left\{1 - \tfrac{1}{46} + \tfrac{1}{33}\right\}^s = 2N, \text{ i.e. } \left(1 + \tfrac{13}{1518}\right)^s = 2.$$

Taking logarithms to base e, $s \log_e\left(1 + \frac{13}{1518}\right) = \log_e 2$.

Now $13/1518 = 0{\cdot}0085639\ldots$, $\frac{1}{2}(13/1518)^2 = 0{\cdot}0000367\ldots$

$$\log\left(1 + \tfrac{13}{1518}\right) = \tfrac{13}{1518} - \tfrac{1}{2}\cdot\left(\tfrac{13}{1518}\right)^2 + \tfrac{1}{3}\cdot\left(\tfrac{13}{1518}\right)^3 - \ldots$$

It is easily seen that if we work to four significant figures, the third and higher terms will not affect the fourth significant figure. Thus correct to four significant figures

$$s\,({\cdot}008527) = \log_e 2 = \log_{10} 2/\log_{10} e.$$

Taking $e = 2{\cdot}718$, $s = {\cdot}30103/({\cdot}43423 \times {\cdot}008527) = 81{\cdot}3$ approximately. Hence the population will be doubled in 82 years.

(8) *By using the expansion of*

$$\log\,[\{1-(b-c)x\}\{1-(c-a)x\}\{1-(a-b)x\}]$$

prove that

$$(b-c)^4 + (c-a)^4 + (a-b)^4 = 2\,(bc + ca + ab - a^2 - b^2 - c^2)^2.$$

[*Lond. B.Sc.*]

$$\log[\{1-(b-c)x\}\{1-(c-a)x\}\{1-(a-b)x\}]$$
$$= \log\{1-(b-c)x\} + \log\{1-(c-a)x\} + \log\{1-(a-b)x\}.$$

Each logarithm may be expanded by the logarithmic series provided x is small enough. Thus

$$-\log\{1-(b-c)x\} = \sum_{r=1}^{\infty} (b-c)^r x^r/r,$$

$$-\log\{1-(c-a)x\} = \sum_{r=1}^{\infty} (c-a)^r x^r/r,$$

$$-\log\{1-(a-b)x\} = \sum_{r=1}^{\infty} (a-b)^r x^r/r.$$

Each series is absolutely convergent for $|x|$ sufficiently small and consequently the three series when added together and *rearranged* in ascending powers of x converges to the original function. In this expansion the coefficient of x^4 is

$$\tfrac{1}{4}(b-c)^4 + \tfrac{1}{4}(c-a)^4 + \tfrac{1}{4}(a-b)^4.$$

Now $\{1-(b-c)x\}\{1-(c-a)x\}\{1-(a-b)x\} = 1 - \lambda x^2 - \mu x^3$
where $\lambda = a^2 + b^2 + c^2 - ab - bc - ca,\ \mu = (a-b)(b-c)(c-a).$

If x be sufficiently small, $\log(1-\lambda x^2 - \mu x^3)$ may be expanded by the logarithmic series. Thus

$$-\log(1-\lambda x^2-\mu x^3) = \sum_{r=1}^{\infty} x^{2r}(\lambda+\mu x)^r/r$$
$$= x^2(\lambda+\mu x) + \tfrac{1}{2}x^4(\lambda+\mu x)^2 + \dots$$

The coefficient of x^4 is $\frac{1}{2}\lambda^2$. Now the two expansions which have been obtained are power series which converge to the same function for $|x|$ sufficiently small. Hence the two powers series are identical and the corresponding coefficients equal (Chap. I., § 1·8). Thus

$$\tfrac{1}{2}\lambda^2 = \tfrac{1}{4}(b-c)^4 + \tfrac{1}{4}(c-a)^4 + \tfrac{1}{4}(a-b)^4,$$

giving the required result.

(9) *If* $a+b+c=0$, *prove that*

$$\tfrac{1}{5}(a^5+b^5+c^5) = \tfrac{1}{3}(a^3+b^3+c^3) \times \tfrac{1}{2}(a^2+b^2+c^2).$$

Now
$$-\log(1-ax) = \sum_{r=1}^{\infty}\frac{a^r x^r}{r},\quad |ax|<1.$$
$$-\log(1-bx) \quad \sum^{\infty} \quad \frac{b^r x^r}{} \quad |bx|<1.$$
$$-\log(1-cx) = \sum_{r=1}^{\infty}\frac{c^r x^r}{r},\quad |cx|<1.$$

Whatever the value of a, b, c it is clear that by taking x small enough, all the series will converge. Thus

$$-\log\{(1-ax)(1-bx)(1-cx)\} = \sum_{r=1}^{\infty}\frac{a^r+b^r+c^r}{r}x^r.$$

The coefficient of x^5 in the expansion is $\frac{1}{5}(a^5+b^5+c^5)$. Now

$$-\log\{(1-ax)(1-bx)(1-cx)\}$$
$$= -\log\{1-(a+b+c)x+(ab+bc+ca)x^2-abcx^3\}.$$
$$= -\log\{1-x^2(abcx-ab-bc-ca)\},$$

since $a+b+c=0$.

Again for sufficiently small values of x

$$-\log\{1-x^2(abc\,x-ab-bc-ca)\}$$
$$= x^2(abc\,x-ab-bc-ca) + \tfrac{1}{2}x^4(abc\,x-ab-bc-ca)^2 + \dots$$

The coefficient of x^5 in this expansion is $-abc(ab+bc+ca)$.

Since both expansions represent the same function for sufficiently small values of x,

$$- abc\,(ab + bc + ca) = \tfrac{1}{5}\,(a^5 + b^5 + c^5).$$

Since $a + b + c = 0$, we have

$$a^2 + b^2 + c^2 = -2\,(ab + bc + ca),$$
$$a^3 + b^3 + c^3 = 3\,abc.$$

Substitution gives $\tfrac{1}{2}\,(a^2 + b^2 + c^2) \times \tfrac{1}{3}\,(a^3 + b^3 + c^3) = \tfrac{1}{5}\,(a^5 + b^5 + c^5)$.

(10) *Prove that* $\log 2 = \dfrac{1}{1.2} + \dfrac{1}{3.4} + \dfrac{1}{5.6} + \dots$

Since the logarithmic series is valid for $x = 1$ (§ 5·5)

$$\begin{aligned}\log 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \dots \\ &= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \dots\end{aligned}$$

Note the groupings of the terms in this way has not altered the arrangement of the terms in the original series. The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}/n$ is only conditionally convergent and a rearrangement of the terms may produce a series with a different sum. The series obtained by grouping the terms is *absolutely* convergent, for all its terms are positive. Thus the two series are not equivalent, one being conditionally convergent and the other absolutely convergent, but they have the same sum.

(11) *Prove that*

$$\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \dots \textit{ to infinity} = \log_e 2 - \tfrac{1}{2}.$$

[*Camb. Sch.*]

$$\begin{aligned}2 \log 2 &= 2\,(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots) \\ &= 2 - \tfrac{2}{2} + \tfrac{2}{3} - \tfrac{2}{4} + \tfrac{2}{5} - \tfrac{2}{6} + \dots \\ &= 1 + 1 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{6} - \tfrac{1}{6} + \dots\end{aligned}$$

This is a new series which is different to that in the line above, but it is clear that they converge to the same value. Bracketing the terms without altering the given order.

$$\begin{aligned}2 \log 2 &= 1 + (1 - \tfrac{1}{2}) - (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) - (\tfrac{1}{4} - \tfrac{1}{5}) + (\tfrac{1}{5} - \tfrac{1}{6}) - \dots \\ &= 1 + \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \frac{1}{5.6} - \dots\end{aligned}$$

Bracketing the terms again but still keeping the same order we obtain

$$1 + \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{3.4} - \frac{1}{4.5}\right) + \left(\frac{1}{5.6} - \frac{1}{6.7}\right) + \dots$$

$$1 + \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7} + \dots$$

Dividing throughout by 2 we obtain the required result

(12) *Find the sum to infinity of the series*

$$\frac{1}{1.2.3}+\frac{5}{3.4.5}+\frac{9}{5.6.7}+\frac{13}{7.8.9}+\ldots$$

If u_n denote the nth term of the series

$$u_n=\frac{4n-3}{(2n-1)\,2n\,(2n+1)}=\frac{2(2n+1)-5}{(2n-1)\,2n\,(2n+1)}$$

$$=\frac{2}{(2n-1)\,2n}-\frac{5}{(2n-1)\,2n\,(2n+1)}.$$

$$\text{Hence}\quad\sum_{n=1}^{\infty}u_n=2\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n}-5\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n\,(2n+1)},$$

provided the series on the right converge. It is easily seen that they are both absolutely convergent. From Exx. 10, 11,

$$\log 2=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n},\quad \log 2-\tfrac{1}{2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n\,(2n+1)}.$$

$$\text{Hence}\quad\sum_{n=1}^{\infty}u_n=2\log 2-5\,(\log 2-\tfrac{1}{2})=\tfrac{5}{2}-3\log 2.$$

5·9. Construction of a Table of Common Logarithms

Using the series (2) of § 5·6 or one of the related series we can calculate the Napierian logarithm of a number, *i.e.* the logarithm to base e, to any desired degree of accuracy. Then to obtain logarithms to base 10 we would multiply by $1/\log_e 10$, *i.e.* by $\log_{10}e$. This number whose value is ·434,294, . . . , and which may be calculated by one of the series indicated is usually denoted by μ and is called the **modulus** of the common system.

In practice it is more convenient to proceed directly. Thus suppose we wish to construct a seven-figure logarithm table of numbers from 1 to 100,000. First of all we observe that it is sufficient to consider numbers from 10,000 to 100,000; for an earlier number will only differ from one of the later numbers in its characteristic. Thus, *e.g.* the mantissa for 53 will be the same as that for 53,000, that for 6394 will be the same as that for 63,940, and so on.

We take the expansion

$$\log\frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\},$$

what is known to be valid for $n>0$. (§ 5·6.) The error due to neglecting all terms except the first is now estimated, when n is not small.

$$2\left\{\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\}$$

$$<2\left\{\frac{1}{3(2n+1)^3}+\frac{1}{3(2n+1)^5}+\ldots\right\}$$

$$=\frac{2}{3}\frac{1}{(2n+1)^3}\Big/\left\{1-\frac{1}{(2n+1)^2}\right\}$$

$$=\frac{1}{6}\frac{1}{(2n+1)\,n\,(n+1)}<\frac{1}{10n^3}.$$

If $n \geqslant 10{,}000$ the error term is less than 10^{-13} and so cannot, *e.g.* affect the eighth decimal place. Thus if we retain only the first term of the series

$$\log_e\frac{n+1}{n}=\frac{2}{2n+1},\quad \text{i.e. } \log_{10}\frac{n+1}{n}=\frac{2\mu}{2n+1}$$

$$\text{or}\quad \log_{10}(n+1)=\log_{10}n+\frac{2\mu}{2n+1}.$$

Now $\log 10{,}000 = 4$ and hence

$$\log 10{,}001=\log 10{,}000+\frac{2\mu}{20{,}001}=4+\frac{2\mu}{20{,}001}$$

$$\log 10{,}002=\log 10{,}001+\frac{2\mu}{20{,}003},\ \text{and so on.}$$

5·91. Proof of the Method of Interpolation

We first show that if $h < N$ numerically, then

$$\log_{10}(N+h)-\log_{10}N=\frac{\mu h}{N}\left[1-\frac{1}{2}\frac{h}{N}+\frac{1}{3}\frac{h^2}{N^2}-\frac{1}{4}\frac{h^3}{N^3}+\ldots\right].$$

For $\log_{10}(N+h)-\log_{10}N=\log_{10}\left(1+\frac{h}{N}\right)=\mu\log_e\left(1+\frac{h}{N}\right)$

$$=\mu\left[\frac{h}{N}-\frac{1}{2}\frac{h^2}{N^2}+\frac{1}{3}\frac{h^3}{N^3}-\ldots\right].$$

Next we estimate the magnitude of the error obtained by neglecting all terms of the series after the first. Write $\left|\frac{h}{N}\right| = u$. Then the absolute value of the terms is less than

$$\tfrac{1}{2}\mu u^2\left[1+\tfrac{2}{3}u+\tfrac{2}{4}u^2+\ldots\right]<\tfrac{1}{2}\mu u^2\left[1+u+u^2+\ldots\right]$$

$$=\tfrac{1}{2}\mu u^2/(1-u).$$

If $u < \frac{1}{10}$, $\mu/(1 - u) < \cdot 45/\cdot 9$, i.e. $\frac{1}{2}\mu u^2/(1 - u) < \frac{1}{4}u^2$.
Hence the error in writing

$$\log_{10}(N + h) - \log_{10}N = \mu h/N$$

is less than $\frac{1}{4}h^2/N^2$, provided $|h/N| < \frac{1}{10}$.

Now suppose that $0 < h \leqslant 10$ and $N \geqslant 2000$. Then

$$\tfrac{1}{4}h^2/N^2 < \tfrac{1}{16}.10^{-4}.$$

and thus correct to four decimal places

$$\log_{10}(N + h) - \log_{10}N = \mu h/N.$$

If, on the other hand, we take $N \geqslant 10{,}000$ and $0 < h \leqslant 1$, then

$$\tfrac{1}{4}h^2/N^2 < \tfrac{1}{4}.10^{-8}$$

so that the equation would be true correct to eight places of decimals.

For practical purposes it is convenient to represent the result in slightly different form. If k is a number which satisfies the same inequality as h,

$$\log_{10}(N + k) - \log_{10}N = \mu k/N.$$

Then if $0 < h \leqslant 10$, $0 < k \leqslant 10$, $N \geqslant 2000$,

$$\frac{\log_{10}(N + h) - \log_{10}N}{\log_{10}(N + k) - \log_{10}N} = \frac{\mu h/N}{\mu k/N} = \frac{h}{k},$$

correct to four decimal places. Again, if

$$0 < h \leqslant 1, \quad 0 < k \leqslant 1, \quad n \geqslant 10{,}000,$$

$$\frac{\log_{10}(N + h) - \log_{10}N}{\log_{10}(N + k) - \log_{10}N} = \frac{h}{k}.$$

correct to seven places of decimals.

In particular, if we write $k = 1$, the equation becomes

$$\frac{\log_{10}(N + h) - \log_{10}N}{\log_{10}(N + 1) - \log_{10}N} = h.$$

Suppose that in a table of seven-figure logarithms it is known that $\log 10421 = 4{\cdot}0179094$, $\log 10422 = 4{\cdot}0179511$, and we require the logarithm of $10421{\cdot}31$. Write $h = 0{\cdot}31$ in the formula. Then

$$\frac{\log 10421{\cdot}31 - \log 10421}{\log 10422 - \log 10421} = 0{\cdot}31,$$

i.e. $\log 10421{\cdot}31 = 4{\cdot}0179094 + 0{\cdot}31 \times 0{\cdot}0000417$
$= 4{\cdot}0179223.$

Example.—*Given that* $\log_{10}2400 = 3\cdot3802$, *calculate the values of* $\log_{10}2403$, $\log_{10}2407$, *and* $\log_{10}2410$ *correct to four decimal places.* ($\mu = 0\cdot4343$.)

We know that if $N > 2000$, $0 \leqslant h \leqslant 10$,

$\log_{10}(N + h) = \log_{10}N + \mu h/N$, correct to four decimal places.

Put $N = 2400$, and $h = 3, 7, 10$ in succession. Then

$$\log_{10}2403 = 3\cdot3802 + 3 \times 0\cdot4343/2400 = 3\cdot3807.$$
$$\log_{10}2407 = 3\cdot3802 + 7 \times 0\cdot4343/2400 = 3\cdot3815.$$
$$\log_{10}2410 = 3\cdot3802 + 10 \times 0\cdot4343/2400 = 3\cdot3820.$$

The results could be calculated directly from the logarithmic series. Thus

$$\begin{aligned}\log_{10}2403 &= \log_{10}\{2400\,(1 + 3/2400)\}\\ &= \log_{10}2400 + \mu\log_e(1 + 3/2400)\\ &= 3\cdot3802 + \cdot4343\,\{\tfrac{3}{2400} - \tfrac{1}{2}(\tfrac{3}{2400})^2 + \ldots\}.\end{aligned}$$

Clearly the second term in the expansion and higher terms will not affect the fourth decimal place. Thus $\log_{10}2403 = 3\cdot3802 + 0\cdot4343 \times 3/2400$, as before.

EXERCISES V

1. Expand $e^{a + bx + cx^2}$ in ascending powers of x up to, and including the term in x^3. If the first three terms of the expansion are $p + qx + rx^2$, where p, q, r are supposed given, express a, b, c in terms of p, q, r. Hence find a, b, c so that $e^{a + bx + cx^2} = 2 + 3x + 4x^2 + \ldots$

2. If $f(t) = \dfrac{t}{e^t - 1} + \frac{1}{2}t$, prove that $f(t) = f(-t)$.

3. Prove that, if c_n be the coefficient of $(x + 1)^n$ in the expansion of $e^{x^2 + 2x}/(x^2 + 2x + 2)^3$ in a series of positive powers of $(x + 1)$, then $c_n = 0$ if n be odd, while

$$= \frac{1}{2e}\sum_{m=0}^{k}(-1)^m\frac{(m + 1)(m + 2)}{(k - m)!},\ (k = 0, 1, 2, \ldots).$$

[*Camb. Sch.*]

4. If x be very small, show that $e^{e^x} = e\,(1 + x)$ very nearly.

5. Show that the coefficient of x^n in the infinite series

$$1 + \frac{a + bx}{1!} + \frac{(a + bx)^2}{2!} + \ldots + \frac{(a + bx)^n}{n!} + \ldots \text{ is } e^a b^n/n!$$

6. Find the value correct to three decimal places of $\frac{1}{2}(e^x - e^{-x})$ when $x = 0\cdot5$.

7. Calculate the value of e^{-x^2} when $x = 0, x = 1, x = 2$, and $x = 3$; and sketch the graph of e^{-x^2} for all values of x, positive and negative. [Take $e = 2\cdot718$.]

8. Show that $\frac{1}{e} = \frac{1}{1.3} + \frac{1}{1.2.3.5} + \frac{1}{1.2.3.4.5.7} + \ldots$ to infinity.

9. Show that $e^{-1} = 2\left(\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots\right)$.

10. Sum to infinity $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots$

11. Prove that $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots = 2e$. [*Lond. B.A.*]

12. Sum to infinity $\frac{1.2}{1!} + \frac{2.3}{2!} + \frac{3.4}{3!} + \frac{4.5}{4!} + \ldots$

13. Sum to infinity $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots$

14. Given that $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ prove that $\cosh 2x = \cosh^2 x + \sinh^2 x$.

Show that $e^{x+x^2} = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \ldots$ and obtain the coefficient of x^4.

15. Show that the difference of the coefficients of x^n and x^{n-1} in the expansion of $e^x/(1 - x)$ is $1/n!$

16. Find the coefficient of x^4 in the expansion of $1/\{(1 + x)\, e^x\}$.

17. (*a*) Prove that $e^{x+x^2} = a_0 + a_1x + a_2x^2 + \ldots$, where

$$a_n = \frac{1}{n!0!} + \frac{1}{(n-2)!\,1!} + \frac{1}{(n-4)!\,2!} + \ldots,$$

the last summation extending over all terms $\frac{1}{(n-2r)!\,r!}$, such that $r \geqslant 0$, $n - 2r \geqslant 0$.

(*b*) By combining expansions of trigonometrical and exponential functions, or otherwise, sum the series

(i) $1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + \ldots$ (ii) $\frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \ldots$. [*Lond. B.Sc.*]

18. By equating the coefficients of x^n in the expansions of

$$(e^x - 1)^n = (x + x^2/2! + x^3/3! + \ldots)^n,$$

prove that $n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots = n!$

19. Find the coefficient of x^r in the expansion of

$$\log(1 + ax) + \log(1 + bx) + \log(1 + cx).$$

Deduce that if $a + b + c = 0$, when $2(bc + ca + ab)^2 = a^4 + b^4 + c^4$. [*Lond. B.Sc.*]

20. If $a = -\log(1 - \frac{1}{10})$, $b = -\log(1 - \frac{4}{100})$, $c = \log(1 + \frac{1}{80})$, show that $\log 2 = 7a - 2b + 3c$, $\log 5 = 16a - 4b + 7c$. Hence use the logarithmic series to calculate $\log_e 10$ to four significant figures. [*M.T.*]

21. Prove that, if x be positive, $e > x^{1/x}$.

22. Prove that $2\log_e m - \log_e(m + 1) - \log_e(m - 1)$

$$= 2\left\{\frac{1}{2m^2 - 1} + \frac{1}{3(2m^2 - 1)^3} + \frac{1}{5(2m^2 - 1)^5} + \ldots\right\}.$$

23. Prove that $\log_e (n + 1) - \log_e (n - 1) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots\right)$.

24. Given $\log_e 2 = 0{\cdot}69315$, $\log_e 5 = 1{\cdot}60944$, use the series

$$\log \frac{1 + x}{1 - x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right\}$$

to calculate $\log_e 7$ and $\log_{10} 7$ to five places of decimals.

25. Write down the expansion of $\log_e \left(1 + \frac{1}{n}\right)$ in ascending powers of $1/(2n + 1)$, and state the range of values of n for which the expansion is valid.

Prove that, if n is positive, the value of $\log_e\left(1 + \frac{1}{n}\right)$ lies between $(2n + 1)/2n\,(n + 1)$ and $2/(2n + 1)$. [*M.T.*]

26. When x is small, expand $\log (1 + 2x + 2x^2)$ in ascending powers of x up to the term containing x^3.

27. Prove that if $-1 <$ $+1$,

$$\log_e \left\{\frac{(1 + x)^2}{1 + x^2}\right\} = 2\left(x - x^2 + \frac{x^3}{3} + \frac{}{5} - \frac{}{3} + \dots\right).$$

By putting $x = 1/7$ calculate $\log_e 1{\cdot}28$ to four figures.

28. Expand $\log_e (1 + 4x + 3x^2)$ in ascending powers of x, and state for what values of x the expansion is valid. Find the value of the function when $x = \frac{1}{25}$ correct to three places of decimals.

29. Using the identity $\log 5 = 4 \log \frac{3}{2} - \log \frac{81}{80}$, calculate $\log_e 5$ correct to four places of decimals.

30. Use the series for e^x in ascending powers of x to calculate the 10th root of e correct to six places of decimals. Show that, if $\log_e x = 2/(2 + p)$, where p is small, x is given approximately by the equation $x = e\,(1 - \frac{1}{2}p)$.

31. Prove that

$$\tfrac{1}{3}x^3 \qquad 1 - x + \frac{}{} \qquad \frac{x^5}{20} \qquad \text{to } \infty.$$

32. Assuming the logarithmic series and the conditions for its convergency, obtain the expansion for $\log x$ in ascending powers of $(x - 1)/(x + 1)$, giving the range of values of x for which it is valid. Prove that for this range of values of x, the sum of the infinite series, whose rth term is

$$\frac{r}{4r^2 - 1}\left(\frac{x - 1}{x + 1}\right)^{2r-1}$$

is $[(x^2 + 1) \log_e x - x^2 + 1]/4\,(x - 1)^2$. [*Lond. B.Sc.*]

33. Given $\log_e 10 = 2{\cdot}30258509 \dots$, calculate $\log_e 101$ correct to seven decimal places.

34. Prove that $\log_e (1 + x)^{\frac{1}{2}(x + 1)} + \log_e (1 - x)^{\frac{1}{2}(1 - x)}$

$$\frac{}{1{\cdot}2} + \frac{}{3{\cdot}4} + \frac{}{5{\cdot}6} + \dots$$

35. If m and n be the roots of $x^2 + px + q = 0$, show that
$$\log_e(1 - px + qx^2) = (m + n)x - \tfrac{1}{2}(m^2 + n^2)x^2 + \tfrac{1}{3}(m^3 + n^3)x^3 - \ldots$$

36. If $a = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$, prove that
$$x = a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots$$

37. Expand $\log_e\left(\dfrac{1 + x + x^2}{1 - x + x^2}\right)$ in a series of ascending powers of x.

38. Assuming the expansion of $\log_e(1 + x)$ prove that
$$\log_e \frac{p}{q} = 2\left\{\frac{p - q}{p + q} + \frac{1}{3}\left(\frac{p - q}{p + q}\right)^3 + \frac{1}{5}\left(\frac{p - q}{p + q}\right)^5 + \ldots\right\}.$$
It being given that $\log_e 2 = 0{\cdot}69315$, obtain the values of $\log_e 3$ and $\log_{10}$ correct to four places of decimals.

39. Show that, when x is positive and less than unity, $-\log_e(1 - x)$ exceeds $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots + x^n/n$ by less than $\dfrac{x^{n+1}}{n + 1}(1 - x)^{-1}$.

40. What is the least number of terms of the series
$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$
necessary to give an approximation to e correct to seven decimal places Give reasons.

41. Expand $e^{(1 + x)\log(1 + x)}$, where $|x| < 1$, in ascending powers of x as far as the term containing x^4. If x is small enough for x^5 and higher powers to be neglected, show that to this approximation
$$(1 + x)^{1+x} = 1 + x - x\log_e(1 - x). \qquad [\textit{Lond. B.Sc.}$$

42. If $\log_e \dfrac{(2x + 1)^{\frac{3}{2}}}{3^{\frac{3}{2}}x}$ be expanded in ascending powers of $x - 1$, prove that the first term in the expansion will be $\frac{1}{3}(x - 1)^2$.

43. Show that $\log \dfrac{4}{e} = \dfrac{1}{1.2} - \dfrac{1}{2.3} + \dfrac{1}{3.4} - \dfrac{1}{4.5} + \ldots$

44. Sum to infinity $\dfrac{1}{2.3} + \dfrac{1}{4.5} + \dfrac{1}{6.7} + \ldots$

45. Sum to infinity $\dfrac{1}{2.3.4} + \dfrac{1}{4.5.6} + \dfrac{1}{6.7.8} + \ldots$

46. A sum of money £P is invested at r per cent. per annum compound interest. Find an expression for the amount after n years if interest is added quarterly; in how many years will the principal double itself if the rate is $3\frac{1}{2}$ per cent. per annum.

47. It is given that y is the positive value of $(1 + x + x^2)^{x^{-2}}$. By means of the expansions of $\log_e(1 + x)$ and e^x, prove that when x is small,

$$y = (1 - \tfrac{2}{3}x \ldots)e^{\frac{1}{x} + \frac{1}{2}} \qquad [M.T.]$$

48. Show that $\left(1 + \dfrac{1}{x}\right) \log_e (1 + x) - 1 = \dfrac{x}{1.2} - \dfrac{x^2}{2.3} + \dfrac{x^3}{3\cdot 4} - \dfrac{x^4}{4\cdot 5} + \ldots$

49. Prove that the coefficient of x^n in the expansion of $\{\log_e (1 + x)\}^2$ is

$$2\frac{(-1)^n}{n}\left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n-1}\right].$$

50. Show, by equating the coefficients of x^n in the expansions of $2 \log_e(1 - x)$ and $\log_e(1 - 2x + x^2)$, that

$$-n.2^{n-2} + \frac{n(n-3)}{1.2}2^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}2^{n-6} + \ldots$$

51. Show that $\quad \frac{1}{7}(a^7 + b^7 + c^7) = \frac{1}{5}(a^5 + b^5 + c^5) \times \frac{1}{2}(a^2 + b^2 + c^2)$ if $a + b + c = 0$.

52. Prove that, when n is large, $\left(1 + \dfrac{1}{n}\right)^{n + \frac{1}{2}} = e^{1 + \frac{1}{12n^2}}$ approx.

53. Prove, by taking the logarithms of both the sides, that

$$\sqrt[x]{\frac{1+x}{1-x}} > \sqrt[y]{\frac{1+y}{1-y}},$$ if x and y are proper fractions and $x > y$.

54. Find the value when x tends to the limit 1, of the expression

$$\log (x^{\frac{5}{3}} - 1) - \log (x^{\frac{3}{2}} - 1).$$

55. Show that $\quad \lim\limits_{x \to 0} \left\{\dfrac{e^x - 1 - \log (1 + x)}{x^2}\right\} = 1.$

56. Find the limit as $x \to 0$ of

$$\{\log (1 + \tfrac{1}{3}x) - (1 + x)^{\frac{1}{3}} + 1\}/x^2.$$

57. If $n > 1$, show that

$$\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots$$

58. Evaluate $\lim\limits_{x \to a} \dfrac{a^n - x^n}{\log_{10}a - \log_{10}x}$. [Camb. Sch.]

59. Prove that the limiting value of $(a^h - 1)/h$, as h tends to 0, is $\log_e a$. By putting $(1 + x)$ for a, deduce the expansion of $\log_e (1 + x)$ in powers of x, stating between what values of x it holds true. Assuming this expansion, and being given that $\log_e 10 = 2{\cdot}30258$, find $\log_e 11$ to 4 decimal places.

60. Draw carefully the graph of $y = \log_{10}(1 + 2x^2)$ for values of x between 0 and 5. Hence find an approximate value for the positive root of the equation

$$\log_e (1 + 2x^2) = 2x - 3, \quad (\log_{10}e = \cdot 4343).$$

By using the series for $\log_e (1 + z)$ or by any other method, obtain from the approximate value found from the graph a closer approximation to the root of the equation. [*Camb. Sch.*]

61. Show that $\sum\limits_{r=1}^{2n} (-1)^{r-1}r^{-1} < \log_e 2 < \sum\limits_{r=1}^{2n+1} (-1)^{r-1}r^{-1}$. [*M.T.*]

62. Find the sum of the infinite series whose nth term is $(n + 1)x^n/n$, x being numerically less than unity.

63. Prove that $\dfrac{1}{3!} + \dfrac{2}{5!} + \dfrac{3}{7!} + \ldots$ to $\infty = \dfrac{1}{2e}$.

If $s_r = e^{rx} + e^{-rx}$, express $(e^x + e^{-x})^n$ in terms of s_n, s_{n-2}, etc.

64. Prove that, if n is large,

$$\left(n - \frac{1}{3n}\right) \log \frac{n+1}{n-1} = 2 + \frac{8}{45n^4} + \ldots$$

and $$\left(\frac{n+1}{n-1}\right)^{n - \frac{1}{3n}} = e^{2\left(1 + \frac{8}{45n^4} + \ldots\right)}.$$ [*Lond. B.Sc.*]

65. If p is small, so that p^3 is negligible, prove that an approximation to a solution of the equation $x^{2+p} = a^2$ is

$$x = a - \tfrac{1}{2}ap \log_e a + \tfrac{1}{8}ap^2 (2 + \log_e a) \log_e a.$$ [*Camb. Sch.*]

66. Show that, if $1 > x > 0$ the remainder after two terms in the expansion of $\log_e\{1/(1 - x)\}$ is less than $x^3/3(1 - x)$ and greater than $x^3/3(1 - \frac{3}{4}x)$. [*Lond. B.Sc.*]

67. If a, b, c, d are four real quantities whose sum is zero, show that

$$\frac{a^5 + b^5 + c^5 + d^5}{5} = \frac{a^3 + b^3 + c^3 + d^3}{3} \cdot \frac{a^2 + b^2 + c^2 + d^2}{2}.$$

If d is zero, show further that

$$\frac{a^7 + b^7 + c^7}{7} = \frac{a^5 + b^5 + c^5}{5} \cdot \frac{a^2 + b^2 + c^2}{2} = \frac{a^4 + b^4 + c^4}{2} \cdot \frac{a^3 + b^3 + c^3}{3}.$$

[*Camb. Sch.*]

68. Taking the identity $(1 - ax)(1 - bx) = 1 - px + qx^2$, where $p = a + b, q = ab$, expand the logarithms of both sides of this identity in powers of x as far as x^4, and hence write down the values of $a^3 + b^3$, and $a^4 + b^4$, in terms of p, q.

69. From the identity $2 \log (1 - x) = \log (1 - 2x + x^2)$, prove that

$$2^n - n.2^{n-2} + \frac{n(n-3)}{1.2} 2^{n-4} - \frac{n(n-4)(n-5)}{1.2.3} 2^{n-6} + \ldots = 2.$$

[*Camb. Sch.*]

70. Find the coefficient of x^7 in the expansion of $\log (1 + 6x + 11x^2 + 6x^3)$. [*N.Sc.*]

CHAPTER VI

FUNCTIONS FROM EMPIRICAL DATA

IN this chapter we first elaborate some properties of a linear graph. Then we consider functions which by suitable transformations of the variables may be reduced to linear functions. These methods are then applied to determine functions which fit certain empirical results.

6·1. The Equation of a Straight Line

It is known that the equation

$$ax + by = c \quad \text{.......................(i)}$$

represents a straight line. The letters a, b, c represent constants, x and y are the variables, each variable occurring to the first degree only. We may regard x as the independent variable and y as the dependent variable, or vice versa.

If $a = 0$, the equation reduces to $y = c/b$. This represents a straight line which is parallel to the x-axis and distant c/b from it. If $c/b > 0$ the line is above the x-axis, while if $c/b < 0$ the line is below it. If in addition $c = 0$, then the line becomes $y = 0$, i.e. the x-axis.

Again, if $b = 0$, (i) becomes $x = c/a$. This represents a line parallel to the y-axis and distant c/a from it, the line being to the right or left of the y-axis according as c/a is positive or negative. If in addition $c = 0$, the line becomes $x = 0$, i.e. the y-axis.

Now suppose that a, b are different from zero. Then if $c = 0$, the equation is satisfied by the coordinates $(0, 0)$, *i.e.* the line passes through the origin.

Suppose then that $c \neq 0$ and let the line cut the x-axis in A, the y-axis in B (Fig. 10). Suppose further, that the line makes an angle ψ with *the positive direction* of the x-axis, where $0 \leqslant \psi < \pi$.

To find the coordinates of A, B, put $y = 0$, $x = 0$ in succession in (i). Then when $y = 0$, $x = c/a$ and when $x = 0$, $y = c/b$. Thus A is the point $(c/a, 0)$, B the point $(0, c/b)$.

If $c/a > 0$, A lies to the right of the origin O, while $c/a < 0$, A is to the left as in Fig. 10. Similarly B is above or below O according as $c/b > 0$ or < 0. In Fig. 10, $c/b > 0$. OA and OB

are called the **intercepts** on the x-axis and y-axis respectively, and may be positive or negative.

If $\psi = 0$, the line will not meet the x-axis at a finite distance unless it coincides with it entirely. It will be observed that $\psi = 0$ corresponds to $a = 0$.

If $\psi = \frac{1}{2}\pi$ then the line will not meet the y-axis at a finite point or else coincides with it entirely. This value of ψ corresponds to $b = 0$.

Next, suppose that $0 < \psi < \frac{1}{2}\pi$. Then ψ is an acute angle and $\tan\psi > 0$. If $\frac{1}{2}\pi < \psi < \pi$, ψ is an obtuse angle and $\tan\psi$ is negative. The quantity $\tan\psi$ is defined to be the **gradient of the line.**

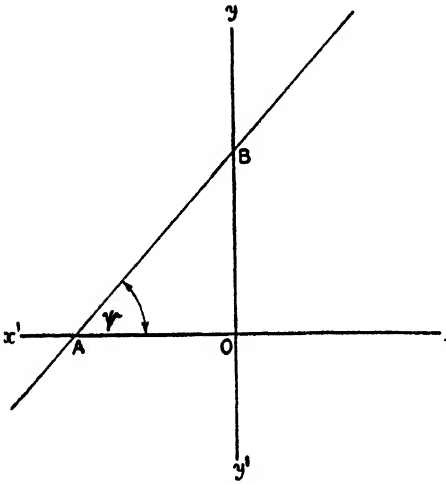


FIG. 10.

As ψ ranges from 0 to π, $\tan\psi$ ranges from 0 to $+\infty$ and from $-\infty$ to 0, so that the gradient may have any value between $+\infty$ and $-\infty$. The gradient is always finite except in the case in which the line is parallel to the y-axis.

Whether ψ is acute or obtuse, it is easily seen that

$$\tan\psi = \frac{OB}{AO} = \frac{OB}{OA}$$

$$= -\frac{c/b}{c/a} = -\frac{a}{b}.$$

Thus if the equation of the line is given in the form $ax + by = c$ its gradient is $-a/b$.

Now if $b \neq 0$ the equation may be written in the form

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

Thus if m denote the gradient, k the intercept on the y-axis,

$$y = mx + k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Note.—In the discussion given above it has been assumed that the *scales of representation* for x and y are *equal.* If this condition is *not* satisfied the angle the line makes with the x-axis will depend on the scales of representation and will not be the angle ψ

considered above. The scales of representation used do not affect the equation of the line, in particular also the gradient and the intercepts on the axes. The distances from the origin to the points where the line cuts the axes will always represent the intercepts when interpreted in accordance with the scales used. A similar remark applies to the gradient.

It will be observed that the equation of any straight line contains essentially only two arbitrary constants—the third apparent constant in (i) being eliminated by division to obtain the two constants m, k in (ii). By giving m and k all possible values we would obtain all the lines in the plane considered. It follows that a straight line will be completely determined by two conditions. Thus, *e.g.* we may determine its equation if we know (*a*) two points on it, or (*b*) one point on it and the gradient.

Consider (*a*) and let the two points be (x_1, y_1), (x_2, y_2).

From (ii) it is easily seen that the equation of the line may be written in the form

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Now consider (*b*): in this case m and (x_1, y_1) are given and the equation of the line can then be written in the form

$$y - y_1 = m(x - x_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

It should be observed that no matter what the scales of representation are, the above results are true provided (x_1, y_1) and (x_2, y_2) are the actual coordinates of two points on the line and are thus independent of the scales of representation.

6·10. Determination of the Constants in the Equation of a Straight Line, from the Graph of the Line

There are several ways in which this may be done.

(*a*) The intercept in the y-axis can be measured giving k directly. The length has to be interpreted in accordance with the scale used to represent y. Thus, *e.g.* suppose that the actual length measured from the origin to the point where the line cuts the y-axis is 1·6 inches and that the scale of representation for y is 1 in. represents 2 units. Then the intercept is $1{\cdot}6 \times 2 = 3{\cdot}2$ units. Thus $k = 3{\cdot}2$.

In determining the gradient we are concerned only with the tangent of an angle and not with the angle itself. Thus it is best to measure m by taking the coordinates of two points on the line, at a convenient distance apart and then calculating m by means

of the relation $m = (y_2 - y_1)/(x_2 - x_1)$. If the angle itself should be wanted, it is best to calculate m first and then determine ψ from trigonometrical tables.

(*b*) The intercept on the x-axis, i.e. $-k/m$ can be measured directly. Thus, *e.g.* if the length measured from the origin to the point where the line cuts the x-axis is $-3{\cdot}5$ in. and the scale of representation for x is 1 in. represents $1{\cdot}5$ units, then the intercept is $-3{\cdot}5 \times 1{\cdot}5 = -5{\cdot}25$. Hence $-k/m = -5{\cdot}25$. Then m can be determined as in (*a*) and so k may be calculated.

(*c*) m and k can both be calculated by taking two points (x_1, y_1) (x_2, y_2) on the line and using the relations

$$m = (y_2 - y_1)/(x_2 - x_1), \quad k = (x_2 y_1 - x_1 y_2)/(x_2 - x_1).$$

In the majority of cases it will be found that (*c*) is the most convenient method. Now in order to obtain accurate results the factor $(x_2 - x_1)$ in the denominators should be taken as large as is convenient, i.e. *the two points* (x_1, y_1), (x_2, y_2) *should be chosen as far apart as possible.*

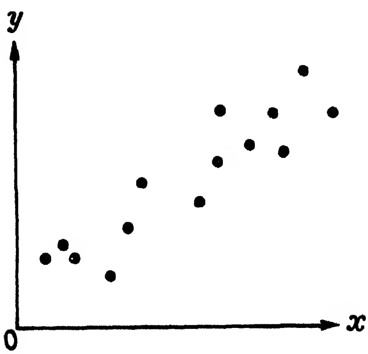


FIG. 11.

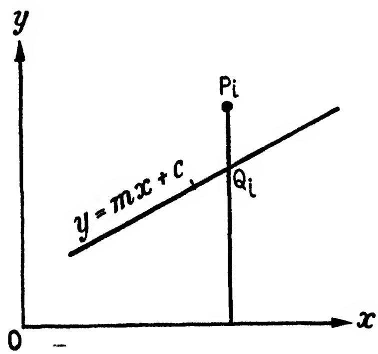


FIG. 12.

6·11. Method of Least Squares

Suppose we are given a set of points in a plane, *i.e.* a *scatter diagram* as in Fig. 11, and we wish to determine by calculation, the line of "best" fit, *i.e.* the straight line which gives the "best" approximation to the set of points. To make the problem explicit, suppose we are given a set of N points. Take an arbitrary point O as origin in the plane with rectangular axes Ox, Oy and let the set

of points be $P_i\{x_i, y_i\}$, $i = 1, 2, \ldots, N$. The problem is to find an equation $y = mx + c$, *i.e.* determine constants m, c such that the equation gives for each value of x, the best average value of y. Take $Q_i(x_i, Y_i)$, Fig. 12, to be the point on $y = mx + c$ corresponding to $P_i(x_i, y_i)$. Then the line of "best" approximation is interpreted in accordance with the principle of the method of least squares, *i.e.* the m, c are to be chosen so that

$$\sum_{i=1}^{N} P_iQ_i^2 = \sum_{i=1}^{N} (y_i - Y_i)^2 = \sum_{i=1}^{N} (y_i - mx_i - c)^2$$

is a minimum. The condition for this is that the partial derivatives with respect to m and c will both be zero. Hence the equations for determining m and c are

$$\frac{\partial}{\partial m} \Sigma (y_i - mx_i - c)^2 = 0, \quad \frac{\partial}{\partial c} \Sigma (y_i - mx_i - c)^2 = 0,$$

or $$\Sigma x_i (y_i - mx_i - c) = 0, \quad \Sigma (y_i - mx_i - c) = 0.$$

Write $\Sigma x_i = N\bar{x}$, $\Sigma y_i = N\bar{y}$ so that $\bar{x}$, $\bar{y}$ are the means of the values of x_i, y_i respectively.

The equation $$\Sigma (y_i - mx_i - c) = 0$$
may be written
$$\Sigma y_i - m\Sigma x_i - Nc = 0,$$
or $$\bar{y} = m\bar{x} + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$
Hence the straight line required passes through $G(\bar{x}, \bar{y})$ the mean of the given set $\{x_i, y_i\}$.

Now transfer the origin from O to G by making the transformation
$$x_i = \bar{x} + \xi_i, \quad y_i = \bar{y} + \eta_i.$$
The equation
$$\Sigma x_i (y_i - mx_i - c) = 0$$
then becomes
$$\begin{aligned} 0 &= \Sigma [(\bar{x} + \xi_i)(\bar{y} + \eta_i - m\bar{x} - m\xi_i - c)] \\ &= \Sigma [\xi_i (\bar{y} - 2m\bar{x} - c) + \eta_i\bar{x} + (\bar{x}\bar{y} - m\bar{x}^2 - \bar{x}c) - m\xi_i^2 + \xi_i\eta_i] \\ &= (\bar{y} - 2m\bar{x} - c)\Sigma\xi_i + \bar{x}\Sigma\eta_i + N(\bar{x}\bar{y} - m\bar{x}^2 - \bar{x}c) - m\Sigma\xi_i^2 + \Sigma\xi_i\eta_i. \end{aligned}$$
Since the new origin is the mean of the values of (ξ_i, η_i),
$$\Sigma\xi_i = 0, \quad \Sigma\eta_i = 0.$$

Write $$\sigma_x^2 = \frac{1}{N}\Sigma\xi_i^2, \quad \mu = \frac{1}{N}\Sigma\xi_i\eta_i.$$

Then the condition becomes

$$\bar{x}\bar{y} - m\bar{x}^2 - c\bar{x} + \mu - \sigma_x^2 = 0 \ldots\ldots\ldots\ldots \text{(ii)}$$

Observe that σ_x^2 is the sum of the squares of the deviation of the values of x from the mean of x and that μ is the sum of the products of the deviations of x and y from the corresponding means. Every individual term in σ_x^2 is positive or zero whereas the terms in μ may be positive, zero, or negative.

Equations (i) and (ii) then determine m and c.

Eliminating c between them, $m = \mu/\sigma_x^2$.

Hence the equation of the required line is

$$y - \bar{y} = \frac{\mu}{\sigma_x^2}(x - \bar{x}).$$

In practice the only problem in determining the constants $\bar{x}$, $\bar{y}$, σ_x^2, μ is the tedious arithmetic which may be involved. The technique of an assumed mean has been evolved to reduce the arithmetic and details may be found in standard books on statistics.

In the above calculation we have assumed that the value of x is accurate and determined a best possible value of y by expressing the condition that the sum of the squares of the deviations for y shall be a minimum. The straight line obtained is called the *line of regression of y on x.*

Alternatively we can approach the problem by assuming that the values of y are accurate and calculate the best possible value of x using the condition that the sum of squares of the deviations for x shall be a minimum. It follows from what has already been done that the equation in this case would be

$$x - \bar{x} = \frac{\mu}{\sigma_y^2}(y - \bar{y}),$$

where $$\sigma_y^2 = \frac{\Sigma\eta_i^2}{N}.$$

This second line is called the *line of regression of x on y.* It is, in general, *different* from the first line but *both* lines pass through the mean $(\bar{x}, \bar{y})$.

The following example illustrates the method of calculation involved in the determination of the best straight line.

Example.—*Use the method of least squares to find a linear relation between the two variables x and y which are found experimentally to be related as set out in the following table.*

$x =$	1	2	3	4
$y =$	0·4	0·7	0·9	1·4
$x =$	5	6	7	8
$y =$	1·7	1·9	2·3	2·7

The table on the right is arranged so that both lines of regression can be calculated. If only y on x is required then the η^2 column can be omitted, if only x on y is required the ξ^2 column can be omitted.

x	y	$\xi = x - \bar{x}$	$\eta = y - \bar{y}$	ξ^2	η^2	$\xi\eta$
1	0·4	−3·5	−1·1	12·25	1·21	+3·85
2	0·7	−2·5	−0·8	6·25	0·64	2·00
3	0·9	−1·5	−0·6	2·25	0·36	0·90
4	1·4	−0·5	−0·1	0·25	0·01	0·05
5	1·7	+0·5	+0·2	0·25	0·04	0·10
6	1·9	1·5	0·4	2·25	0·16	0·60
7	2·3	2·5	0·8	6·25	0·64	2·00
8	2·7	3·5	1·2	12·25	1·44	4·25
$\Sigma x = 36$	$\Sigma y = 12·0$			$\Sigma\xi^2 = 42·0$	$\Sigma\eta^2 = 4·50$	$\Sigma\xi\eta = 13·75$

Number of terms (or total frequency) = 8.

$$\bar{x} = \frac{36}{8} = 4·5; \quad \bar{y} = \frac{12·0}{8} = 1·5.$$

$$\sigma_x^2 = \frac{42·0}{8} = 5·25;$$

$$\sigma_y^2 = \frac{4·50}{8} \simeq 0·56;$$

$$\mu = \frac{13·75}{8} \simeq 1·72.$$

The line of regression of y on x is

$$y - \bar{y} = \frac{\mu}{\sigma_x^2}(x - \bar{x}), \text{ i.e. } y - 1·5 = 0·33\,(x - 4·5).$$

The line of regression of x on y is

$$x - \bar{x} = \frac{\mu}{\sigma_y^2}(y - \bar{y}),$$

$$\text{or} \quad y - \bar{y} = \frac{\sigma_y^2}{\mu}(x - \bar{x}),$$

$$\text{i.e. } y - 1·5 = 0·33\,(x - 4·5).$$

In this particular example, the two equations are the same, correct to the order of accuracy considered.

Observe that in the above calculation a different criterium for line of "best" fit has been used for each line, but that in both, all observations have been given the same weight. It is important to realise that other criteria for line of "best" fit could be postulated.

6·12. The Gradient at a Point of a Non-linear Graph

Let P be any point on a curve, PT the tangent to the curve at P. Then the gradient of the tangent PT is defined to be the **gradient of the curve** at P.

(*a*) Suppose, first, that the equation which represents the curve is known, and let it be expressed in the form $y = f(x)$. The gradient at any point is then found by differentiation.

Let (x, y) be the coordinates of a point P on the curve whose equation is $y = f(x)$. PT is the tangent at P meeting the x axis in T, Q a point on the curve near P, its coordinates being $(x + \delta x, y + \delta y)$. (Fig. 13.) Suppose that the chord PQ meets Ox in S, that M and N are the feet of the perpendiculars from P and Q to Ox respectively, that PT meets QN in U and that R is the foot of the perpendicular from P to QN. Then $PR = \delta x$, $RQ = \delta y$.

Write $\angle NTP = \psi$, $\angle RPQ = \theta$. Then

$$\tan \theta = \frac{RQ}{PR} = \frac{\delta y}{\delta x}.$$

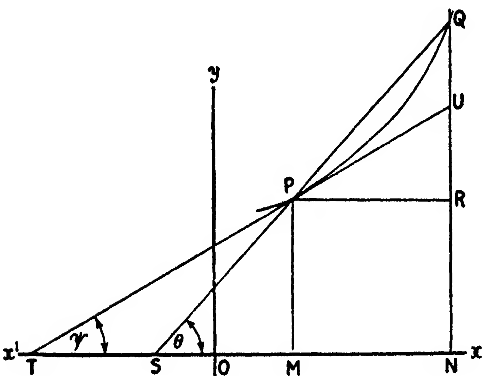


FIG. 13.

The gradient of the curve is

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\theta \to \psi} \tan \theta = \tan \psi.$$

The gradient at (x_0, y_0) is $f'(x_0)$ and the tangent at this point is

$$y - y_0 = f'(x_0)(x - x_0).$$

(*b*) Suppose next that the curve is given but that its equation is not known. Then we may determine the gradient approximately at any point P by drawing the tangent to the curve at P and then by taking two points on the tangent calculate the gradient.

Example.—*A certain function of x is equal to ax^2 for values of x less than 1, and to $bx - ax^2 - 1$ for values of x greater than 1. Find the values of the*

constants a and b in order that there may be no discontinuity or abrupt change of slope in the graph of the function at $x = 1$.

With these values of a and b, find the values of x for which the function is zero. [*Camb. Sch.*]

The function and its differential coefficient are to be continuous at $x = 1$. Differentiating we see that the gradient of the graph is $2ax$ or $b - 2ax$ according to the equation taken. If they are to be the same at $x = 1$,

$$4a = b \qquad \text{(i)}$$

In order that the curves may be continuous at $x = 1$,

$$a = b - a - 1, \text{ i.e. } 2a = b - 1. \qquad \text{(ii)}$$

From (i) and (ii) $b = 2$, $a = \frac{1}{2}$. Thus $y = \frac{1}{2}x^2$, $x \leqslant 1$;

$$\therefore\ y = 2x - \tfrac{1}{2}x^2 - 1,\ x \geqslant 1.$$

Hence the function vanishes when $x = 0$ and when $4x - x^2 - 2 = 0$, i.e. $x = 2 \pm \sqrt{2}$. Since $2 - \sqrt{2} < 1$ the value $x = 2 - \sqrt{2}$ must be excluded. Thus function vanishes when $x = 0$ and when $x = 2 + \sqrt{2}$.

6·2. Functions from Empirical Data

Suppose that we have two variables, x and y, which are connected by a relation which can be represented in the symbolic form $f(x, y) = 0$. Then it may be possible from theoretical considerations to find the form of the relation except for certain constants, say $a_1, a_2, a_3, \ldots, a_n$. Then as the function depends on these constants the relation may be represented more conveniently in the form

$$f(x, y, a_1, a_2, \ldots, a_n) = 0.$$

In order to determine these constants we require n values of x, y. In this way we would obtain n equations involving the n unknowns, $a_1, a_2, \ldots, a_n$.

As the values of x and y are determined by observation and thus subject to experimental error, *more than n* values should be obtained. Taking these values y should be plotted against x and a curve as smooth as possible should be drawn among the points, *i.e.* the curve should be drawn in such a way that the plotted points are distributed equally as far as possible, on both sides of the curve. Then take n points on the curve drawn at convenient distances apart and in this way obtain n equations for the n unknowns.

The precise determination of these n unknown constants and the facility with which this can be done will depend upon the form of the equation, the number of unknowns involved and the way in which they occur in the equation. If the equation is linear in the constants the values can always be found as accurately as is desired.

But if irrational and transcendental functions of the constants occur then the solution of the equations may offer serious difficulty.

There are certain special cases which occur frequently and which can be treated simply. Some of these are indicated below

6·21. Two Constants Involved Linearly

Let $f(x, y)$, $\phi(x, y)$, $\psi(x, y)$ be three functions of x and y. Then the type of equation contemplated can be expressed in the form

$$a_1 f(x, y) + a_2 \phi(x, y) + \psi(x, y) = 0,$$

and this may be written as

$$\frac{\psi(x, y)}{f(x, y)} = - \frac{a_2 \phi(x, y)}{f(x, y)} - a_1.$$

Writing $v = \psi(x, y)/f(x, y)$, $u = \phi(x, y)/f(x, y)$, m $k = - a_1$, this equation becomes

$$v = mu + k,$$

which is the standard linear form for the variables u, v. Using the method of least squares, or a graphical method, we can determine our line of best fit which expresses the linear relation between u and v. Hence m and k, and so a_1 and a_2 can be found.

Examples.—(1) Given $\frac{1}{y} = ax^2 + b$. Then writing $u = x^2$, $v = 1/y$ the equation takes the form $v = au + b$, which is linear in u and v.

(2) $y = ax^n/(c + bx^n)$ where c and n are known. Then

$$\frac{1}{y} = \frac{c + bx^n}{ax^n} = \frac{c}{a} x^{-n} + \frac{b}{a}.$$

Writing $cx^{-n} = u$, $1/y = v$ the equation takes the linear form

$$v = \frac{1}{a} u + \frac{b}{a}.$$

(3) $y = a \cos x + b \sin x + \sec x$. Dividing both sides of the equation by $\cos x$.

$$y \sec x - \sec^2 x = a + b \tan x, \text{ i.e. } v = a + bu$$

where $u = \tan x$, $v = y \sec x - \sec^2 x$.

6·211. Numerical Examples

(1) *It is suspected that two variable quantities denoted by x and y are connected by a relation of the form $y = ax/(1 + bx)$, where a and b are unknown constants, and the following experimental values of x and y are given:*

x	1	2	4	6	8	9	10
y	0·39	0·668	1·04	1·28	1·44	1·5	1·55

Verify the form of the relation, by constructing a suitable linear graph, and deduce the probable values of a and b.

The relation $y = ax/(1 + bx)$ may be written in the form

$$\frac{1}{y} = \frac{1}{a}\cdot\frac{1}{x} + \frac{b}{a}.$$

Writing $1/y = v$, $1/x = u$, the equation takes the linear form

$$v = \frac{1}{a}u + \frac{b}{a}.$$

We first construct a table of values of u and v.

x	1	2	4	6	8	9	10
y	0·39	0·668	1·04	1·28	1·44	1·5	1·55
u	1	0·5	0·25	0·167	0·125	0·111	0·10
v	2·564	1·497	0·962	0·781	0·694	0·667	0·645

The plotted points are found to lie very approximately along a straight line AB which is drawn as evenly as possible between the points. A, B are the points (0, 0·433) and (1, 2·564). The intercept OA on the v-axis is 0·433 and the gradient of AB is $(2·564 - 0·433)/(1 - 0) = 2·131$. Thus $1/a = 2·131$, $b/a = 0·433$. These equations give

$$a = 0·469,\ b = 0·203 \text{ approx.} \quad \text{(See Fig. 14.)}$$

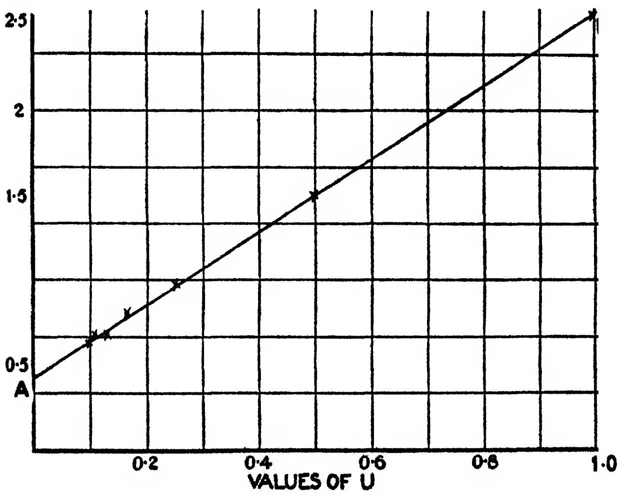


FIG 14.

(2) *Two quantities x and y are measured experimentally and the following values obtained.*

x	8	10	12	14	16	18	20
y	32	60	82	108	138	172	220

It is expected that they are connected by a law of the form $y = a + bx^2$. Test if this is so and find the probable values of a and b.

Which values of y have probably been misread, and what should be their correct values? [*N. Sc.*]

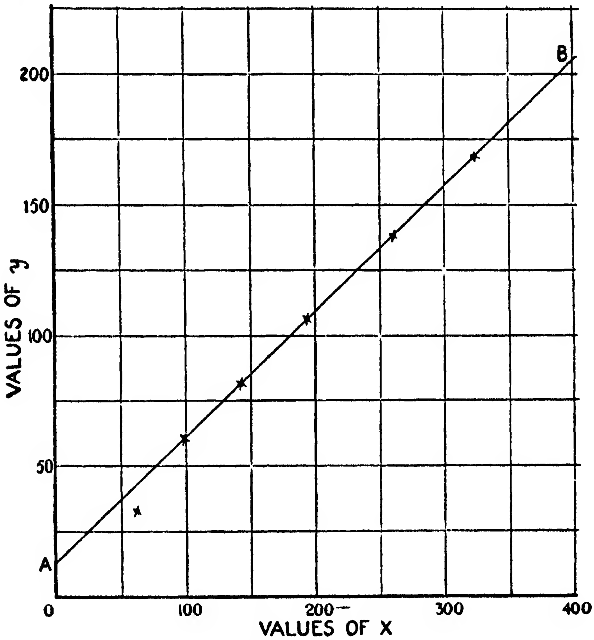


FIG. 15.

The suggested law is $y = a + bx^2$. Write $x^2 = X$ so that the equation takes the form $y = a + bX$, which is a linear relation.

A table of values is first constructed from the given numerical values.

x	8	10	12	14	16	18	20
X	64	100	144	196	256	324	400
y	32	60	82	108	138	172	220

All points except the first and last values lie very approximately along a straight line. (Fig. 15.) The intercept on the y-axis is 10 units. Thus $a = 10$.

To find the gradient of the line consider the two points A, B, whose coordinates are (0, 10) and (400, 210) respectively. Thus the gradient is

$$b = (210 - 10)/(400 - 0) = 0{\cdot}5.$$

It follows that the probable law is $y = 10 + \frac{1}{2}x^2$ .... (i)

The values which have been misread are the first and the last. The correct values may be found by using (1). Thus when $x = 8$, $y = 42$ (instead of 32) and when $x = 20$, $y = 210$ (instead of 220).

6·3. One Constant as an Index

An important relation which occurs frequently is

$$y = ax^n$$

where a and n are the unknown constants. Taking logarithms of both sides of the equation,

$$\log_{10}y = \log_{10}a + n\log_{10}x.$$

If we write $u = \log_{10}x$, $v = \log_{10}y$ the equation takes the form

$$v = nu + b$$

where $b = \log_{10}a$. The equation is linear in the variables u, v.

Examples.—(1) *Show, by drawing a suitable linear graph, that the values tabulated below are consistent with a relation of the form $y = ax^n$, and assuming that such a relation holds good estimate the values of a and n:*

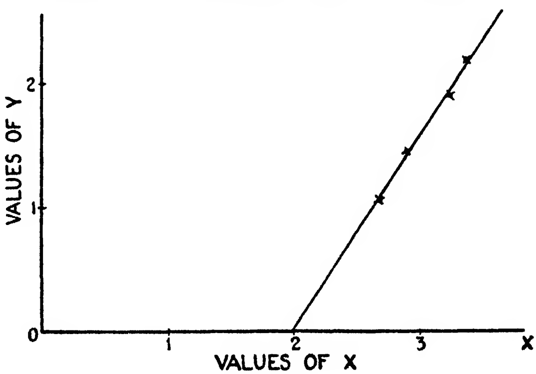


FIG 16.

x	483	870	1754	2750
y	12·0	29·5	83·0	165·0

$y = ax^n$. Thus $\log_{10}y = \log_{10}a + n\log_{10}x$.

Write $Y = \log_{10}y$, $X = \log_{10}x$. Then we have the following table of values:—

x	483	870	1754	2750
y	12·0	29·5	83·0	165·0
X	2·684	2·940	3·244	3·439
Y	1·079	1·470	1·919	2·217

The points lie almost exactly along a straight line. (Fig, 16.) Reading from the graph the intercept on the X-axis is 2 and the gradient of the line is 3·07 approx. Thus

$$n = 1{\cdot}54, \quad -\log_{10}a/n = 2, \text{ i.e. } \log_{10}a = -3{\cdot}08 = \bar{4}{\cdot}92.$$

Thus $a = 0{\cdot}00083$ approx. and the equation is

$$y = 0{\cdot}00083x^{1{\cdot}54}.$$

(2) *The following table gives corresponding values of the pressure and specific volume of dry saturated steam. Show that they are very approximately related by an equation of the form* $pv^n = c$, *and estimate the values of the constants* n *and* c.

p	10	20	50	100	150	200
v	38·4	20·0	8·51	4·44	3·03	2·31

Since $pv^n = c$, $\log_{10}p = \log_{10}c - n\log_{10}v$.

Write $\log_{10}v = x$, $\log_{10}p = y$ and the equation takes the linear form $y = \log_{10}c - nx$.

p	10	20	50	100	150	200
v	38·4	20·0	8·51	4·44	3·03	2·31
x	1·5843	1·3010	0·9299	0·6474	0·4814	0·3636
y	1·0	1·3010	1·6990	2·0	2·1761	2·3010

Convenient scales would be as follows: for x and y, 1 in. = 0·25 units. It is found that the plotted points approximate closely to a straight line. (Fig. 17.) The two points (1·584, 1) (0, 2·7) are found to lie on the line drawn.

Hence the equation of the line is $y = -\dfrac{1{\cdot}7}{1{\cdot}584}x + 2{\cdot}7 = -1{\cdot}073x + 2{\cdot}7$.

Thus $n = 1{\cdot}073$, $c = 501$ approx.

(3) *The following table gives the velocity of reaction between hydrogen and chlorine at different temperatures T:—*

$x = \dfrac{T}{300}$	0·1	0·2	0·35	0·44	0·58	0·76	0·88
y	0·0105	0·0672	0·333	0·687	1·83	6·3	18·18

Do these values fit a curve of the form $y = ax^n/(x-1)$? and if so what values of a and of n do you find the most suitable?
[*Camb. Sch.*]

The suggested relation is

$$y(1-x) = -ax^n,$$

i.e. $\log_{10}y + \log_{10}(1-x)$

$$= \log_{10}(-a) + n\log_{10}x.$$

Write $\log_{10}y + \log_{10}w = v$, $\log_{10}x = u$ where $1 - x = w$ and the equation takes the linear form $v = \log_{10}(-a) + nu$.

We first construct a table of values of v and u from the given values of x and y.

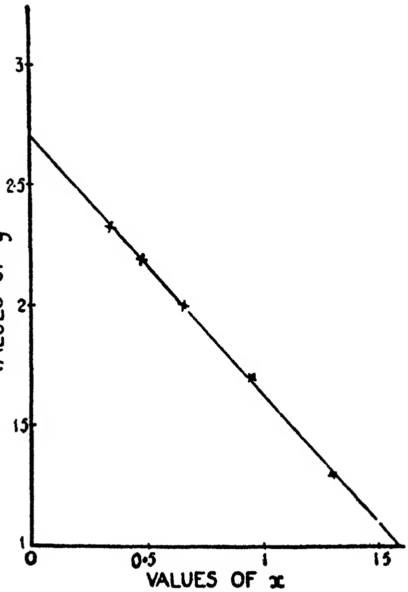


FIG. 17.

$\log_{10}y$	− 1·9788	− 1·1726	− ·4776	− ·1630	·2625	·7993	1·2596
$\log_{10}w$	− ·0458	− ·0969	− ·1871	− ·2518	− ·3768	− ·6198	− ·9208
v	− 2·0246	− 1·2695	− ·6647	− ·4148	− ·1143	·1795	·3388
u	− 1	− ·6990	− ·4559	− ·3566	− ·2366	− ·1192	− ·0555

Convenient scales would be as follows: for u, 1 in. represents 0·2 unit, for v, 1 in. represents 0·3 unit. It is found that the plotted points approximate closely to a straight line. (Fig. 18.) The two points P, Q, whose coordinates

are $(-0{\cdot}186, 0)$, $(-1{\cdot}034, -2{\cdot}1)$ respectively lie on the line drawn. The equation of the line is

$$\frac{v - 0}{-2{\cdot}1 - 0} = \frac{u + 0{\cdot}186}{-1{\cdot}034 + 0{\cdot}186}, \text{ i.e. } v = 2{\cdot}476u + 0{\cdot}4606.$$

Hence $n = 2{\cdot}476$, $\log_{10}(-a) = 0{\cdot}4606$ giving $a = -2{\cdot}8$

6·4. Exponential Graphs

If the equation can be written in the form $y =$,

then on taking logarithms of both sides of the equation,

$$\log_{10} y = \log_{10} a + bx \log_{10} e.$$

Writing $x \log_{10} e = u$, $\log_{10} y = v$, $\log_{10} a = A$ the equation takes the form

$$v = A + bu,$$

which is linear in u and v. Similar, if the given equation has the form

$$y = cn^x$$

where c and n are constants, then

$$\log_{10} y = \log_{10} c + x \log_{10} n.$$

Substituting $v = \log_{10} y$, $C = \log_{10} c$, $N = \log_{10} n$, the equation becomes

$$v = C + Nx,$$

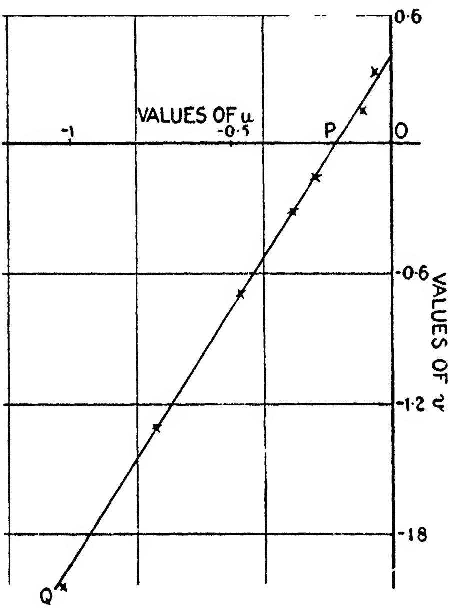


FIG. 18.

which is linear in the variables x, v.

Examples.—(1) *The coefficient of friction* (μ) *between two lubricated surfaces moving with a certain definite relative velocity is measured at different temperatures* (t° C.), *with the following results:—*

t	60	70	80	90	100	110
μ	0·0096	0·0077	0·0063	0·0045	0·0039	0·0030

Show that the numbers are related by an approximate equation of the type $u = ke^{-\lambda t}$, *and determine the best values of* k *and* λ.

[*Lond. B.Sc. Eng.*]

The equation $\mu = ke^{-\lambda t}$ is equivalent to

$$\log_{10}\mu = \log_{10}k - \lambda t \log_{10}e.$$

Writing $\log_{10}\mu = y$, $\log_{10}k = a$, $\lambda \log_{10}e = b$ the equation takes the linear form

$$y = a - bt.$$

From the given values the following table of values is obtained.

t	60	70	80	90	100	110
μ	0·0096	0·0077	0·0063	0·0045	0·0039	0·0030
y	− 2·0177	− 2·1135	− 2·2007	− 2·3468	− 2·4089	− 2·5229

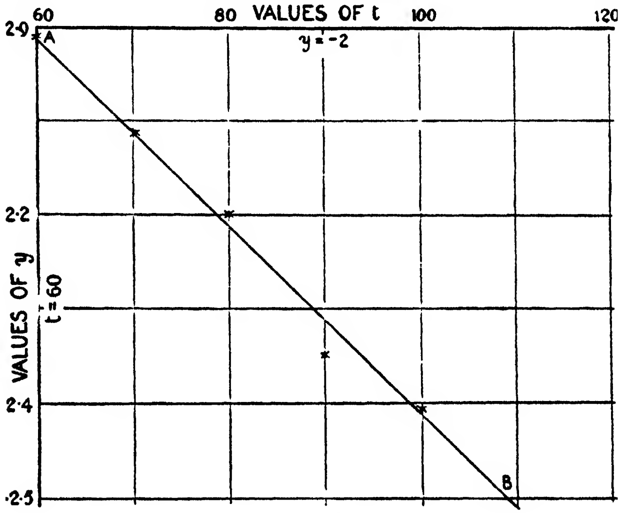


FIG. 19.

The points are plotted as indicated in Fig. 19. Because of the values involved it is not convenient to mark the origin and the axis of t and y in the figure. A line AB is drawn as evenly as possible among the plotted points. In the figure A is taken to be the point (60, − 2·02), B the point (108·5, − 2·5). Thus the equation of the line AB is

$$\frac{y + 2\cdot5}{-2\cdot02 + 2\cdot5} = \frac{t - 108\cdot5}{60 - 108\cdot5},$$

which reduces to $y = -0\cdot009897t - 1\cdot427$.

Thus $a = \log_{10}k = -1\cdot427 = \bar{2}\cdot573$. Hence

$$k = 0\cdot0374 \text{ approx.}$$

Again, $-\lambda \log_{10}e = -0\cdot009897$, $\therefore\ \lambda = 0\cdot009897/0\cdot4343$,

i.e. $\lambda = 0\cdot0228$ approx.

(2) *The table below gives simultaneous values of two quantities x and y. Show that $y = a^{(x + b)}$ is approximately true, and find the value of a and of b.*

x	2	2·5	3	3·5	4	4·5	5
y	1	1·24	1·56	1·93	2·41	3·02	3·77

[*N.Sc.*]

The equation $y = a^{x + b}$ is equivalent to $\log_{10}y = (x + b)\log_{10}a$. Writing $Y = \log_{10}y$, the equation takes the form

$$Y = b \log_{10}a + x \log_{10}a,$$

an equation which is linear in x and Y.

We first construct a table of values and then plot Y against x thus:—

x	2	2·5	3	3·5	4	4·5	5
y	1	1·24	1·56	1·93	2·41	3·02	3·77
Y	0	0·093	0·193	0·286	0·382	0·480	0·576

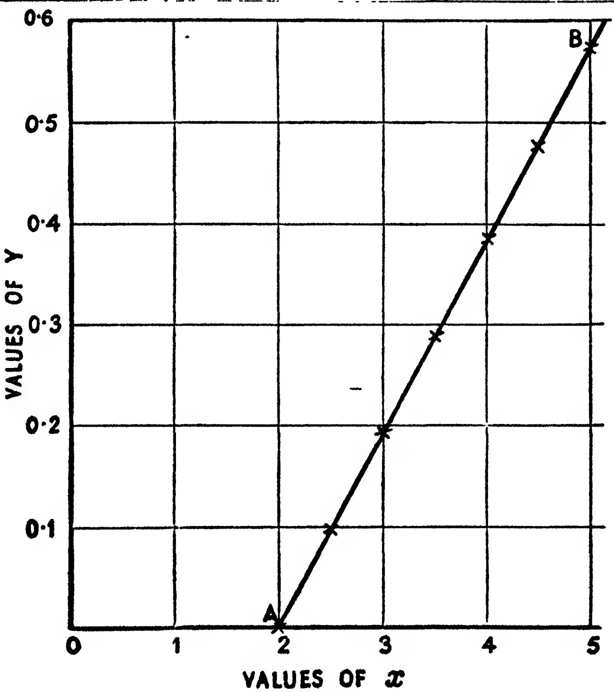


FIG. 20.

Convenient scales would be as follows: for x, 1 in. = 1 unit; for y, 1 in. = 0·1 of a unit. It is found that the points lie almost exactly along a straight line.

The intercept on the x-axis is 2. Thus $b \log_{10}a + 2 \log_{10}a = 0$, giving $b = -2$. (Fig. 20.)

To calculate the gradient of the graph take the two points A, B where A is (2, 0) and B (5, 0·575). Then the gradient is $(0{\cdot}575 - 0)/(5 - 2) = 0{\cdot}192$ approx. Thus $\log_{10}a = 0{\cdot}192$ giving $a = 1{\cdot}56$ approx.

(3) *The temperature T of a chemical substance undergoing a slow change of state was found to vary with the time t as follows:—*

Temp. in degs. F.	14·3	16·7	19·8	21·9	24·2	26·3	27·9	30·6
Time in mins.	0	1	3	5	8	12	16	25

It was thought that the temperature was connected with the time by a formula of the type $T = 15 \log_{10}(a + bt)$. Explain how you would test if such a formula fitted the data and by drawing a suitable graph find the values of a and b that you consider to be most suitable. [*Camb. Sch.*]

The suggested relation may be written in the form $y = a + bt$ where $y = 10^{\frac{T}{15}}$. This equation is linear, so that if the equation is approximately true a linear graph should be obtained when y is plotted against t.

t	0	1	3	5	8	12	16	25
T	14·3	16·7	19·8	21·9	24·2	26·3	27·9	30·6
$\frac{T}{15}$	0·9533	1·113	1·320	1·460	1·613	1·753	1·860	2·040
y	8·981	12·972	20·893	28·840	41·020	56·624	72·444	109·65

Suitable scale of representation would be as follows: for t, 1 in. = 3 min.; for y, 1 in. = 10 units. It is found that the points approximate closely to a straight line so that the suggested relation is verified. (Fig. 21.)

The two points (0, 8·98) (12, 56·62) lie on the line drawn. Taking these values $a = 8{\cdot}98$, $b = (56{\cdot}62 - 8{\cdot}98)/12 = 3{\cdot}97$.

6·5. An Important Property of ae^{kx}

Let $y = ae^{kx}$. Consider a set of values of x which are in arithmetic progression. Let the values be $\lambda, \lambda + d, \lambda + 2d, \lambda + 3d, \ldots$ so that the distance between successive ordinates is d. The corresponding values of y are

$$ae^{k\lambda},\ ae^{k\lambda + kd},\ ae^{k\lambda + 2kd},\ ae^{k\lambda + 3kd}, \ldots$$

Since x_1, x_2, x_3 are in arithmetic progression, $x_1 + x_3 = 2x_2$. Thus

$$(y_1 - c)(y_3 - c) = ae^{bx_1} \times ae^{bx_3} = a^2e^{b(x_1+x_3)}$$
$$= (ae^{bx_2})^2 = (y_2 - c)^2.$$

Thus $c = (y_1 y_3 - y_2^2)/(y_1 - 2y_2 + y_3)$.

6·7. The Equation $y = ax^b + c$

The constant c is first determined by the method of § 6·6. In order to determine the values y_1, y_2, y_3 corresponding to x_1, x_2, x_3 in geometric progression y is plotted against x and a smooth curve drawn so as to approximate to the plotted points as far as possible. Then take three points x_1, x_2, x_3 in geometrical progression and read the corresponding values of y from the graph. Substitution in the formula

$$c = (y_1 y_3 - y_2^2)/(y_1 - 2y_2 + y_3)$$

determines c. In order to obtain accurate results it is important to arrange that the denominator

$$(y_1 - 2y_2 + y_3)$$

be not too small. The given equation may now be written in the form $\log_{10}(y - c)$

$$= \log_{10}a + b \log_{10}x,$$

i.e. $v = A + bu$,

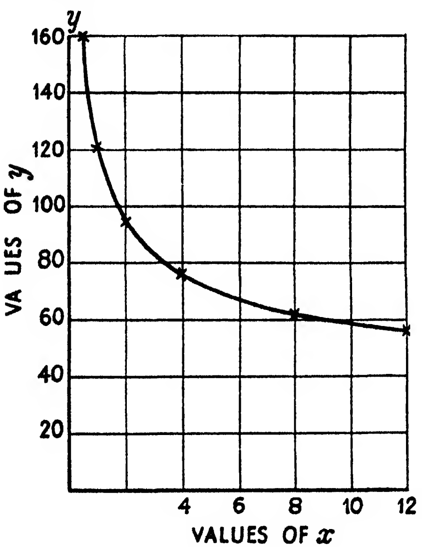


FIG. 22.

where $u = \log_{10}x$, $v = \log_{10}(y - c)$. Thus when v is plotted against u a straight line graph should be obtained, and from this the constants a and b may be determined.

Example.—*Two variables x and y are supposed to be connected by an equation of the form $y = ax^n + c$. Verify that this result is approximately true and find values for a, n and c when corresponding values of x and y are given by the following table:*—

x	0·5	1	2	4	8	12
y	160	120	94	75	62	56

We first plot y against x and obtain the curve in Fig. 22.

To find the value of c we take the three values of x in geometric progression. Convenient values are $x = 0{\cdot}5$, $\sqrt{6} = 2{\cdot}45$ and 12. The corresponding values of y are 160, 88·4, 56, the second value of y being read from the graph.

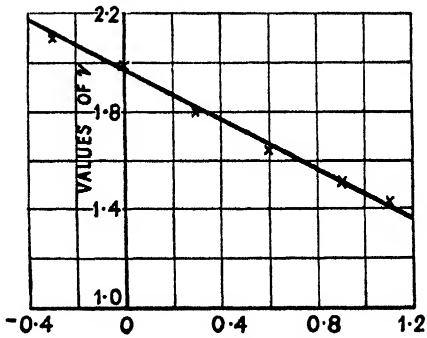
Thus $$c = \frac{160 \times 56 - 88{\cdot}4^2}{160 - 2 \times 88{\cdot}4 + 56} = \frac{1145{\cdot}4}{39{\cdot}2} = 29{\cdot}2 \text{ approx.}$$

To find a and n we use
$$\log_{10}(y - c) = \log_{10}a + n\log_{10}x,$$
and write $u = \log_{10}x$, $v = \log_{10}(y - c)$. We have then the following table of values:—

x	0·5	1	2	4	8	12
$y - c$	130·8	90·8	64·8	45·8	32·8	26·8
u	−0·301	0	0·301	0·602	0·903	1·079
v	2·12	1·96	1·81	1·66	1·52	1·43

Plotting v against u we see that the points approximate closely to a straight line. We take the two points (0, 1·97) (0·9, 1·52) which lie on the line drawn. The intercept on the v-axis is 1·97 and the gradient of the line is
$(1{\cdot}97 - 1{\cdot}52)/(-0{\cdot}9) = -0{\cdot}5$.
(Fig. 23.)

Hence $n = -0{\cdot}5$,
$\log_{10}a = 1{\cdot}97$, $a = 93$ approx.
Thus the relation is
$$y = 93x^{-0\cdot5} + 29{\cdot}2$$



VALUES OF u
FIG. 23.

6·8. The Equation $y = ae^{bx} + c$

Using the result of § 6·6 the constant c is first determined. For this we require the values of y corresponding to three values of x which are in arithmetic progression. We first plot y against x and draw a smooth curve which approximates as closely as possible to the plotted points. This graph will be required in general, for the determination of the values of y corresponding to the values of x chosen.

Having determined c, the equation may then be written in the form
$$v = \log_{10}a + bu,$$

where $v = \log_{10}(y - c)$, $u = x \log_{10}e$. From the linear graph obtained by plotting v against u we obtain the values of a and b.

Example.—*Two variables t and y whose tabulated values, given below, are determined experimentally, are known to be connected by a relation of the form $y = ae^{bt} + c$.*

t	0	1	2	3	5	7	10	15	20
y	52·2	48·8	46·0	43·5	39·7	36·5	33·0	28·7	26·0

Find the best values you can for a, b, c.

First plot y against t. It is then found that the points lie on a smooth curve. Since the values of y corresponding to $t = 0$, 10, 20 lie on the smooth curve and the corresponding values of y are given, nothing is gained by reading from the graph other values of y corresponding to values of t which are in arithmetic progression. Thus

$$c = (52{\cdot}2 \times 26 - 33^2)/(52{\cdot}2 - 66 + 26) = 22 \text{ approx.}$$

$$\text{Again } \log_{10}(y - c) = \log_{10}a + bt \log_{10}e.$$

Write $\log_{10}(y - 22) = v$, $t \times {\cdot}4343 = u$; the equation takes the form:

$$v = \log_{10}a + bu.$$

Next we construct a table of values of u and v.

t	0	1	2	3	5	7	10	15	20
$y - 22$	30·2	26·8	24·0	21·5	17·7	14·5	11·0	6·7	4
u	0	0·434	0·869	1·303	2·172	3·040	4·343	6·51	8·67
v	1·480	1·428	1·380	1·332	1·248	1·161	1·041	0·826	0·602

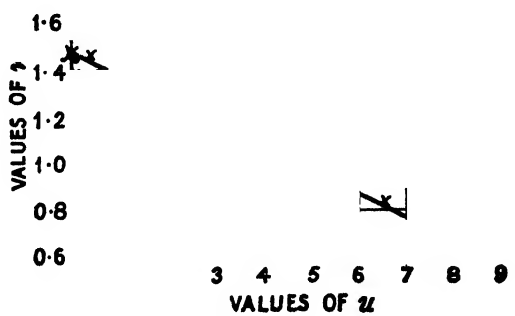


FIG. 24.

The plotted points are found to approximate very closely to a straight line. From the graph we find that the points (0, 1·48), (8·67, 0·6) lie on the line.

(Fig. 24.) Thus the intercept on the v-axis is 1·48 and the gradient of the line is

$$(1{\cdot}48 - 0{\cdot}6)/(-8{\cdot}67) = -0{\cdot}1 \text{ approx.}$$

Hence $\log_{10}a = 1{\cdot}48$, $a = 30$ approx. Thus the relation is

$$y = 30e^{-\frac{1}{10}t} + 22.$$

6·9. The Equation $y = ax^2 + bx + c$

This curve, which is a parabola, may be reduced to a linear graph from which the constants a and b may be determined. Let (h, k) be a *particular* point on the curve. Then

$$k = ah^2 + bh + c.$$

$$y - k = a\,(x^2 - h^2) + b\,(x - h) = (x - h)\,\{a\,(x + h) + b\}.$$

Writing $(y - k)/(x - h) = v$, $x + h = u$, the equation takes the form

$$v = au + b$$

which is linear in u and v. Thus when v is plotted against u a straight line graph is obtained from which a and b can be found. The value of c may be found by taking an average value determined by considering all the given values of x and y. (See Ex.)

Example.—*Two quantities θ and t are measured and corresponding values are given in the following table:—*

t	10	20	30	40	50	60
θ	4·5	7·1	10·5	15·5	20·5	27·1

The quantities are thought to be connected by a relation of the form $\theta = a + bt + ct^2$. Find whether this is the case and if so determine values of a, b, c.

We first plot θ against t to find whether the given points lie on a smooth curve. When this is done (Fig. 25) it is found that all the points except that corresponding to $t = 40$, lie on a good curve. We take (10, 4·5) as the special point. Then substituting in the equation we have

$$4{\cdot}5 = a + 10b + 100c.$$

Substituting from

$\theta = a + bt + ct^2$ we obtain

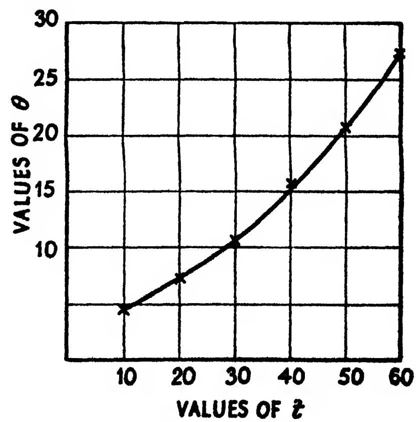


FIG. 25.

$$\theta - 4{\cdot}5 \quad b\,(t - 10) + c\,(t^2 - 100) = (t - 10)\,\{b + ct + 10c\}.$$

Writing $v = (\theta - 4{\cdot}5)/(t - 10)$ the equation takes the form

$$v = ct + b + 10c.$$

Thus if v be plotted against t we should obtain a straight line. We first construct a table of values.

t	10	20	30	40	50	60
$\theta - 4{\cdot}5$	0	2·6	6·0	11·0	16·0	22·6
$t - 10$	0	10	20	30	40	50
v		0·260	0·300	0·367	0·400	0·542

Plotting v against t we find that the points approximate to a straight line, Fig. 26. As may be anticipated from the first graph, the point corresponding to $t = 40$ is further from the straight line than the other points. We find that the points (0, 0·16), (60, 0·452) lie on the line drawn. Thus the intercept on the v-axis is 0·16 and the gradient of the line is $({\cdot}452 - {\cdot}16)/60 = {\cdot}0049$ approx. Hence $c = {\cdot}0049$, and $b + {\cdot}049 = {\cdot}16$ giving $b = {\cdot}111$.

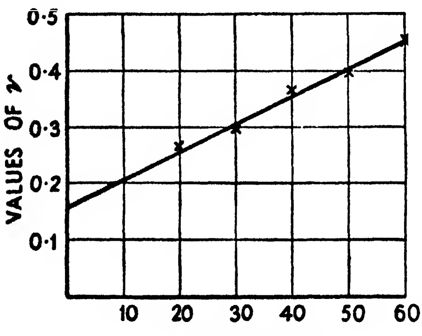


FIG. 26.

To find an average value of a we write down the 6 given values and add. Then $6a = \Sigma\theta - b\,\Sigma t - c\,\Sigma t^2$
$= 85{\cdot}2 - {\cdot}111 \times 210 - {\cdot}0049 \times 9000$;
$a = 3{\cdot}1$ approx., giving
$\theta = 3{\cdot}1 + {\cdot}111t + {\cdot}0049t^2$.

EXERCISES VI

1. A series of values of the variables x and y are given. Explain how you would test whether the variables are related by an equation of one of the following forms: (i) $y + ay\sqrt{x} = bx$; (ii) $y = a \log bx$; (iii) $y = ax + b \log x$.

2. Two quantities x and y are connected by the equation

$$x - axy - by = 0,$$

where a and b are constants. Show that the graph of y/x and y is a straight line. Below are given corresponding values of x and y. Show, by plotting, that they are connected by an equation of the form given above, and determine the constants a and b.

x	0·2	0·5	1	2	5
y	0·5	1	1·5	2	2·5

3. Corresponding values of the observed quantities x and y are given in the following table:—

x	1	2	3	4	5	6
y	6	7·2	9·6	12	14·8	17·4

These quantities are connected by the relation $xy = ax^2 + b$. Find the best values you can of a and b by plotting xy against x^2.

4. The following table gives corresponding pairs of observed values of two related variables x (in degrees) and y:—

x	38°	85°	117°	201°	293°
y	12·2	8·0	4·9	1·9	9·7

It is believed that x and y are related by a law of the form $y = a + b \cos x$. Verify that the tabulated values confirm this, by drawing a suitable linear graph, and estimate the values of a and b.

5. Two quantities x and y are connected by the equation $y = ax^2 + bx^3$, where a and b are constants. Show that the values of x and y given in the table below satisfy an equation of the above form, and determine the constants a and b.

x	1	2	3	4
y	1·65	9·2	27·0	59·2

6. It is believed that two variable quantities w and I are connected by a relation of the form $w = b + a/I$, where a, b are numerical constants.

In an experiment to verify this, the following values of w and I were obtained:—

I	36·8	31·5	26·3	21·0	15·8	12·6	8·4
w	12·5	12·9	13·1	13·3	14·1	14·5	16·3

Use a graphical method to verify that the suggested relation is correct; use the method of least squares to determine suitable values of a and b.

7. Corresponding values of two variables x and y are observed as follows:

x	1	1·3	2	3	4	6
y	3·7	3·4	2·8	2·5	2·4	2·1

If the variables are known to be connected by a relation of the form $y^2 = a + b/x$, explain how to use the above data to obtain a straight line graph. Plot the graph and from it obtain the best values you can for a and b.

8. Assuming that the following experimental values of x and y may be approximately represented by a formula of the type $y = ax + b10^x$, find values for the constants a and b and show in a table the values of y corresponding to the given values of x as given by the formula with the ascertained values of the constants.

x	1	2	3	4
y	0·30	0·64	1·32	5·20

[*Lond. B.Sc. Eng.*]

9. Two quantities of y and x are connected by the relation $y = ax^n$, where a and n are unknown. By a graphical method, find the most probable values of a and n from the experimental figures given below.

y	2·51	5·01	10·0	12·5	16·9	20·0
x	1000	2500	5000	7500	10,000	12,500

10. A force of P lb. per ton is required to pull a canal boat at a speed of V miles per hour, and corresponding values of P and V are given in the following table. Show by plotting log P and log V that there is a relation of the form $P = aV^n$ and find approximately the values of a and n.

P	1·58	2·08	2·62	3·20
V	2·5	3	3·5	4

[*Camb. Sch.*]

11. A number of simultaneous values of two quantities x and y are given: explain how it may be found whether they satisfy either of the forms

(i) $y = ax^n$, (ii) $y = ax/(1 + bx)$.

The safe distributed load for different lengths of girder are given in the table:

Length feet (l)	30	39	48	60
Load cwts. per foot run (w)	27·4	16	10·6	6·6

Show that $w = al^n$ approximately, and find the best values of a and n.
[*N.Sc.*]

12. The times taken, y seconds, to run a distance x yards are given in the following table:—

y secs.	19·4	30·6	48·4	111·6	185·4	253·8	549·3	851·5	1149
x yds.	200	300	440	880	1320	1 mile	2 miles	3 miles	4 miles

By plotting log y against log x, show that x and y are connected approximately by a relation of the form $y = ax^n$, and find the best values y can for a and n.

13. The following table gives the pressure P and the volume V as determined experimentally at various instants during the expansion of steam in a cylinder. Show the equation of the expansion is of the form $PV^n = C$, and find approximately the values of n and C

P lb. per sq. in.	200	100	50	30	20	10
V cub. ft. ..	1·00	1·70	2·89	4·30	5·88	10·00

14. If a series of corresponding values of two connected physical quantities, x and y, are measured, what test can be applied to determine which of the following equations best represents the connection?

(i) $y = ax + b$, (ii) $y = ax^n$, (iii) $y = an^x$,

a and n being constants. Assuming equation (iii) applies to the following, find the probable values of a and n.

x	1	3	5	7	10
y	5	14·8	47·9	141	708

[*N.Sc.*]

15. Show that the points of which the coordinates are given below lie very nearly on a curve represented by the equation $y = ab^x$, and find the probable values for the constants a and b.

x	5	6·25	7·5	8·75	10
y	100	118·9	141·4	168·2	200

[*N.Sc.*]

16. Two variables x and y are related by a law of the form $y = a + bx^n$. If x_1, x_2, x_3 are three values of x in geometric progression and y_1, y_2, y_3 are the corresponding values of y, prove that $a = (y_1 y_3 - y_2^2)/(y_1 - 2y_2 + y_3)$.

The following pairs of values of x and y are known to be related by a law of the above form:

x	1	4	16
y	17	11	8

Find the values of a, b, and n.

17. Two variables x and y are thought to be connected by an equation of the form $y = ax^n + b$. Corresponding values are given in the following table:—

x	1	2	4	6		16
y	15·0	44·9	165	364	1004	2564

Find the best values you can for a, n and b.

18. Values of p and v are given in the following table. Show graphically that the approximate law connecting p and v is $p = ae^{bv} + c$ and find values for a, b and c.

v	1	2	6	8	11
p	12·71	12·46	11·65	11·34	10·99

CHAPTER VII

FURTHER THEOREMS ON CONVERGENCE

IN this chapter we first prove some further tests for ordinary convergence and uniform convergence of series and then pass to the consideration of infinite products.

7·1. Cauchy's Condensation Test

Let Σu_n denote a series of positive terms such that u_n is a steadily decreasing function of n, i.e. $u_{n+1} \leqslant u_n$. *Then if k denote any positive integer greater than unity, the series Σu_n and $\Sigma k^n u_{k^n}$ are both convergent or both divergent.*

The terms of Σu_n may be grouped as follows:

$$\{u_1 + u_2 + \ldots + u_k\} + \{u_{k+1} + u_{k+2} + \ldots + u_{k^2}\}$$
$$+ \{u_{k^2+1} + u_{k^2+2} + \ldots + u_{k^3}\} + \ldots$$
$$+ \{u_{k^r+1} + u_{k^r+2} + \ldots + u_{k^n}\} + \ldots$$

where $r = n - 1$.

$$= v_1 + v_2 + v_3 + \ldots + v_n + \ldots$$

where v_n is the sum of the terms in the nth group.

Since $k^n = k^{n-1} + (k^n - k^{n-1})$ it follows that v_n contains $k^n - k^{n-1}$ terms. Also, since u_n steadily decreases as n increases it follows that each term of $v_n \leqslant u_{k^{n-1}}$ and $\geqslant u_{k^n}$. Hence

$$(k^n - k^{n-1})\, u_{k^n} \leqslant v_n \leqslant (k^n - k^{n-1})\, u_{k^{n-1}},$$

$$\text{i.e.}\quad \left(1 - \frac{1}{k}\right) k^n u_{k^n} \leqslant v_n \leqslant (k-1)\, k^{n-1} u_{k^{n-1}}.$$

Hence if $\Sigma u_n = \Sigma v_n$ converges, so does $\Sigma k^n u_{k^n}$, and conversely if $\Sigma k^n u_{k^n}$ converges so also does Σv_n, i.e. Σu_n. Also if Σv_n diverges so also does $\Sigma k^n u_{k^n}$, and conversely.

In particular, if $k = 2$, the test asserts that if u_n is a steadily decreasing positive function of n then $\sum\limits_{n=1}^{\infty} u_n$ converges or diverges with $\sum\limits_{n=1} 2^n u_{2^n}$.

Example.—*Test the convergence of the series whose* $(n-1)$*th term is* $(\log n)^{-p}$.

It is sufficient to consider $p > 0$.

Write
$$u_n = [\log (n+1)]^{-p}.$$

Then
$$\frac{u_n}{u_{n-1}} = \frac{[\log (n+1)]^{-p}}{[\log n]^{-p}} = \left[\frac{\log n}{\log (n+1)}\right]^p.$$

For $n \geqslant 2$, $\log (n+1) > \log n$ and hence $u_n < u_{n-1}$, so that u_n is a decreasing positive function of n.

Hence the series $\sum\limits_{n=2}^{\infty} (\log n)^{-p}$ will converge or diverge with
$$\sum_{n=2}^{\infty} 2^n (\log 2^n)^{-p}.$$

Now $2^n (\log 2^n)^{-p} = 2^n n^{-p} (\log 2)^{-p}$. Hence it it is sufficient to consider the series Σv_n where $v_n = 2^n n^{-p}$, since $(\log 2)^{-p}$ is a constant factor of each term.

Now Σn^{-p} diverges, $0 < p \leqslant 1$; hence Σv_n and $\Sigma (\log n)^{-p}$ must also diverge for $0 < p \leqslant 1$.

Consider then $p > 1$.
$$\frac{v_{n+1}}{v_n} = \frac{2^{n+1}}{(n+1)^p}\cdot\frac{n^p}{2^n} = 2\left(\frac{n}{n+1}\right)^p.$$

Now the sequence $w_n = \left(\dfrac{n}{n+1}\right)^p$ is monotonic increasing and $\lim\limits_{n\to\infty} w_n = 1$. Hence we can find a number n_0 such that for all $n > n_0$, $w_n > \frac{2}{3}$. It would be sufficient to take n_0 to be the first positive integer greater than $\lambda/(1-\lambda)$ where $\lambda = (\frac{2}{3})^{1/p}$. With such a value n of n_0,
$$\left(\frac{n}{n+1}\right)^p > \left[\frac{\lambda/(1-\lambda)}{1+\lambda/(1-\lambda)}\right]^p = \lambda^p = \tfrac{2}{3}.$$

Hence for $n > n_0$, $\dfrac{v_n+1}{v_n} > 2\cdot\frac{2}{3} = \frac{4}{3} > 1$.

Thus Σv_n diverges and so also $\Sigma (\log n)^{-p}$, for $p > 1$. It follows that the series $\Sigma (\log n)^{-p}$ does not converge for any finite value of p.

7·11. The Series $\Sigma\, 1/n (\log n)^p$

It follows from the theorem of § 7·1 that $\Sigma 1/n (\log n)^p$ *converges if* $p > 1$ *and diverges if* $p \leqslant 1$. For let k be any positive integer greater than unity: then the series
$$\Sigma \frac{1}{n(\log n)^p} \quad\text{and}\quad \Sigma \frac{k^n}{k^n (\log k^n)^p}$$

diverge or converge together. The second series is $\frac{1}{(\log k)^p} \Sigma \frac{1}{n^p}$. This converges if $p > 1$ and diverges if $p \leqslant 1$. (Chap. I., § 1·41.)

7·2. The Maclaurin Integral Test

Let $\sum_{n=1}^{\infty} u_n$ denote a series whose terms are all positive and such that u_n steadily decreases as n increases. Write $f(n) = u_n$ so that $f(n)$ is a function of n which is defined for integral values of n. We now replace the integral variable n by the continuous variable x and consider the function $f(x)$. It is assumed that the function so represented is defined for all values of x greater than or equal to unity, and further, that $f(x)$ is a monotonic decreasing function of x. Thus, e.g.

$$f(n-1) \geqslant f(x) \geqslant f(n), \text{ i.e. } u_{n-1} \geqslant f(x) \geqslant u_n,$$

where $n - 1 \leqslant x \leqslant n$.

From the definition of an integral it follows that

$$\int_{n-1}^{n} u_{n-1} dx \geqslant \int_{n-1}^{n} f(x)\, dx \geqslant \int_{n-1}^{n} u_n dx,$$

$$\text{i.e. } u_{n-1} \geqslant \int^{n} f(x)\, dx \geqslant u_n.$$

$$\begin{aligned} \text{Thus } u_1 &\geqslant \int_1^2 f(x)\, dx \geqslant u_2, \\ u_2 &\geqslant \int_2^3 f(x)\, dx \geqslant u_3, \\ &\cdots\cdots\cdots\cdots \\ u_{n-1} &\geqslant \int_{n-1}^{n} f(x)\, dx \geqslant u_n. \end{aligned}$$

Adding, and writing $s_n = \sum_{r=1}^{n} u_r$, it follows that

$$s_n - u_n \geqslant \int_1^n f(x)\, dx \geqslant s_n - u_1,$$

$$\text{i.e. } u_1 \geqslant s_n - \int_1^n f(x)\,dx \geqslant u_n > 0.$$

Write $F(n) = s_n - \int_1^n f(x)\,dx > 0$ for all values of n. Then

$$F(n) - F(n-1) = s_n - s_{n-1} - \int_1^n f(x)\,dx + \int_1^{n-1} f(x)\,dx$$

$$= u_n - \int_{n-1}^n f(x)\,dx \leqslant 0,$$

since $u_n \leqslant f(x)$, for $n - 1 \leqslant x \leqslant n$.

Hence $F(n) \leqslant F(n-1)$, i.e. $F(n)$ is a monotonic decreasing sequence which is bounded below. Thus $\lim\limits_{n\to\infty} F(n)$ exists. (Chap. I., § 1·23.)

Further, if l denote the limit, it follows from $u_1 \geqslant F(n) > 0$ that

$$u_1 \geqslant l \geqslant 0.$$

There are now two possibilities. Either s_n tends to a limit as $n \to \infty$ or $s_n \to \infty$ as $n \to \infty$. In the former case the series converges to some number s. Thus

$$u_1 \geqslant s - \int_1^\infty f(x)\,dx \geqslant 0,$$

$$\text{i.e. } u_1 + \int_1^\infty f(x)\,dx \geqslant s \geqslant \int_1^\infty f(x)\,dx,$$

and the integral must always be finite.

Conversely, if the integral exists it follows that $\lim\limits_{n\to\infty} s_n$ must exist, *i.e.* the series must converge. If I be the value of the integral then the sum s lies between I and $u_1 + I$.

On the other hand, if $\lim\limits_{n\to\infty} s_n = \infty$, the integral must also be infinite and conversely. In this case we can assert, however, that

$$\lim_{n\to\infty} \left\{ s_n - \int_1^n f(x)\,dx \right\}$$

is always finite and lies between 0 and u_1.

Combining the results we have the following theorem. *The series* $\sum_{n=1}^{\infty} u_n$ *converges or diverges with the integral* $\int_1^{\infty} f(x)\,dx$; *if convergent the sum of the series differs from the integral by less than* u_1; *if divergent* $\lim_{n\to\infty} \left\{ s_n - \int_1^n f(x)\,dx \right\}$ *exist and lies between* 0 *and* u_1.

Examples.—(1) *Consider the series* $\sum_{n=1} 1/n^p$.

$$\text{Here } f(x) = 1/x^p.$$

$$\int_1^n f(x)\,dx = \frac{1}{1-p}\Big[x^{1-p}\Big]_1^n = \frac{1}{1-p}\left\{n^{1-p} - 1\right\},\ p \lessgtr 1.$$

$$\text{If } p = 1,\ \int_1^n f(x)\,dx = \Big[\log x\Big]_1^n = \log n.$$

$$\text{If } p > 1,\ \lim_{n\to\infty} \frac{1}{1-p}\left\{n^{1-p} - 1\right\} = -\frac{1}{1-p} = \frac{1}{p-1}.$$

$$\text{If } p < 1,\ \lim_{n\to\infty} \frac{1}{1-p}\left\{n^{1-p} - 1\right\} = \infty.$$

$$\text{If } p = 1,\ \lim_{n\to\infty} \log n = \infty.$$

Hence the integral exists if $p > 1$ and tends to infinity if $p < 1$. Hence also the series converges if $p > 1$ and diverges if $p < 1$. (*cf.* Chap. I., § 1·41.) In the former case we infer that

$$\frac{1}{p-1} < \sum_{n=1}^{\infty} 1/n^p < \frac{1}{p-1} + 1.$$

The case $p = 1$ gives an important result. In this case $F(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$, so that

$$\lim_{n\to\infty} \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$$

exists and lies between 0 and 1. This number is called *Euler's constant* and is frequently denoted by γ. Its value is 0·577 ....

(2) *Consider the series* $\sum_{n=2} 1/n(\log n)^p$.

$$\text{Here } f(x) = 1/x(\log x)^p.$$

In order to evaluate the integral $\int^n f(x)\,dx$, write $\log x = y$, $dx/x = dy$.

$$\text{Then } \int_2^n \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\log n} \frac{dy}{y^p} = \frac{1}{1-p}\Big[y^{1-p}\Big]_{\log 2}^{\log n},$$

$$= \frac{1}{1-p}\left\{(\log n)^{1-p} - (\log 2)^{1-p}\right\}.$$

If $p > 1$, $(\log n)^{1-p} \to 0$ while if $p < 1$, $(\log n)^{1-p} \to \infty$. Thus the integral $\int_2^\infty \frac{dx}{x(\log x)^p}$ exists and is equal to $(\log 2)^{1-p}/(p-1)$ if $p > 1$ while the integral is infinite if $p < 1$.

$$\text{If } p = 1,\ \int_2^n \frac{dx}{x\log x} = \int_{\log 2}^{\log n} \frac{dy}{y} = \Big[\log y\Big]_{\log 2}^{\log n}$$

$$= \log(\log n) - \log(\log 2) = \log\{\log n/\log 2\}.$$

As $n \to \infty$ this expression tends to infinity. Hence the series converges if $p > 1$ and diverges if $p \leqslant 1$. [*cf.* § 7·11.]

(3) *Prove that if* $k > 0$, then $\sum_{n=0}^{\infty} 1/(k+n)^2 < (k+1)/k^2$.

Consider the function $f(x) = 1/(k+x)^2$. Then

$$\int_0^n \frac{dx}{(k+x)^2} = -\left[\frac{1}{k+x}\right]_0^n = \frac{1}{k} - \frac{1}{k+n}.$$

$$\text{Hence } \lim_{n\to\infty} \int_0^n \frac{dx}{(k+x)^2} = \frac{1}{k}.$$

$$\text{Thus if } s = \sum_{n=0}^{\infty} 1/(k+n)^2, \quad \frac{1}{k^2} > s - \frac{1}{k} > 0.$$

$$\text{i.e. } s < \frac{1}{k} + \frac{1}{k^2} = \frac{(k+1)}{k^2}.$$

(4) *By expressing* $\lim_{n\to\infty}\left\{\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}\right\}$ *as an integral, or otherwise, prove that the value of the expression is* $\log_e 2$ [*Lond. B.Sc.*]

Now $$\frac{1}{r} \geqslant \frac{1}{x} \geqslant \frac{1}{r+1}, \; r \leqslant x \leqslant r+1.$$

Hence $$\int_r^{r+1} \frac{dx}{r} \geqslant \int_r^{r+1} \frac{dx}{x} \geqslant \int_r^{r+1} \frac{dx}{r+1}.$$

Applying this inequality for $r = n, n + 1, \ldots, 2n$ and adding,

$$\sum_{r=n}^{2n} \frac{1}{r} \geqslant \int_n^{2n+1} \frac{dx}{x} \geqslant \sum_{r=n}^{2n} \frac{1}{r+1},$$

or $$\sum_{r=n}^{2n} \frac{1}{r} \geqslant \log \frac{2n+1}{n} \geqslant \sum_{r=n}^{2n} \frac{1}{r} - \frac{1}{n} + \frac{1}{2n+1}.$$

Hence $$0 \geqslant \log \frac{2n+1}{n} - \sum_{r=n}^{2n} \frac{1}{r} \geqslant \frac{1}{2n+1} - \frac{1}{n}.$$

Letting $n \to \infty$ it follows that

$$\lim_{n \to \infty} \left[\log \frac{2n+1}{n} - \sum_{r=n}^{2n} \frac{1}{r} \right] = 0.$$

Thus $\lim\limits_{n \to \infty} \sum\limits_{r=n}^{2n} \frac{1}{r} = \lim\limits_{n \to \infty} \log \frac{2n+1}{n}$, provided the latter limit exists. As $n \to \infty$, $(2n + 1)/n \to 2$ and hence $\lim\limits_{n \to \infty} \sum\limits_{r=n}^{2n} \frac{1}{r} = \log 2.$

In applying the Maclaurin integral test it is necessary to make use of the property that a function $f(x)$ is monotonic decreasing. If the function is differentiable a simple method which may frequently be used to decide this point is the consideration of the sign of the differential coefficient. Thus if $f'(x)$ denote the differential coefficient of $f(x)$ then $f'(x) \leqslant 0$ for $x \geqslant a$ is sufficient to ensure that $f(x)$ is monotonic decreasing for $x \geqslant a$.

Thus, *e.g.* in Ex. 3, $f(x) = 1/(k + x)^2$, $f'(x) = -2/(k + x)^3 < 0$ for $k > 0$, $x \geqslant 0$.

7·3. Kummer's Test

Let Σu_n, Σd_n^{-1} denote two series of positive terms, the latter series Σd_n^{-1} being divergent, and write $v_n = u_n d_n / u_{n+1} - d_{n+1}$. *If there exists a number p such that for all $n \geqslant p$, $v_n \geqslant k > 0$ then the series Σu_n converges. If there exists a number q such that for all $n \geqslant q$, $v_n \leqslant l < 0$, then the series Σu_n diverges.*

(i) $u_n d_n/u_{n+1} - d_{n+1} \geqslant k$, all $n \geqslant p$. Then

$$u_p d_p - u_{p+1} d_{p+1} \geqslant k\, u_{p+1}$$
$$u_{p+1} d_{p+1} - u_{p+2} d_{p+2} \geqslant k\, u_{p+2}$$
$$u_{p+2} d_{p+2} - u_{p+3} d_{p+3} \geqslant k\, u_{p+3}$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$u_{\nu-1} d_{\nu-1} - u_\nu d_\nu \geqslant k\, u_\nu.$$

Adding $u_p d_p - u_\nu d_\nu \geqslant k\,(u_{p+1} + u_{p+2} + \ldots + u_\nu)$.

Since $u_\nu d_\nu > 0$, $k > 0$ it follows that

$$u_{p+1} + u_{p+2} + \ldots + u_\nu < u_p d_p/k.$$

Since $u_p d_p$ is independent of ν, and all the terms are positive, it follows that

$$\lim_{\nu \to \infty} (u_{p+1} + u_{p+2} + \ldots + u_\nu)$$

exists and is less than or equal to $u_p d_p/k$.

Hence the series Σu_n converges, for the addition of a finite number of terms will not affect convergence.

(ii) $u_n d_n/u_{n+1} - d_{n+1} \leqslant l < 0$, $n \geqslant q$. It follows that

$$u_n d_n < u_{n+1} d_{n+1},\ n \geqslant q.$$

$$\text{Thus } u_q d_q < u_{q+1} d_{q+1} < \ldots < u_\nu d_\nu,$$

$$\text{i.e. } u_\nu > u_q d_q d_\nu^{-1}.$$

Since $u_q d_q$ is a constant and $\Sigma\, d_\nu^{-1}$ diverges it follows that $\Sigma\, u_\nu$ diverges.

A particular case occurs when $\lim_{n \to \infty} v_n$ exists. *If the limit be λ then the series Σu_n converges if $\lambda > 0$ and diverges if $\lambda < 0$.*

7·41. Ratio Tests

From the results of § 7·3 may be deduced a number of special tests which are of frequent use in the discussion of the convergence of series. We assume in the statements given below that $\lim_{n \to \infty} v_n$ exists. If this condition is not satisfied the statements are readily modified accordingly.

D'ALEMBERT'S TEST.—Write $d_n = 1$ so that $v_n = u_n/u_{n+1} - 1$. Then Σu_n converges if $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} > 1$, and diverges if $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} < 1$. (*cf.* Chap. I., § 1·381.)

7·42. Raabe's Test

Write $d_n = n$ so that $v_n = nu_n/u_{n+1} - n - 1$. Then the series Σu_n converges if $\lim\limits_{n \to \infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) > 1$, and diverges if $\lim\limits_{n \to \infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) < 1$. This test should be applied when D'Alembert's test fails.

Example.—*Prove that the series*

$$1 + \frac{1}{2} \cdot \frac{a}{b} + \frac{1.3}{2.4} \cdot \frac{a(a+1)}{b(b+1)} + \frac{1.3.5}{2.4.6} \cdot \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$$

is convergent if $a > 0$, $b > 0$ *and* $b > a + \frac{1}{2}$.

Let u_n denote the nth term of the series. Then for $n > 1$

$$u_n = \frac{1.3.5 \ldots (2n-3)}{2.4.6 \ldots (2n-2)} \cdot \frac{a(a+1)(a+2) \ldots (a+n-2)}{b(b+1)(b+2) \ldots (b+n-2)}.$$

$$\text{Hence } \frac{u_n}{u_{n+1}} = \frac{2n(n-1+b)}{(2n-1)(n-1+a)}.$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} \quad \lim_{n \to \infty} \frac{1 + (b-1)/n}{(1 - 1/2n)\{1 + (a-1)/n\}}$$

Thus D'Alembert's test fails.

$$\text{Again } \frac{u_n}{u_{n+1}} - 1 = \frac{(2b - 2a + 1) + (a-1)/n}{n(2 - 1/n)\{1 + (a-1)/n\}}$$

$$\text{Hence } \lim_{n \to \infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = b - a + \tfrac{1}{2}.$$

Thus the series will converge if $b - a + \frac{1}{2} > 1$, i.e. $b > a + \frac{1}{2}$.

7·43. de Morgan's and Betrand's Test

If the limits used in §§ 7·41, 7·42 are both unity it is necessary to use a more delicate test, *i.e.* we take $1/d_n$ to be the nth term of a series which diverges more slowly than $\Sigma 1/n$. Thus, *e.g.* if we take $d_n = n \log n$, $v_n = u_n n \log n/u_{n+1} - (n+1)\log(n+1)$.

We can deduce the following test. *If* u_n/u_{n+1} *be expressed in*

the form $1 + 1/n + w_n/n \log n$ *then the series will converge or diverge according as* $\lim\limits_{n \to \infty} w_n$ *is greater than or less than unity.*

This test may be expressed in a slightly different form which is sometimes useful. Taking logarithms:

$$\log \frac{u_n}{u_{n+1}} = \log \left(1 + \frac{1}{n} + \frac{w_n}{n \log n}\right.$$

$$= \frac{1}{n} + \frac{w_n}{n \log n} + \text{higher powers of } \frac{1}{n}.$$

$$= \frac{1}{n} + \frac{\rho_n}{n \log n},$$

where $\lim\limits_{n \to \infty} \rho_n = \lim\limits_{n \to \infty} w_n$, the limit being assumed to exist. *Hence if* $\log (u_n/u_{n+1})$ *be expressed in the form* $1/n + \rho_n/n \log n$ *the series will converge or diverge according as* $\lim\limits_{n \to \infty} \rho_n$ *is greater than or less than unity.*

Alternatively we may express the conditions for convergence in the following forms:

$$(a) \quad \lim_{n \to \infty} \left[\log n \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \right] > 1.$$

$$(b) \quad \lim_{n \to \infty} \left[\left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right] > 1.$$

Examples.—(1) *Investigate the convergence of the series* $\sum\limits_{n=1}^{\infty} a_n x^n$ *for positive values of* x, *where* $a_n = \{1^2.3^2.5^2 \ldots (2n-1)^2\}/\{2^2.4^2.6^2 \ldots (2n)^2\}$.

$$\text{Write } u_n = a_n x^n. \text{ Then } \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x},$$

$$\text{Hence } \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Thus the series converges if $x < 1$ and diverges if $x > 1$.

$$\text{When } x = 1, \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{(2n+1+1)^2}{(2n+1)^2},$$

$$\text{i.e. } \frac{u_n}{u_{n+1}} = 1 + \frac{2}{2n+1} + \frac{1}{(2n+1)^2}.$$

Hence $\lim\limits_{n \to \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = 1$ and Raabe's test fails.

Again $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{1}{n+\frac{1}{2}} + \dfrac{1}{(2n+1)^2}$

$$= 1 + \frac{1}{n}\left(1 + \frac{1}{2n}\right)^{-1} + \frac{1}{4n^2}\left(1 + \frac{1}{2n}\right)^{-2}$$

$$= 1 + \frac{1}{n}\left(1 - \frac{1}{2n} + \frac{1}{4n^2} - \dots\right) + \frac{1}{4n^2}\left(1 - \frac{2}{2n} + \dots\right)$$

i.e. $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{1}{n} - \dfrac{1}{4n^2}(1 + \lambda_n)$

where λ_n involves $1/n$ as a factor and tends to zero as $n \to \infty$. Thus

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{n \log n}\left\{-\frac{1}{4n}(1 + \lambda_n)\log n\right\}.$$

In the notation of § 7·43, $w_n = -\dfrac{\log n}{n} \cdot \dfrac{1 + \lambda_n}{4}$.

Now $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0$ (Chap. V., § 5·3). Hence $\lim w_n = 0$ and the series diverges when $x = 1$.

(2) *Discuss the convergence of the series* $\sum\limits_{n=1}^{\infty} n^n x^n/n!$, *for positive values of* x.

Let u_n denote the nth term of the series. Then

$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \cdot \frac{1}{x}.$$

Thus $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{ex}$.

Hence the series converges for $x < 1/e$ and diverges when $x > 1/e$.

When $x = 1/e$, $\dfrac{u_n}{u_{n+1}} = \dfrac{e}{\left(1 + \frac{1}{n}\right)^n}$.

$$\log \frac{u_n}{u_{n+1}} = 1 - n \log\left(1 + \frac{1}{n}\right) = 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \text{higher powers of } \frac{1}{n}\right)$$

$$= \frac{1}{2n} + O\left(\frac{1}{n^2}\right).$$

$$n \log \frac{u_n}{u_{n-1}} - 1 = -\tfrac{1}{2} + O\left(\frac{1}{n}\right).$$

Hence $\lim\limits_{n \to \infty} \left(n \log \dfrac{u_n}{u_{n-1}} - 1\right)\log n \Big] = -\infty$.

Thus the series diverges when $x = 1/e$.

7·44. The Rule of Gauss

Let Σu_n denote a series of positive terms. Then if u_n/u_{n+1} *can be expressed in the form* $1 + \mu/n + O(1/n^p)$ where $p > 1$ *then the series* Σu_n *converges if* $\mu > 1$ *and diverges if* $\mu \leqslant 1$.

Suppose first that $\mu \neq 1$. Then $\lim\limits_{n \to \infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = \mu,$ since $p > 1$. From Raabe's test it follows that the series converges if $\mu > 1$ and diverges if $\mu < 1$.

Next suppose that $\mu = 1$ so that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + O\left(\frac{1}{n^p}\right).$$

Then $\left\{ n\left(\frac{u_n}{u_{n-1}} - 1\right) - 1 \right\} \log n = (\log n)\, O\left(\frac{1}{n^{p-1}}\right).$

Now $\log n = o(n^\delta)$, where δ is any positive number. (Chap. V., § 5·3.) Since $p - 1 > 0$ it follows that

$$\lim_{n \to \infty} \{(\log n)\, O(1/n^{p-1})\} = 0.$$

Hence from § 7·43(a) the series diverges when $\mu = 1$.

Example.—*Find for what values of p the series whose nth term is*

$$\left\{\frac{1.5.9 \ldots (4n+1)}{3.7.11 \ldots (4n+3)}\right\}^p \quad \textit{converges.}$$

If u_n denotes the nth term, then

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \left(\frac{4n+7}{4n+5}\right)^p = \left(1 + \frac{7}{4n}\right)^p \left(1 + \frac{5}{4n}\right)^{-p} \\ &= \left\{1 + \frac{7p}{4n} + O\left(\frac{1}{n^2}\right)\right\} \left\{1 - \frac{5p}{4n} + O\left(\frac{1}{n^2}\right)\right\} \\ &= 1 + \frac{p}{2n} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Hence the series converges if $p > 2$ and diverges if $p \leqslant 2$.

7·45. Note on the Ratio Tests

It should be observed that *d'Alembert's test*, § 7·41, *does not imply convergence if all we know is that* $u_n/u_{n+1} > 1$ *for all values of* n. Thus consider, *e.g.* the harmonic series $\Sigma\, 1/n$. Then

$$u_n/u_{n+1} = (n+1)/n = 1 + \frac{1}{n} > 1$$

for all n. But the series diverges.

Again, *the convergence of a series of positive terms does not imply the existence of the limit* u_n/u_{n+1} *as* $n \to \infty$. For since the terms of the series are all *positive* the series is *absolutely* convergent. Thus the order of the terms may be changed without affecting the convergent property. (Chap. I., § 1·34.) This rearrangement of the terms will in general, however, affect the value of u_n/u_{n+1}.

Consider, *e.g.* the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$ For this series $u_n/u_{n+1} = 2$ so that $\lim\limits_{n \to \infty} (u_n/u_{n+1}) = 2$.

Now rearrange the terms as follows:

$$\frac{1}{2} + 1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^6} + \ldots$$

In this case $u_n/u_{n+1} = \frac{1}{2}$ or 8 according as n is odd or even. Thus $\lim\limits_{n \to \infty} (u_n/u_{n+1})$ does not exist.

7·5. Series whose Terms are Positive and Negative

In Chapter I., § 1·5, the case in which the terms are *alternately positive and negative* has been considered. It has been shown that if $u_n > 0$ and $u_n \geqslant u_{n+1}$, so that $\Sigma(-1)^{n-1}u_n$ is a series whose terms are alternately positive and negative, then the series will converge provided $\lim\limits_{n \to \infty} u_n = 0$. It is clear that the same result holds if there exists a term of the series *beyond which* the properties are true.

A convenient form of expressing the test is as follows. *If* u_n/u_{n+1} *can be expressed in the form*

$$\frac{}{u_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right), \quad p > 1$$

then the series $\Sigma(-1)^{n-1} u_n$ *is convergent if* $\mu > 0$ *and oscillatory if* $\mu \leqslant 0$.

We first prove that if $\frac{u_n}{u_{n+1}} = 1 + \frac{\rho_n}{n}$ where $\rho_n \to \rho > 0$, as $n \to \infty$, then $\lim\limits_{n \to \infty} u_n = 0$.

Since $\rho_n \to \rho$ we can find a number m such that

$$\text{for all } n \geqslant m, \ \rho_n > \tfrac{1}{2}\rho.$$

$$\text{Hence } \frac{u_m}{u_{m+1}} > \left(1 + \frac{\rho}{2m}\right), \ \frac{u_{m+1}}{u_{m+2}} > \left(1 + \frac{\rho}{2(m+1)}\right) \ldots$$

$$> 1 + \frac{}{2(n-1)}.$$

Multiplying the corresponding sides of the inequalities, we get

$$\frac{u_m}{u_n} > \left(1 + \frac{\rho}{2m}\right)\left(1 + \frac{\rho}{2m+2}\right) \ldots \left(1 + \frac{\rho}{2n-2}\right)$$

$$> 1 + \tfrac{1}{2}\rho\left(\frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{n-1}\right).$$

Since the series $\Sigma\, 1/n$ diverges to $+\infty$ it follows that

$$\lim_{n\to\infty}\left(\frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{n-1}\right) = \infty.$$

Hence $\dfrac{u_m}{u_n} \to \infty$ as $n \to \infty$. Since u_m is a fixed number, this can only be the case if $u_n \to 0$ as $n \to \infty$.

We can now show that if $\mu > 0$ the series converges. For n sufficiently great $\dfrac{\mu}{n} > O\left(\dfrac{1}{n^p}\right)$ and hence $u_n > u_{n+1}$. Further it follows from the argument just given that $\lim\limits_{n\to\infty} u_n = 0$. Hence the series converges.

Next suppose that $\mu \leqslant 0$. We can show in this case that $\lim\limits_{n\to\infty} u_n$ is not zero, so that the series must oscillate.

If $\mu < 0$ it is clear that for n sufficiently great, $u_n < u_{n+1}$ and so u_n cannot tend to zero.

Finally suppose that $\mu = 0$. Then

$$\frac{u_n}{u_{n+1}} = 1 + O\left(\frac{1}{n^p}\right), \text{ i.e. } \frac{u_n}{u_{n+1}} = 1 + \frac{\lambda_n}{n^p},$$

where $|\lambda_n| < k$, k being a positive constant, *i.e* is independent of n.

Let m be a fixed value of n. Then arguing as before we see that if $n > m$,

$$\frac{u_m}{u_n} = \left\{1 + \frac{\lambda_m}{m^p}\right\}\left\{1 + \frac{\lambda_{m+1}}{(m+1)^p}\right\}\left\{1 + \frac{\lambda_{m+2}}{(m+2)^p}\right\} \ldots \left\{1 + \frac{\lambda_{n-1}}{(n-1)^p}\right\};$$

$$\therefore\ \frac{u_m}{u_n} < \left\{1 + \frac{k}{m^p}\right\}\left\{1 + \frac{k}{(m+1)^p}\right\}\left\{1 + \frac{k}{(m+2)^p}\right\} \ldots \left\{1 + \frac{k}{(n-1)^p}\right\},$$

$$\text{i.e. } \frac{u_m}{u_n} < \prod_{r=m}^{n-1}\left\{1 + \frac{k}{r^p}\right\}.$$

Now if a be any positive number less than unity,

$$1 + a = (1 - a^2)/(1 - a) < 1/(1 - a).$$

Thus if m is sufficiently great

$$\prod_{r=m}^{n-1} (1 + k/r^p) < \prod_{r=m}^{n-1} \{1 - k/r^p\}^{-1}.$$

Again, if b is another positive number less than unity

$$(1 - a)(1 - b) = 1 - (a + b) + ab > 1 - (a + b).$$

It follows that $\prod_{r=m}^{n-1} (1 - k/r^p) > 1 - \sum_{r=m}^{n-1} k/r^p$. Hence

$$\prod_{r=m}^{n-1} \{1 - k/r^p\}^{-1} < 1/\{1 - \sum_{r=m}^{n-1} k/r^p\}, \text{ provided}$$

$$\sum_{r=m}^{n-1} k/r^p < 1.$$

Since $\Sigma\, 1/r^p$ is a convergent series we can find a number m such that $\sum_{r=m}^{\infty} 1/r^p < 1/2k$. Hence by a suitable choice of m $\prod_{r=m}^{n-1} \{1 - k/r^p\}^{-1} < 2$. Thus with this value of m,

$$\frac{u_m}{u_n} < 2.$$

Since u_m/u_n is always positive, and u_m is fixed it follows that u_m/u_n is finite. Thus u_n cannot tend to zero as $n \to \infty$.

It follows that if $\dfrac{u_n}{u_{n+1}} = 1 + O\left(\dfrac{1}{n^p}\right)$, the series $\Sigma\,(-1)^{n-1}u_n$ must oscillate.

7·51. The Hypergeometric Series

The series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\,\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots,$$

is known as the **hypergeometric** series.

If α or β are negative integers the series terminates so that there are only a finite number of terms. If γ is a negative integer all terms would be infinite beyond a certain point. We assume then that α, β, γ are not negative integers so that the series contains an infinite number of terms. Then we have the following properties:

(i) *the series converges absolutely if* $|x| < 1$ *and diverges if* $|x| > 1$;

(ii) *when* $x = 1$, *the series converges if, and only if,* $\gamma > \alpha + \beta$;

(iii) *when* $x = -1$, *the series converges if, and only if,*

$$\gamma + 1 > \alpha + \beta.$$

Let u_n denote the nth term of the series. Then u_{n+1} is

$$\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)\,\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)}{1.2.3\ldots n.\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}x^n,$$

(i) $$\frac{u_n}{u_{n+1}} = \frac{n(\gamma+n-1)}{(\alpha+n-1)(\beta+n-1)}\cdot\frac{1}{x}$$

$$= \frac{1 + \dfrac{\gamma-1}{n}}{\left(1+\dfrac{\alpha-1}{n}\right)\left(1+\dfrac{\beta-1}{n}\right)}\cdot\frac{1}{x}.$$

Hence $$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \frac{1}{|x|}.$$

Thus the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$.

(ii) $x = 1$. Then $$\frac{u_n}{u_{n+1}} = \frac{n(n+\gamma-1)}{(n+\alpha-1)(n+\beta-1)}.$$

For n sufficiently great, it is clear that this ratio is always positive, so that beyond a certain point in the series all the terms will have the same sign.

Now $$\frac{u_n}{u_{n+1}} = \left(1+\frac{\gamma-1}{n}\right)\left(1+\frac{\alpha-1}{n}\right)^{-1}\left(1+\frac{\beta-1}{n}\right)^{-1}$$

$$= \left(1+\frac{\gamma-1}{n}\right)\left\{1-\frac{\alpha-1}{n}+O\left(\frac{1}{n^2}\right)\right\}\left\{1-\frac{\beta-1}{n}+O\left(\frac{1}{n^2}\right)\right\}$$

$$= \left(1+\frac{\gamma-1}{n}\right)\left\{1-\frac{\alpha+\beta-2}{n}+O\left(\frac{1}{n^2}\right)\right\}$$

$$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + O\left(\frac{1}{n^2}\right).$$

From the rule of Gauss it follows that the series will converge if, and only if, $\gamma + 1 - \alpha - \beta > 1$, i.e. $\gamma > \alpha + \beta$.

(iii) $x = -1$. In this case $\dfrac{u_n}{u_{n+1}} \to -1$ as $n \to \infty$ so that beyond a certain point the terms are alternately positive and negative. Then

$$\left|\frac{u_n}{u_{n+1}}\right| = 1 + \frac{\gamma + 1 - \alpha - \beta}{n} + O\left(\frac{1}{n^2}\right).$$

The series will converge if

$$\gamma + 1 - \alpha - \beta > 0, \text{ i.e. } \gamma + 1 > \alpha + \beta$$

and will oscillate if $\gamma + 1 \leqslant \alpha + \beta$. (§ 7·5.)

7·61. Abel's Lemma

Let $\{u_r\}$ *denote a sequence of positive numbers such that* $u_r \geqslant u_{r+1}$ *for all values of* r. *Suppose, further, that* $a_1, a_2, \ldots$ *is a sequence of real numbers such that*

$$k < \sum_{r=1}^{p} a_r < K,$$

where $p = 1, 2, 3, \ldots n$. *Then*

$$ku_1 < \sum_{r=1}^{n} a_r u_r < Ku_1.$$

Writing $s_p = \sum_{r=1}^{p} a_r$. Then

$$\sum_{r=1}^{n} a_r u_r = \sum_{r=1}^{n} (s_r - s_{r-1}) u_r, \quad s_0 = 0,$$

$$= s_1 u_1 + (s_2 - s_1) u_2 + (s_3 - s_2) u_3 + \ldots + (s_n - s_{n-1}) u_n$$

$$= s_1 (u_1 - u_2) + s_2 (u_2 - u_3) + \ldots + s_{n-1} (u_{n-1} - u_n) + s_n u_n.$$

Now $u_r - u_{r+1} \geqslant 0$, for all values of r. Hence the greatest value of $\sum_{r=1}^{n} a_r u_r$ will be obtained by replacing s_r by K, for each r. Similarly the least value will be obtained by replacing s_r by its least value k. It follows that

$$ku_1 < \sum_{r=1} a_r u_r < Ku_1.$$

Writing H equal to the greater of $|k|$ and $|K|$ it follows that

$$\left|\sum_{r=1}^{n} a_r u_r\right| < Hu_1, \text{ where } \left|\sum_{r=1}^{p} a_r\right| < H,\ p = 1, 2, 3, \ldots n.$$

7·62. Abel's Test

This is a comparison test for a series *which need not converge absolutely. Let* Σa_n *be a convergent series,* $\{v_n\}$ *a monotonic sequence such that* $|v_n| < k$, *a fixed positive constant, for all* n. *Then* $\Sigma a_n v_n$ *converges.*

Since $\{v_n\}$ is a *bounded monotonic* sequence, $\lim\limits_{n \to \infty} v_n$ exists (Chap. I., § 1·23). Let the limit be denoted by v. Then if $\{v_n\}$ is monotonic increasing, write $u_n = v - v_n$, while if $\{v_n\}$ is monotonic decreasing write $u_n = v_n - v$. In each case $\{u_n\}$ is a sequence of terms such that $u_n \geqslant u_{n+1}$, for all n. Further, $\lim\limits_{n \to \infty} u_n = 0$.

Now $\Sigma a_n v_n = \Sigma a_n (v - u_n)$ or $= \Sigma a_n (u_n + v)$ according as $\{v_n\}$ is monotonic increasing or decreasing. Also

$$\Sigma a_n v = v \, \Sigma a_n,$$

which is convergent since Σa_n is convergent. Thus in both cases it is sufficient to consider the convergence of the series $\Sigma a_n u_n$.

To prove that this series is convergent it is sufficient to show that there exists an integer $\nu(\epsilon)$ such that the partial sum

$$\left|\sum_{n=\nu+1}^{\nu+p} a_n u_n\right| < \epsilon,$$

where p is any positive integer. (Chap. I., § 1·32.)

Let H denote the greatest of the sums

$$\left|\sum_{n=\nu+1}^{\nu+m} a_n\right|, \quad m = 1, 2, 3, \ldots p.$$

Then from Abel's lemma,

$$\left|\sum_{n=\nu+1}^{\nu+p} a_n u_n\right| < Hu_{\nu+1} < Hu_1.$$

Now let ϵ be any positive number. Then there exists an integer ν, depending on ϵ such that

$$\left|\sum_{n=\nu+1}^{\nu+p} a_n\right| < \epsilon/u_1,$$

for every positive integer p.

Hence $H < \epsilon/u_1$, i.e. $Hu_1 < \epsilon$. Thus $\left|\sum_{n=\nu+1}^{\nu+p} a_n u_n\right| < \epsilon$, for every positive integer p and the result follows.

7·7. Dirichlet's Test

If the series Σa_n converges or oscillates between finite limits, and $\{u_n\}$ is a monotonic decreasing sequence which tends to zero as n tends to infinity, then the series $\Sigma a_n u_n$ converges.

Since Σa_n always lies between finite limits there exists a constant H such that

$$\left|\sum_{n=\nu+1}^{\nu+p} a_n\right| < H,$$

for all positive integral values of ν and p.

From Abel's lemma it follows that $\left|\sum_{n=\nu+1}^{\nu+p} a_n u_n\right| < Hu_\nu$

Since $\lim. \ u_n = 0$, there exists a positive integer ν depending on the arbitrary positive number ϵ such that $u_{\nu+1} < \epsilon/H$. Thus

$$\left|\sum_{n=\nu+1}^{\nu+p} a_n u_n\right| < \epsilon,$$

for every positive integer p. Thus the series $\Sigma a_n u_n$ converges.

Examples.—(1) Consider the series $\sum_{n=1}^{\infty} (-1)^{n-1}$. The sum oscillates between 1 and 0. Hence if u_n denote any monotonic sequence of numbers which tend to zero as a limit the series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges. (Compare § 7·5.)

(2) Consider the series $\Sigma u_n \cos n\theta$, $\Sigma u_n \sin n\theta$, where u_n is a monotonic decreasing sequence such that $\lim_{n \to \infty} u_n = 0$. Then both series will converge provided

$$\sum_{n=\nu+1}^{\nu+p} \cos n\theta, \qquad \sum_{n=\nu+1}^{\nu+p} \sin n\theta$$

are finite values for all values of p.

Consider the first sum and denote it by ρ. Then

$$2\rho \sin \tfrac{1}{2}\theta = \sum_{n=\nu+1}^{\nu+p} 2 \sin \tfrac{1}{2}\theta \cos n\theta$$

$$= \sum_{n=\nu+1}^{\nu+p} \{\sin (n + \tfrac{1}{2})\theta - \sin (n - \tfrac{1}{2})\theta\}$$

$$= \{\sin (\nu + \tfrac{3}{2})\theta - \sin (\nu + \tfrac{1}{2})\theta\}$$
$$+ \{\sin (\nu + \tfrac{5}{2})\theta - \sin (\nu + \tfrac{3}{2})\theta\} + \ldots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \{\sin (\nu + p + \tfrac{1}{2})\theta - \sin (\nu + p - \tfrac{1}{2})\theta\}$$
$$= \sin (\nu + p + \tfrac{1}{2})\theta - \sin (\nu + \tfrac{1}{2})\theta$$
$$= 2 \cos \{\nu + \tfrac{1}{2}(p + 1)\}\theta . \sin \tfrac{1}{2}p\theta.$$

Hence $\rho = \cos \{\nu + \frac{1}{2}(p + 1)\}\theta . \sin \frac{1}{2}p\theta . \operatorname{cosec} \frac{1}{2}\theta.$

Similarly, $\displaystyle\sum_{n=\nu}^{\nu+p} \sin n\theta = \sin \{\nu + \tfrac{1}{2}(p + 1)\}\theta . \sin \tfrac{1}{2}p\theta . \operatorname{cosec}$

Now these sums are bounded provided $\operatorname{cosec} \frac{1}{2}\theta$ is bounded, *i.e.* θ is not 0 or a multiple of 2π. Since the sine and cosine cannot exceed unity, θ being real, it follows that

$$\sum_{n=\nu+1}^{\nu+p} \cos n\theta \qquad \sum_{n=\nu+1}^{\nu+p} \sin n\theta$$

cannot exceed $|\operatorname{cosec} \frac{1}{2}\theta|$. Hence if $\theta \neq 0$ or $2m\pi$, m being an integer, the series $\Sigma u_n \cos n\theta$, $\Sigma u_n \sin n\theta$ are both convergent.

When $\theta = 0$ or $2m\pi$ the first series becomes Σu_n, so that convergence will depend on the form of u_n. The second series becomes $0 + 0 + 0 + \ldots$ which converges to zero.

In particular, it follows that the series $\Sigma \frac{1}{n} \cos n\theta$, $\Sigma \frac{1}{n} \sin n\theta$ are both convergent for all values of θ, with the exception that the first series diverges when $\theta = 0$ or a multiple of 2π.

(3) *Prove that the series* $\Sigma \frac{1}{n} \cos n\theta$, $\Sigma \frac{1}{n} \sin n\theta$ *are never absolutely convergent except in one special case.*

Consider the first series. In order to discuss absolute convergence it is necessary to take the series $\Sigma \frac{1}{n} |\cos n\theta|$

$$\text{Now} \quad |\cos n\theta| \geqslant \cos^2 n\theta = \tfrac{1}{2}(1 + \cos 2n\theta).$$

Let s_p denote the sum to p terms of the series $\Sigma \frac{1}{n} |\cos n\theta|$. Then

$$s_p > \sum^{p} \frac{1}{2n}(1 + \cos 2n\theta)$$

$$\sum_{n=1}^{p} \frac{1}{n} + \sum_{n=1}^{p} \frac{1}{2n} \cos 2n\theta.$$

Now the series $\sum_{n=1}^{\infty} \frac{1}{2n} \cos 2n\theta$ converges for all values of θ, except $\theta = 0$ of a multiple of π. Hence with these exceptions

$$\sum \frac{1}{\cdot} \cos 2n\theta < K,$$

for all values of p, where K is a fixed positive constant. Thus

$$s_p > \tfrac{1}{2} \sum_{n=1}^{p} \frac{1}{n} - K.$$

But as $p \to \infty$, $\sum_{n=1}^{p} \frac{1}{n} \to \infty$. Hence $s_p \to \infty$ as $p \to \infty$.

If $\theta = 0$ or a multiple of π the series $\Sigma \frac{1}{n} \mid \cos n\theta \mid$ becomes $\Sigma \frac{1}{n}$ which is divergent. Thus $\Sigma \frac{1}{n} \mid \cos n\theta \mid$ diverges for all values of θ, i.e. $\Sigma \frac{1}{n} \cos n\theta$ is never absolutely convergent.

The series $\Sigma \frac{1}{n} \mid \sin n\theta \mid$ may be discussed in a similar way. But when $\theta = 0$ or a multiple of π, the series $\Sigma \frac{1}{n} \mid \sin n\theta \mid$ becomes $0 + 0 + 0 + \ldots$ which converges to the sum zero. Hence the sine series is never absolutely convergent except in the trivial case in which $\theta = 0$ or a multiple of π.

7·8. Uniform Convergence

In Chapter IV., the fundamental ideas on uniform convergence have been introduced. We now consider further tests for uniform convergence which are more delicate than the M-test considered in Chap. IV., § 4·33. This test requires the series to be absolutely convergent, a property which need not be satisfied in the case of the tests considered in §§ 7·81, 7·82.

We first introduce the notion of a function tending to a limit uniformly. Let $f_n(x)$ be a function of the variable x and suppose that *for each value* of x in the range $a \leqslant x \leqslant b$, $f_n(x) \to f(x)$ as $n \to \infty$. Then $f_n(x)$ tends to $f(x)$ **uniformly** in the interval $a \leqslant x \leqslant b$ if corresponding to the arbitrary positive ϵ, there exists a number n_0, depending on ϵ but independent of x such that

$$|f_n(x) - f(x)| < \epsilon, \quad n \geqslant n_0.$$

The point to be observed is that the fixed number n_0 is to be the ***same for all values of x in the interval.***

7·81. Abel's Test for Uniform Convergence

Let $\sum_{n=1}^{\infty} a_n(x)$ *be a series which converges uniformly in the interval* $a \leqslant x \leqslant b$. *Suppose also that for each* x *in* (a, b), $u_n(x)$ *is a positive monotonic decreasing function of* n *and that* $u_1(x) < k$, *for all* x *in* (a, b), *where* k *is fixed number. Then the series* $\Sigma a_n(x)\, u_n(x)$ *converges uniformly in* (a, b).

Since $\Sigma a_n(x)$ converges uniformly there exists an integer ν independent of x such that

$$\left|\sum_{n=\nu+1}^{\nu+p} a_n(x)\right| < \epsilon/k,$$

for every positive integer p. From Abel's lemma it follows that

$$\left|\sum_{n=\nu+1}^{p} a_n(x)\, u_n(x)\right| \leqslant u_{\nu+1}(x)\,.\,\epsilon/k < u_1(x)\,.\,\epsilon/k < \epsilon.$$

Since ν is independent of x, it follows that $\Sigma a_n(x)\, u_n(x)$ converges uniformly in (a, b).

Note 1.—In this test $\Sigma a_n(x)$ need not converge absolutely. If it did the result would follow immediately from the M-test.

Note 2.—The most important cases occur when either $a_n(x)$ or $u_n(x)$ is independent of x.

Abel's theorem on power series, stated on p. 154, is an immediate deduction. Thus suppose the interval of convergence of $\Sigma a_n x^n$ is $(-\rho, \rho)$, and that the series is convergent at one of the end points, say $x = \rho$. Then $\Sigma a_n \rho^n$ is convergent.

In the above test write $a_n(x) = a_n\rho^n$, $u_n(x) = (x/\rho)^n$, $0 \leqslant x \leqslant \rho$. Then $\Sigma a_n(x)\, u_n(x) = \Sigma a_n x^n$ and the series converges uniformly in $0 \leqslant x \leqslant \rho$.

Example.—Consider the series

$$\sum_{n=1}^{\infty} (-1)^n x^n/n^p(1 + x^n),\ p > 0,\ 0 \leqslant x \leqslant 1.$$

The series $\sum_{n=1}^{\infty} (-1)^n/n^p$ converges absolutely for $p > 1$. In this case we can deduce uniform convergence from the M-test.

For if $0 \leqslant x \leqslant 1$, $0 \leqslant x^n \leqslant 1$, so that $x^n/(1 + x^n) < 1$. Hence $|(-1)^n x^n/n^p(1 + x^n)| < 1/n^p$, and the result follows.

For $0 < p \leqslant 1$, the series $\Sigma(-1)^n/n^p$ is only conditionally convergent. But $\Sigma(-1)^n/n^p$ converges uniformly respect to x since its terms are independent of x. Thus the given series will be uniformly convergent if (i) for each x

in $0 \leqslant x \leqslant 1$, $x^n/(1 + x^n)$ is a positive monotonic decreasing function of n; (ii) $x/(1 + x) < k$, where k is a fixed number.

The condition (ii) is obviously satisfied, for we can take $k = 1$. Further, $x^n/(1 + x^n)$ is never negative in $(0, 1)$.

It remains to show that the function is a monotonic decreasing function of n. Now

$$\frac{x^n}{1 + x^n} - \frac{x^{n+1}}{1 + x^{n+1}} = \frac{x^n (1 - x)}{(1 + x^n)(1 + x^{n+1})} > 0, \quad 0 < x < 1.$$

since each factor is positive. Thus the function steadily decreases as n increases.

7·82. Dirichlet's Test for Uniform Convergence

Let (i) $\Sigma a_n(x)$ *be a series which oscillates finitely*; (ii) *for each* x *in* (a, b), $u_n(x)$ *be a positive monotonic decreasing function of* n; (iii) $u_n(x)$ *tend to zero* **uniformly** *for* $a \leqslant x \leqslant b$, *as* n *tends to infinity. Then the series* $\Sigma a_n(x)\, u_n(x)$ *converges uniformly in* (a, b).

Since $\Sigma a_n(x)$ oscillates finitely there exists a fixed number k such that

$$\left| \sum_{n=\nu+1}^{\nu+p} a_n(x) \right| < k,$$

for every positive integral value of ν and p.

Since $u_n(x) \to 0$ *uniformly*, we can find a number ν *independent of* x such that $u_\nu(x) < \epsilon/k$.

From Abel's lemma it follows that

$$\left| \sum_{n=\nu+1}^{\nu+p} a_n(x)\, u_n(x) \right| < \epsilon,$$

the same ν serving for all values of x in the interval. Thus the series $\Sigma a_n(x)\, u_n(x)$ converges uniformly in (a, b).

Example.—*Prove that the series* $\Sigma n^{-p} \sin n\theta$ *has the following properties*:

(i) *converges uniformly for all real* θ *if* $p > 1$;

(ii) *is not uniformly convergent in any interval containing* $\theta = 0$ *if* $p \leqslant 1$.

(iii) *is uniformly convergent for all* θ *such that*

$$0 < \delta \leqslant \theta \leqslant 2\pi - \delta, \text{ if } p > 0.$$

Dirichlet's test (§ 7·7, Ex. 2), shows that the series converges if $p > 0$.

(i) $p > 1$. $|n^{-p} \sin n\theta| \leqslant n^{-p}$, all real θ. Since Σn^{-p} converges, uniform convergence follows from the M-test.

(ii) $p \leqslant 1$. If the series converges uniformly then corresponding to ϵ there exists a number ν, *independent of* θ, such that

$$\left| \sum_{n=\nu+1}^{\nu+q} n^{-p} \sin n\theta \right| < \epsilon,$$

for every positive integer of q. This inequality is to be true for all θ.

Write $\theta = \pi/4\nu$, so that $n\theta = n\pi/4\nu$. Using the inequality

$$\sin \phi \geqslant 2\,\phi/\pi, \quad 0 < \phi \leqslant \tfrac{1}{2}\pi,$$

it follows that

$$\sin n\theta \geqslant n/2\nu \text{ provided } n\theta < \tfrac{1}{2}\pi, \text{ i.e. } n < 2\nu.$$

Writing $q = \nu$, we have

$$\sum_{n=\nu+1}^{\nu+q} n^{-p} \sin n\theta \quad \sum_{n=\nu+1}^{2\nu} n^{-p+1}/2\nu > \sum_{n=\nu+1}^{2\nu} 1/2\nu \text{ since } p < 1.$$
$$= \tfrac{1}{2}.$$

Since ϵ is arbitrary this contradicts the original inequality.

As ν tends to infinity the value $\theta = \pi/4\nu$ tends to zero. Hence the series is not uniformly convergent in any interval which contains $\theta = 0$.

(iii) To obtain this result we use Dirichlet's test for uniform convergence.

$$\sum_{n=\nu+1}^{\nu+p} \sin n\theta = \sin \{\nu + \tfrac{1}{2}(p+1)\}\,\theta . \sin \tfrac{1}{2}p\theta . \operatorname{cosec} \tfrac{1}{2}\theta. \quad (\S\ 7\cdot 7, \text{ Ex. } 2.)$$

$$\text{Hence} \quad \left|\sum_{n=\nu+1}^{\nu+p} \sin n\theta\right| \leqslant \operatorname{cosec} \tfrac{1}{2}\delta.$$

Further, if $p > 0$, n^{-p} is positive, a monotonic decreasing function of n which tends to zero uniformly for all values of θ (for n^{-p} is independent of θ).

Hence $\Sigma\, n^{-p} \sin n\theta$ converges uniformly in the interval

$$0 < \delta \leqslant x \leqslant 2\pi - \delta.$$

7·9. Infinite Products

Let $a_1, a_2, a_3, \ldots a_n, \ldots$ denote a sequence of real numbers, all of which are different from zero, and consider the product

$$P_n = a_1 a_2 a_3 \ldots a_n = \prod_{r=1}^{n} a_r.$$

If P_n *tends to a limit* P different from zero, *as* $n \to \infty$, *then the infinite product*

$$a_1 a_2 a_3 \ldots \textit{to } \infty = \prod_{n=1}^{\infty} a_n$$

is said to converge to the value P. That is

$$P = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \prod_{r=1}^{n} a_r.$$

If $P \to \infty$ or to $-\infty$ or to 0 the infinite product is said to *diverge*. If P_n oscillates as $n \to \infty$ the infinite product is said to oscillate.

To say that an infinite product *diverges to zero* is a matter of convention. It is introduced in order to obtain a close parallelism between infinite series and infinite products. Thus, suppose $a_n > 0$

for all values of n and that P_n converges to P. Then adopting the convention of divergence to zero it is easily seen that *the infinite product* $\Pi\, a_n$ *and the infinite series* $\Sigma \log a_n$ *behave in the same way,* i.e. *they both converge, both diverge or both oscillate.*

For $\log P_n = \sum_{r=1}^{n} \log a_r$. Hence if $P_n \to P$ different from zero, $\log P_n \to \log P$ which is finite, *i.e.* $\sum_{r=1}^{\infty} \log a_r$ is finite.

Again, if the infinite product diverges, *i.e.* tends to $\pm\infty$ or 0, $\log P_n \to \pm\infty$, since $\log 0 = -\infty$.

WEIERSTRASS' INEQUALITIES.—The following inequalities (i) and (ii) are of importance in connection with the theory of infinite products. *Let* s_n *denote the sum of the* n *numbers* $u_1, u_2, u_3, \ldots u_n$, *each of which is positive and less than unity. Then*

$$\text{(i)}\quad 1 - s_n < \prod_{r=1}^{n} (1 - u_r) < 1/(1 + s_n),$$

$$\text{(ii)}\quad 1 + s_n < \prod_{r=1}^{n} (1 + u_r) < 1/(1 - s_n),$$

where in (ii) $s_n < 1$.

Now $(1 - u_1)(1 - u_2) = 1 - (u_1 + u_2) + u_1u_2 > 1 - (u_1 + u_2)$, and since $(1 - u_3) > 0$,

$$(1 - u_1)(1 - u_2)(1 - u_3) > \{1 - (u_1 + u_2)\}(1 - u_3)$$
$$> 1 - (u_1 + u_2 + u_3).$$

Proceeding in this way we obtain $\prod_{r=1}^{n} (1 - u_r) > 1 - \sum_{r=1}^{n} u_r$.

In the same way it may be proved that

$$\prod_{r=1}^{n} (1 + u_r) > 1 + \sum_{r=1}^{n} u_r.$$

Again $1 - u_r = (1 - u_r^2)/(1 + u_r) < 1/(1 + u_r)$, $0 < u_r < 1$.

Hence $\prod_{r=1}^{n} (1 - u_r) < 1\Big/\prod_{r=1}^{n} (1 + u_r) < 1/(1 + \sum_{r=1}^{n} u_r)$.

Finally $1 + u_r = (1 - u_r^2)/(1 - u_r) < 1/(1 - u_r)$, $0 < u_r < 1$.

Hence $\prod_{r=1}^{n} (1 + u_r) < 1\Big/\prod_{r=1}^{n} (1 - u_r) < 1/(1 - \sum_{r=1}^{n} u_r)$,

provided $1 - \sum_{r=1}^{n} u_r > 0$.

In discussing infinite products it is convenient to write each factor in the form $1 + u_r$. Further, we may assume without loss of generality that $|u_r| < 1$. For it is clear that if an infinite product is to converge there can only be a finite number of factors which are numerically greater than two. Further, a finite number of factors can be suppressed without affecting convergence.

The most important cases occur when the u's all have the same sign. Thus it is convenient to distinguish two forms $\Pi (1 - u_n)$, $\Pi (1 + u_n)$ where $0 < u_n < 1$ for all values of n.

7·91. Relations between Σu_n and $\Pi (1 \pm u_n)$

If $\{u_n\}$ denote a sequence of numbers such that $0 < u_n < 1$ then the convergence of the series Σu_n is necessary and sufficient for the convergence of the infinite products $\Pi (1 + u_n)$, $\Pi (1 - u_n)$.

(i) *The condition is sufficient.* Suppose that Σu_n converges. Then there exists a number m such that $\sum_{n=m}^{\infty} u_n < 1$. Since the omission of a finite number of terms will not affect convergence we may suppose without loss of generality that the sum s of the series Σu_n is less than unity.

$$\text{Write } P_n = (1 + u_1)(1 + u_2) \ldots (1 + u_n),$$
$$Q_n = (1 - u_1)(1 - u_2) \ldots (1 - u_n).$$

Then if s_n denote the sum of the first n terms of Σu_n,

$$P_n < 1/(1 - s_n) < 1/(1 - s), \text{ since } s_n < s < 1,$$
$$Q_n > 1 - s_n > 1 - s. \quad (\S\ 7\cdot 9.)$$

Since $P_{n+1} = (1 + u_{n+1}) P_n > P_n$ it follows that P_n is a monotonic increasing function of n which is bounded above by $1/(1 - s)$. Hence $\lim_{n \to \infty} P_n$ exists and is positive. Thus the infinite product $\Pi (1 + u_n)$ converges.

Again $Q_{n+1} = (1 - u_{n+1}) Q_n < Q_n$. Hence Q_n is a monotonic decreasing function of n which is bounded below by $1 - s$. Thus $\lim_{n \to \infty} Q_n$ exists and is positive, *i.e.* the infinite product $\Pi (1 - u_n)$ converges to positive value.

Thus the convergence of Σu_n is sufficient to ensure the convergence of the two infinite products.

(ii) *The condition is necessary.* In order to prove this we assume that Σu_n diverges and deduce that under this condition the two infinite products must diverge.

Now $P_n > 1 + s_n$. (§ 7·9.) Since

$$s_n \to \infty \text{ as } n \to \infty, \; P_n \to \infty \text{ as } n \to \infty,$$

i.e. the infinite product diverges to ∞.

Again $Q_n < 1/(1 + s_n)$. (§ 7·9.) Hence $Q_n \to 0$ as $n \to \infty$, *i.e.* the infinite product diverges to zero.

Examples.—(1) *Prove that*

(i) $(1 + \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{4}) \ldots \to \infty$; (ii) $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \ldots \to 0$.

Since $\Sigma 1/n$ diverges these results follow immediately from the preceding theorem. In these cases it is easy to see the actual form of P_n and Q_n.

$$P_n = (1 + \tfrac{1}{2})(1 + \tfrac{1}{3})(1 + \tfrac{1}{4}) \ldots \left(1 + \frac{1}{n+1}\right)$$

$$= \tfrac{3}{2} \cdot \tfrac{4}{3} \cdot \tfrac{5}{4} \ldots \frac{n+2}{n+1} = \tfrac{1}{2}(n+2) \to \infty \text{ as } n \to \infty.$$

$$Q_n = \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{3}{4} \ldots \frac{n}{n+1} = 1/(n+1) \to 0 \text{ as } n \to \infty.$$

(2) *Prove that infinite product* $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots$ *converges to the value* $\frac{1}{2}$.

Since $\Sigma 1/n^2$ converges it follows that $\Pi (1 - 1/n^2)$ converges to a positive value.

Let Q_{n-1} denote the product of the first $(n - 1)$ factors. Then

$$Q_{n-1} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots \left(1 - \frac{1}{n^2}\right)$$

$$= (1 - \tfrac{1}{2})(1 + \tfrac{1}{2})(1 - \tfrac{1}{3})(1 + \tfrac{1}{3}) \ldots \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)$$

$$= \tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \tfrac{2}{3} \cdot \tfrac{4}{3} \ldots \frac{n-1}{n} \cdot \frac{n+1}{n}$$

$$= \tfrac{1}{2} \frac{n+1}{n}.$$

$$\text{Hence} \quad \lim_{n \to \infty} Q_{n-1} = \tfrac{1}{2}.$$

7·92. The Order of the Factors

It is easy to see from first principles that if $0 < u_n < 1$ *the values of the infinite products*

$$P = \prod_{n=1}^{\infty} (1 + u_n), \; Q = \prod_{n=1}^{\infty} (1 - u_n)$$

are independent of the order of the factors.

Since P_n is monotonic increasing, there exists a number ν depending on the arbitrary positive number ϵ such that

$$P > P_n > P - \epsilon \text{ all } n \geqslant \nu.$$

Let P'_m be the product of the first m factors in any rearrangement. Then there exists a number N such that P_N contains all the factors of P_n. Hence if $m \geqslant N$

$$P > P'_m > P_n > P - \epsilon.$$

$$\text{Thus } P'_m \to P \text{ as } m \to \infty.$$

Similarly for the infinite product Q.

7·93. Absolute Convergence

Let $\{u_n\}$ denote a sequence of real numbers such that $|u_n| = a_n < 1$. Then the infinite product

$$P = \prod_{n=1}^{\infty} (1 + u_n)$$

is said to converge absolutely if the infinite product

$$\prod_{n=1}^{\infty} (1 + a_n) \text{ converges.}$$

Now we know that $\prod_{n=1}^{\infty} (1 + a_n)$ will converge if and only if Σa_n, i.e. $\Sigma |u_n|$ converges. Hence a necessary and sufficient condition that $\Pi (1 + u_n)$ should converge absolutely is that Σu_n converge absolutely.

7·94. The General Principle of Convergence

In the case of infinite products this principle may be stated as follows: *Let ϵ be an arbitrary positive number. Then there exists a positive integer n_0 such that for all $n \geqslant n_0$ and for every positive integer p*

$$\left| \frac{P_{n+p}}{P_n} - 1 \right| < \epsilon.$$

Using this principle it is easily seen that *absolute convergence implies convergence.*

Using the notation of § 7·93 it follows that if the infinite product $\Pi (1 + a_n)$ converges, then there exists a number n_0 such that for all $n \geqslant n_0$

$$0 < (1 + a_{n+1})(1 + a_{n+2}) \ldots (1 + a_{n+p}) - 1 < \epsilon, \; p = 1, 2, 3, \ldots$$

$$\begin{aligned}
\text{Now } & |(1 + u_{n+1})(1 + u_{n+2}) \ldots (1 + u_{n+p}) - 1| \\
& = |\Sigma u_r u_s + \Sigma u_r u_s u_t + \ldots|,
\end{aligned}$$

on expanding the factors,

$$\begin{aligned}
& \leqslant |\Sigma u_r u_s| + |\Sigma u_r u_s u_t| + \ldots \\
& = \Sigma |u_r| |u_s| + \Sigma |u_r| |u_s| |u_t| + \ldots \\
& = \Sigma a_r a_s + \Sigma a_r a_s a_t + \ldots \\
& = (1 + a_{n+1})(1 + a_{n+2}) + \ldots (1 + a_{n+p}) - 1 \\
& < \epsilon.
\end{aligned}$$

Thus $\Pi(1 + u_n)$ converges.

7·95. Rearrangement of the Factors

If the infinite product $P = \Pi(1 + u_n)$ converges absolutely and if the factors are rearranged in any way, the resulting infinite product is absolutely convergent and converges to the value P.

Consider the infinite product $Q = \Pi(1 + v_n)$, which is obtained by a rearrangement of the factors of P, so that $u_r = v_s$, for some integral value of r and s. Since P converges absolutely so does Σu_n. But Σv_n is a rearrangement of Σu_n. Thus Σv_n converges absolutely and hence Q is absolutely convergent.

It is now necessary to show that $P = Q$. Write

$$P_n = \prod_{r=1}^{n} (1 + u_r), \quad Q_n = \prod_{r}^{n} (1 + v_r).$$

Then corresponding to any positive integral value n there exists an integer N such that all the factors of P_n are included in Q_N.

Hence $Q_N/P_n = (1 + u_p)(1 + u_q) \ldots (1 + u_t)$,

where $p, q, \ldots t$ are all greater than n.

Now let $n \to \infty$, so that $N \to \infty$. Then from the general principle of convergence applied to P it follows that

$$|(1 + u_p)(1 + u_q) \ldots (1 + u_t) - 1| \to 0, \text{ i.e. } \frac{Q_N}{P_n} \to 1.$$

But $Q_N \to Q$, $P_n \to P$ so that $P = Q$.

It is easy to see by an example that a rearrangement of the order of the factors of an infinite product which does not converge absolutely may alter the value of the infinite product. Write

$$P = (1 - \tfrac{1}{2})(1 + \tfrac{1}{3})(1 - \tfrac{1}{4})(1 + \tfrac{1}{5})(1 - \tfrac{1}{6}) \ldots,$$
$$Q = (1 - \tfrac{1}{2})(1 - \tfrac{1}{4})(1 + \tfrac{1}{3})(1 - \tfrac{1}{6})(1 - \tfrac{1}{8}) \ldots,$$

where Q is obtained from P by taking two negative signs followed by one positive positive sign in the factors.

$$\text{Then } P_{2n-1} = \tfrac{1}{2}\cdot\tfrac{4}{3}\cdot\tfrac{3}{4}\cdot\tfrac{6}{5}\cdot\tfrac{5}{6}\cdots\frac{2n}{2n-1}\cdot\frac{2n-1}{2n} = \tfrac{1}{2}$$

$$P_{2n} = \tfrac{1}{2}\cdot\tfrac{4}{3}\cdot\tfrac{3}{4}\cdot\tfrac{6}{5}\cdot\tfrac{5}{6}\cdots\frac{2n}{2n-1}\cdot\frac{2n-1}{2n}\cdot\frac{2n+2}{2n+1}$$

$$= \tfrac{1}{2}\cdot\frac{2n+2}{2n+1}\to\tfrac{1}{2} \text{ as } n\to\infty.$$

Hence $P = \frac{1}{2}$.

Again $Q_{3n} = P_{2n}\cdot\lambda_n$ where

$$\lambda_n = \left(1-\frac{1}{2n+2}\right)\left(1-\frac{1}{2n+4}\right)\cdots\left(1-\frac{1}{4n}\right).$$

Now from § 7·9, $1-\mu_n < \lambda_n < 1/(1+\mu_n)$ where

$$\mu_n = \frac{1}{2n+2}+\frac{1}{2n+4}+\dots+\frac{1}{4n} = \sum_{r=n+1}^{2n}\frac{1}{2r}.$$

$$\text{Now } \frac{1}{2n+2}\geqslant\frac{1}{2r}\geqslant\frac{1}{4n},\quad r = n+1,\ n+2,\ \dots,\ 2n.$$

$$\text{Hence } \frac{n}{2n+2} > \sum_{r=n+1}^{2n}\frac{1}{2r} > \frac{n}{4n},$$

i.e. $\frac{1}{2} > \mu_n > \frac{1}{4}$.

It follows that $\frac{1}{2} < Q_{3n}/P_{2n} < \frac{4}{5}$.

Writing $P_{2n} = \frac{1}{2}$, we see that $1 < Q_{3n} < 8/5$.

Hence $\lim\limits_{n\to\infty} Q_{3n}$ cannot be equal to P which is $\frac{1}{2}$.

Thus the rearrangement of the factors has altered the value of the infinite product.

7·96. Infinite Products with u_n Positive or Negative

THEOREM.—*If $\Sigma\, u_n^2$ is convergent then the infinite product $\Pi\,(1+u_n)$ converges if $\Sigma\, u_n$ converges, diverges to ∞ if $\Sigma\, u_n$ diverges to ∞, diverges to 0 if $\Sigma\, u_n$ diverges to $-\infty$ and oscillates if $\Sigma\, u_n$ oscillates.*

The u's may have positive or negative values, but we may suppose without loss of generality that $|u_n| < 1$.

It has been proved in Chapter V., § 5·51 that

$$0 < x - \log(1+x) < \tfrac{1}{2}x^2,\quad 0 < x < 1, \text{ and}$$

$$0 < x - \log(1+x) < \tfrac{1}{2}x^2/(1+x),\quad -1 < x < 0.$$

Now $\log P_n = \log(1 + u_1) + \log(1 + u_2) + \ldots + \log(1 + u_n)$

$$= \sum_{r=1}^{n} \log(1 + u_r).$$

Hence the infinite product P and the infinite series $\Sigma \log(1 + u_n)$ both converge, both diverge or both oscillate.

Let λ be the least value of the numbers

$$1,\ 1 + u_1,\ 1 + u_2,\ \ldots\ 1 + u_n,\ \ldots$$

Then applying the above inequalities

$$0 < u_n + u_{n+1} + \ldots + u_{n+p} - \log\{(1 + u_n)(1 + u_{n+1}) \ldots (1 + u_{n+p})\} < \tfrac{1}{2}(u_n^2 + u^2_{n+1} + \ldots + u^2_{n+p})/\lambda.$$

Since Σu_n^2 converges, there exists an integer n_0, such that for all $n \geqslant n_0$, $\sum_{r=n}^{n+p} u_r^2 < 2\lambda\epsilon$, where ϵ is an arbitrary positive number. Thus

$$0 < \sum_{r=n}^{n+p} u_r - \log\left\{\prod_{r=n}^{n+p} (1 + u_r)\right\} < \epsilon$$

and the theorem is proved.

Examples.—(1) *Prove that the infinite product* $\prod_{n=1}^{\infty}\left\{\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}}\right\}$ *is absolutely convergent for all real values of* x.

The infinite product will converge absolutely if the series

$$\sum_{n=1}^{\infty}\left\{\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}} - 1\right\}$$

converges absolutely.

$$\text{Now } \left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}} - 1 = \left(1 + \frac{x}{n}\right)\left\{1 - \frac{x}{n} + \frac{x^2}{2n^2} + O\left(\frac{1}{n^3}\right)\right\} - 1$$

$$= -\tfrac{1}{2}\frac{x^2}{n^2} + O\left(\frac{1}{n^3}\right)$$

Comparison with the series Σn^{-2} shows that the series

$$\Sigma\left\{\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}} - 1\right\}$$

converges absolutely provided x is finite. Thus the infinite product converges absolutely for all finite values of x.

Note.—The Gamma-function $\Gamma(x)$ has been defined by Weierstrass by means of this infinite product. Thus if γ denote Euler's constant, *i.e.*

$$\gamma = \lim_{n \to \infty} \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$$

then the Gamma-function is defined by the equation

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left\{\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}}\right\}.$$

It will be observed that if $x = 0$ or a negative integer one factor of the product becomes zero, *i.e.* for such a value of x, $\Gamma(x)$ is infinite.

(2) *Prove that unless x is a negative integer*

$$\lim_{n \to \infty} \frac{n^x n!}{(x + 1)(x + 2) \ldots (x + n)} \text{ is finite.}$$

The limit can be written as the infinite product

$$\left\{\frac{1}{x + 1}\right\} \left\{\frac{2}{x + 2}\cdot\left(\frac{2}{1}\right)^x\right\} \left\{\frac{3}{x + 3}\cdot\left(\frac{3}{2}\right)^x\right\} \cdots \left\{\frac{n}{x + n}\cdot\left(\frac{n}{n - 1}\right)^x\right\} \cdots$$

The nth factor is $\dfrac{n}{x + n}\cdot\left(\dfrac{n}{n - 1}\right)^x = \left(1 + \dfrac{x}{n}\right)^{-1}\left(1 - \dfrac{1}{n}\right)^{-x}$

$$= \left\{1 - \frac{x}{n} + \frac{x^2}{n^2} + O\left(\frac{1}{n^3}\right)\right\} \left\{1 + \frac{x}{n} + \frac{x(x + 1)}{2n^2} + O\left(\frac{1}{n^3}\right)\right\}$$

$$= 1 + \frac{x(x + 1)}{n^2} + O\left(\frac{1}{n^3}\right).$$

Thus the infinite product will be absolutely convergent provided

$$\Sigma\left\{\frac{x(x + 1)}{n^2} + O\left(\frac{1}{n^3}\right)\right\} \text{ is absolutely convergent.}$$

Comparison with the series Σn^{-2} shows that this is the case.

When x is a negative integer the expression does not exist because one of the factors in the denominator vanishes.

Note.—The limit considered above may be expressed in terms of the Gamma-function.

Write $F(x) = \lim\limits_{n \to \infty} \dfrac{n^x . n!}{(x + 1)(x + 2) \ldots (x + n)}$. Then

$$1/F(x) = \lim_{n \to \infty} \left\{(1 + x)\left(1 + \frac{x}{2}\right) \ldots \left(1 + \frac{x}{n}\right) n^{-x}\right\}$$

$$= \lim_{n \to \infty} \left\{(1 + x)(1 + \tfrac{1}{2}x) \ldots \left(1 + \frac{x}{n}\right)e^{-x \log n}\right\}$$

$$= \lim_{n \to \infty} \Big[\{(1 + x)e^{-x}\} \left\{(1 + \tfrac{1}{2}x)e^{-\frac{x}{2}}\right\} \ldots$$

$$\left\{\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}}\right\} . e^{x(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n)}\Big]$$

$$= \left[\lim_{n\to\infty} \left\{(1+x)e^{-x}\right\} \left\{(1+\tfrac{1}{2}x)e^{-\frac{1}{2}x}\right\} \dots \left\{\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right\}\right]$$

$$\times \left[\lim_{n\to\infty} e^{x\left(1+\frac{1}{2}\dots+\frac{1}{n}-\log n\right)}\right]$$

provided the limits exist.

Now the first limit is the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}$ which converges absolutely for all finite values of x.

The second limit is $e^{\gamma x}$, where γ is Euler's constant.

$$\text{Hence } 1/F(x) = e^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}.$$

Since $1/\Gamma(x) = xe^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}$ it follows that

$$x/F(x) = 1/\Gamma(x), \text{ i.e. } \Gamma(x) = F(x)/x.$$

$$\text{Hence } \Gamma(x) = \lim_{n\to\infty} \frac{n^x . n!}{x(x+1)(x+2)\dots(x+n)}.$$

(3) *Discuss the convergence of the infinite product*

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\dots$$

The infinite product may be written in the form

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

The series $\frac{x^2}{n^2\pi^2} = \frac{x^2}{\pi^2}\Sigma\frac{1}{n^2}$ which is absolutely convergent. Thus the infinite product converges absolutely for all finite values of x. It may be proved by trigonometry that the product converges to $(\sin x)/x$.

Again the infinite product may be written in the form

$$\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\dots$$

When written in this form the corresponding series is

$$\frac{x}{\pi}\{-1+1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}\dots\}.$$

This series is not absolutely convergent for

$$1+1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{3}+\dots$$

is divergent, as is seen by comparison with $\Sigma 1/n$.

Thus the infinite product when written in the second form is no longer absolutely convergent and its value may be altered by a rearrangement of the order of the factors.

The infinite product can, however, be made absolutely convergent again by introducing factors. Thus

$$\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right)=\left\{\left(1-\frac{x}{n\pi}\right)e^{\frac{x}{n\pi}}\right\}\left\{\left(1+\frac{x}{n\pi}\right)e^{-\frac{x}{n\pi}}\right\}$$

The general term of the product now has one of the forms

$$\left(1\mp\frac{x}{n\pi}\right)e^{\pm\frac{x}{n\pi}}=1+O\left(\frac{1}{n^2}\right). \quad \text{(See Ex. 1.)}$$

Since $\Sigma\,(1/n^2)$ is absolutely convergent, the infinite product is absolutely convergent, so the value of the function cannot now be altered by a rearrangement of the order of the factors.

7·97. A Useful Rule for Sequences

Let $\{u_n\}$ denote a sequence of positive numbers such that the ratio u_n/u_{n+1} can be expressed in the form

$$\frac{u_n}{u_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^p}\right),\ p>1.$$

Then if $\mu>0$, $u_n\to 0$ *steadily as* $n\to\infty$, *while if* $\mu\leqslant 0$, $u_n\not\to 0$.

Write $\dfrac{u_n}{u_{n+1}}=1+v_n$. We use Weierstrass's inequalities proved in § 7·9 and assume that a finite number of terms of the sequence $\{u_n\}$ may be omitted if necessary.

(i) $\mu>0$. In this case $v_n>0$ and $v_n=O\left(\dfrac{1}{n}\right)$ so that Σv_n is a divergent series of positive terms.

Now $\dfrac{u_1}{u_2}=1+v_1,\ \dfrac{u_2}{u_3}=1+v_2\ \ldots,\ \dfrac{u_{n-1}}{u_n}=1+v_{n-1}.$

Hence $\dfrac{u_1}{u_n}=\prod\limits_{r=1}^{n-1}(1+v_r)>1+\sum\limits_{r=1}^{n-1}v_r$, from § 7·9, since we may assume $v_r<1$.

Since $\sum\limits_{r=1}^{n-1}v_r\to\infty$ as $n\to\infty$, it follows that $u_1/u_n\to\infty$, and hence $u_n\to 0$. Also it is clear that for n large, $u_n>u_{n+1}$. Thus $u_n\to 0$ *steadily.*

(ii) $\mu<0$. In this case $v_n<0$ and $v_n=O\left(\dfrac{1}{n}\right)$ so that Σv_n is a divergent series of negative terms. Write $v_n=-w_n$ so that

$w_n > 0$ and Σw_n is a divergent series of positive terms. Proceeding as before

$$\frac{u_1}{u_n} = \prod_{r=1}^{n-1} (1 - w_r) < 1/(1 + \sum_{r=1}^{n-1} w_r), \text{ from § 7·9.}$$

Since $\sum_{r=1}^{n-1} w_r \to \infty$ as $n \to \infty$, it follows that $u_1/u_n \to 0$. Hence $u_n \to \infty$.

(iii) $\mu = 0$. In this case $v_n = O\left(\frac{1}{n^p}\right)$, $p > 1$ so that Σv_n is an absolutely convergent series whose terms may be positive or negative. Write $w_n = | v_n |$ so that Σw_n is an absolutely convergent series whose terms are all positive. Thus corresponding to the arbitrary positive number ϵ there exists a fixed number m (depending on ϵ) such that for all $n > m$, $\sum_m^n w_n < \epsilon$.

Now $\frac{u_m}{u_{m+1}} = 1 + v_m$, $\frac{u_{m+1}}{u_{m+2}} = 1 + v_{m+1}$, $\ldots$, $\frac{u_{n-1}}{u_n} = 1 + v_{n-1}$.

Thus $$\frac{u_m}{u_n} = \prod_{r=m}^{n-1} (1 + v_r) \leqslant \prod_{r=m}^{n-1} (1+w_r).$$

Now we may assume $w_r < 1$ so that $1 + w_r < 1/(1 - w_r)$.

Hence $$\frac{u_m}{u_n} < 1/1 \prod_{r=m}^{n-1} (1 - w_r) < 1/(1 - \sum_{r=m}^{n-1} w_r),$$

or $$\frac{u_n}{u_m} > 1 - \sum_{r=m}^{n-1} w_r > 1 - \epsilon.$$

Since u_m is fixed it follows that u_n cannot tend to zero as $n \to \infty$.

Using the above result we can now restate the test of convergence given in § 1·5 for series whose terms are alternately positive and negative.

If u_n/u_{n+1} can be expressed in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right), \ p > 1$$

then the series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is converged if $\mu > 0$ and oscillatory if $\mu \leqslant 0$.

Example.—*Discuss the convergence of the series*

$$1 - \frac{2 \cdot 3!}{x} + \frac{3 \cdot 4!}{x(x+1)} - \frac{4 \cdot 5!}{x(x+1)(x+2)} + \frac{5 \cdot 6!}{x(x+1)(x+2)(x+3)} - \dots$$

The series is $1 + \sum_{n=1}^{\infty} (-1)^n u_n$ where $u_n = \dfrac{(n+1)(n+2)!}{x(x+1)\dots(x+n)}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(x+n)}{(n+2)(n+3)} = \left(1+\frac{1}{n}\right)\left(1+\frac{x}{n}\right)\left(1+\frac{2}{n}\right)^{-1}\left(1+\frac{3}{n}\right)^{-1}$$

$$= 1 + \frac{1+x-2-3}{n} + O\left(\frac{1}{n^2}\right) = 1 + \frac{x-4}{n} + O\left(\frac{1}{n^2}\right).$$

Hence the series is absolutely convergent if $x - 4 > 1$, i.e. $x > 5$. (Rule of Gauss § 7·44), convergent but not absolutely if $5 \geqslant x > 4$ and oscillatory for $x \leqslant 4$.

EXERCISES VII

1. Prove that if $u_n > 0$ and Σu_n^2 converges, then $\Sigma u_n/n$ also converges.

2. If $\{u_n\}$ is a monotonic decreasing function which tends to zero and Σu_n converges, prove that $\Sigma n(u_n - u_{n+1})$ also converges.

3. Prove that if u_n is a positive monotonic decreasing function of n the condition $nu_n \to 0$ as $n \to \infty$ is necessary for the convergence of Σu_n.

4. Prove that if $u_n > 0$, $v_n > 0$ and Σu_n^2 and Σv_n^2 are convergent, so also is $\Sigma u_n v_n$.

5. Discuss the convergence of the series whose nth terms are

   (i) $(2+n)/(2+n^2)$, (ii) $n^{a+\frac{b}{n}}$, (iii) $(\log n)^{-n}$, (iv) $n^{\log x}$.

6. Test the convergence of the series Σa_n, where a_n is given by the following expressions:

   (i) $(1+n^2)^{-1}$, (ii) $(1+n)/(1+n^2)$, (iii) $1/\{n(\log n)^p\}$.

   [*M.T.*]

7. Discuss the convergence of the series whose nth terms are

   (i) $\dfrac{1}{\sqrt{n}\sqrt{(n+1)}}$, (ii) $\sqrt{\left(\dfrac{n}{n+1}\right)}$, (iii) $\left(\dfrac{n}{n+a}\right)^n$, where a is real.

   [*M.T.*]

8. Determine whether the following series is convergent or divergent:

$$\frac{2}{1+1^2} + \frac{4}{1+2^2} + \frac{6}{1+3^2} + \frac{8}{1+4^2} + \dots$$

   [*Madras, B.Sc.*]

9. Discuss the convergence of the series

$$\frac{1}{\sqrt{1}} + \frac{x}{\sqrt{2}} + \frac{x^2}{\sqrt{3}} + \dots \text{ to } \infty.$$

   [*Madras, B.Sc.*]

10. Test for convergence the series $\sum_{n=1}^{\infty} \left\{\frac{x^n}{n!}\right\}^2$, where x is any real number. [*Madras, B.A.*]

11. Prove that the series $1 + \frac{11}{14} + \frac{11.13}{14.16} + \frac{11.13.15}{14.16.18} + \ldots$ is convergent.

12. In two series of positive terms, if the ratio of corresponding terms tends to a limit other than zero, show that the series will both be convergent, or both be divergent. Discuss the convergency of

$$\frac{(1+a)(1+b)}{1.2.3} + \frac{(2+a)(2+b)}{2.3.4} + \ldots + \frac{(n+a)(n+b)}{n(n+1)(n+2)} + \ldots \textit{ ad inf.}$$

[*Madras, B.Sc.*]

13. Discuss the convergence of the series

$$1 + \frac{a+1}{b+1} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)} + \ldots$$

where a and b are positive.

14. Discuss the convergence of the series whose nth term is

$$\{a(a+1)(a+2)\ldots(a+m)\}/m!$$

where a is real and (i) $m = n$, (ii) m is the greatest integer whose square is less than n.

15. Prove that the series whose nth term is

$$\frac{a(a+1)\ldots(a+n-1).\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1).\delta(\delta+1)\ldots(\delta+n-1)}$$

converges if $\gamma + \delta - a - \beta > 1$.

16. Discuss the convergence of the series whose nth term is

$$(-1)^n (\log n)^p/n.$$

17. Show that, if $u_0, u_1, \ldots, u_n, \ldots$ form a diminishing sequence of positive numbers, and u_n tends to zero as n tends to infinity, the series $\sum_{n=0}^{\infty} (-1)^n u_n$ is convergent.

State carefully, giving a proof, the corresponding result when u_n tends to a positive limit u.

Discuss the series $\sum_{n=0}^{\infty} (-1)^n \{(n^2+2)^{\frac{1}{2}} - (n^2+1)^{\frac{1}{2}}\}$. [*Lond. B.Sc.*]

18. Show that for a certain range of real values of x the two series

$$2\left(\frac{1-x}{1+x}\right) + \frac{2}{3}\left(\frac{1-x}{1+x}\right)^3 + \frac{2}{5}\left(\frac{1-x}{1+x}\right)^5 + \ldots$$

$$(1-x) + \tfrac{1}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3 + \ldots$$

represent the same function, and determine that range. [*Lond. B.Sc.*]

19. Prove that if $0 < x \leqslant \frac{2}{5}$, the series

$$1 + (2x \cos\theta - x^2) + (2x\cos\theta - x^2)^2 + (2x\cos\theta - x^2)^3 + \ldots$$

can be arranged in powers of x, without altering its value.

20. If $p = 2^q$, where q is an integer, prove that

$$\sum_{r=2}^{p} \frac{1}{r \log r} > \frac{\log(q+1)}{2 \log 2} \qquad \text{[Camb. Sch.]}$$

21. Prove that $\sum_{n=1}^{\infty} \lambda^2 + n^2 < \frac{1}{2}\pi.$

22. If $\phi(n) = \sum_{r=1} 1/(2r - 1)$ prove that

$$\phi(n) \to \tfrac{1}{2}\log n + \log 2 + \tfrac{1}{2}\gamma,$$

where γ denotes Euler's constant.

23. By comparison with the corresponding integral prove that the series $\sum_{n=1}^{\infty} 1/n(n+1)$ converges and that its sum lies between $\log 2$ and $\frac{1}{2} + \log 2$. Verify this result by finding the precise sum of the series.

24. Prove that if Σu_n is a convergent series, then $\Sigma u_n x^n/(1 + x^{2n})$ converges uniformly in $0 \leqslant x \leqslant 1$.

25. Show that $\dfrac{\sin nx}{n}$ tends to zero uniformly for all positive values of x, as $n \to \infty$.

26. Prove that if n is a positive integer

$$1 - x^{2n} \geqslant 2nx^n(1 - x), \quad 0 \leqslant x \leqslant 1,$$

Deduce that $\Sigma \dfrac{x^n(1-x)}{(\log n)^2(1 - x^n)}$ converges uniformly for $0 \leqslant x \leqslant 1$.

27. Prove that

$$\lim_{n\to\infty} (1 + \tfrac{1}{2})(1 - \tfrac{1}{3})(1 + \tfrac{1}{4})(1 - \tfrac{1}{5})\ldots\left(1 \pm \frac{1}{n}\right) = 1.$$

28. Prove that if $|x| < 1$,

$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\ldots = 1/(1 - x).$$

29. Prove that if $|x| < 1$, the infinite product $\Pi\left\{1 - \left(\dfrac{nx}{n+1}\right)^n\right\}$ is absolutely convergent.

30. Show that $\dfrac{\sin x}{x} = \cos\dfrac{x}{2} . \cos\dfrac{x}{2^2} . \cos\dfrac{x}{2^3} \ldots$ to ∞.

31. Prove that $\Pi\left(1 + \dfrac{x}{c+n}\right)e^{-\frac{x}{n}}$ converges absolutely for any value of x, provided c is not a negative integer.

32. If $f(x)$ is a function of x which is positive and continuous for all $x > 1$ and decreases steadily as x increases, show that $\sum_{x=1}^{\infty} f(x)$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge. If they both converge, show that $\sum_{x=1}^{\infty} f(x) \leqslant f(1) + \int_1^{\infty} f(x)\,dx$. Use this theorem to show that $\sum_{n=1}^{\infty} \frac{1}{n(1+n^2)}$ converges, and show that

$$\sum_{n=1} \frac{2}{n(1+n^2)} < 1 + \log 2.$$

[*Lond. B. Sc.*]

CHAPTER VIII

COMPLEX NUMBERS

A COMPLEX number has been defined to be an expression of the form $x + iy$ where x and y are real and $i = \sqrt{-1}$. Thus a complex number is the sum or difference of a real and purely imaginary number. It has been shown in Vol. I., Chapter II., § 2·9 that the necessary and sufficient condition that two complex numbers be equal is that the real and purely imaginary parts are separately equal. The process of equating corresponding parts is called *equating real and imaginary parts.*

8·1. Fundamental Laws

We now consider the elementary operations of algebra as applied to complex numbers.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, where x_1, x_2, y_1, y_2 are real, be two complex numbers. Then the *fundamental laws* are as follows:

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2), \quad \textit{(addition)}.$$

$$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2), \quad \textit{(subtraction)}.$$

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1), \quad \textit{(multiplication)}.$$

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}, \quad \textit{(division)}.$$

These laws are consistent with the ordinary laws of algebraic manipulation. Thus in the case of multiplication:

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$
$$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$

Again, in the case of division:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$
$$= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 - i^2y_2^2},$$

$$\text{i.e. } \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

It will be seen that *we operate with complex numbers in precisely the same way as we do with real numbers. The quantity i is treated*

as if it were an ordinary number and every time its square occurs we replace it by the real number -1.

Thus, *e.g.* if we require the cube of the complex number $(x + iy)$ we obtain in the usual way

$$\begin{aligned}(x + iy)^3 &= x^3 + 3x^2.iy + 3x.i^2y^2 + i^3y^3\\ &= x^3 + 3ix^2y - 3xy^2 - iy^3\\ &= x^3 - 3xy^2 + i\,(3x^2y - y^3).\end{aligned}$$

The new number so obtained is another complex number.

8·10. Alternative Approach to Complex Numbers.

In this section we shall use small letters, $a, b, c, \ldots$, to denote real numbers. Such numbers possess the following properties.

1. There exist unique real numbers p, q such that

(i) $a + b = p$.

(ii) $ab = q$.

2. The associative law holds for both addition and multiplication, *i.e.*

(i) $(a + b) + c = a + (b + c)$.

(ii) $(ab)\,c = a\,(bc)$.

3. The commutative law holds for both addition and multiplication, *i.e.*

(i) $a + b = b + a$.

(ii) $ab = ba$.

4. The distributive law holds, *i.e.*

$$a\,(b + c) = ab + ac.$$

5. There exist (i) an identity number $e = 1$, such that

$$ae = ea = a.$$

(ii) A null number, *i.e.* 0 which has the property that

$$a.0 = 0.a = 0.$$

It is pertinent to ask whether, in fact, there are numbers other than real numbers which have the same properties. Consider numbers defined by pairs of real numbers, these new numbers being denoted by capital letters $A, B, C, \ldots$. Thus write $A = (a, b)$, $B = (c, d)$, $C = (e, f)$ where a, b, c, d, e, f are real numbers. In order to include real numbers in our system the

number pair $(a, 0)$ is defined to mean the real number a. We next define the operations of addition and multiplication. Thus

(i) $A + B = (a, b) + (c, d) = (a + c, b + d) =$

(ii) $AB = (a, b)(c, d) = (ac - bd, ad + bc) =$

Then clearly the numbers P, Q belong to our new system of numbers since they are defined in each case, by a pair of real numbers.

We now show that from these definitions, using the laws for real numbers, that (1) the associative, (2) the commutative, and (3) the distributive laws hold for the new system.

(1) (i)
$$\begin{aligned}(A + B) + C &= (a + c,\ b + d) + (e, f)\\ &= (a + c + e,\ b + d + f)\\ &= [a + (c + e),\ b + (d + f)]\\ &= A + (B + C).\end{aligned}$$

(ii)
$$\begin{aligned}(AB)\,C &= (ac - bd,\ ad + bc)\,(e, f)\\ &= (ace - bde - adf - bcf,\\ &\qquad acf - bdf + ade + bce).\\ A\,(BC) &= (a, b)\,(ce - df, cf + de)\\ &= (ace - adf - bcf - bde,\\ &\qquad acf + ade + bce - bdf)\\ &= (ace - bde - adf - bcf,\\ &\qquad acf - bdf + ade + bce)\\ &= (AB)\,C.\end{aligned}$$

(2) (i)
$$\begin{aligned}A + B &= (a, b) + (c, d) = (a + c, b + d)\\ &= (c + a, d + b) = B + A.\end{aligned}$$

(ii)
$$\begin{aligned}AB &= (a, b)(c, d) = (ac - bd, ad + bc)\\ &= (ca - db, da + cb) = BA.\end{aligned}$$

(3)
$$\begin{aligned}A\,(B + C) &= (a, b)\,(c + e, d + f)\\ &= (ac + ae - bd - bf, ad + af +\\ &\qquad bc + be)\\ &= [(ac - bd) + (ae - bf),\\ &\qquad (ad + bc) + (af + be)]\\ &= (ac - bd, ad + bc) + (ae - bf,\\ &\qquad af + be)\\ &= AB + AC.\end{aligned}$$

Now consider the identity element E. It must possess the property that $EA = AE = A$.

Write $E = (g, h)$. Then

$$EA = (g, h)(a, b) = (ga - hb, gb + ah).$$

Since $EA = A = (a, b)$ we have $ga - hb = a$, $gb + ah = b$. Solving these two equations for g and h we find that $g = 1$, $h = 0$ provided a and b are *not both* zero. Thus $E = (1, 0)$. Since the real number a has been defined to be $(a, 0)$ we see that $E = 1$.

If $a = 0$, $b = 0$, A becomes the null element $O = (0, 0)$. This satisfies the condition $A.O = O.A = O$. For

$$(a, b)(0, 0) = (a.0 - b.0, a.0 + b.0) = (0, 0).$$

Consider the number $E_1 = (0, 1)$. We now prove that it is not a real number. Using the notation for indices

$$E_1.E_1 = E_1^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$

Thus E_1 cannot be real for the square of a real number must be positive.

We now take $E_1A = AE_1$, where $A = (a, 0) = a$, a real number. $aE_1 = (a, 0)(0, 1) = (0, a)$, which is not a real number.

$$E_1^2a = E_1(aE_1) = (0, 1)(0, a) = (-a, 0) = -a.$$

$E_1^3 = E_1(E_1^2a) = (0, 1)(-a, 0) = (0, -a)$, which is not a real number.

$$E_1^4a = E_1(E_1^3a) = (0, 1)(0, -a) = (a, 0) = a.$$

Thus the combined multiplication of a real number by E_1 changes real into complex, then into real with changed sign, then into complex, the fourth application producing the original real number. This suggests that it will be advantageous to have a special symbol to denote $E_1 = (0, 1)$. The letter i or j is that usually used. Thus

$$E_1^2 = E_1.E_1 = i.i = -1, \quad E_1^3 = E_1.E_1^2 = i(-1) = -i.$$
$$E_1^4 = E_1(E_1^3) = i(-i) = -i^2 = +1.$$

The introduction of the identity element E enables division to be carried out. Thus provided A is not $(0, 0)$ the reciprocal of A is A^{-1} where $AA^{-1} = E$.

Write $A = (a, b)$, $A^{-1} = (c, d)$ where a, b are not both zero. Then $(a, b)(c, d) = (1, 0)$, i.e. $(ac - bd, ad + bc) = (1, 0)$.

Hence $$ac - bd = 1, \quad ad + bc = 0.$$

Solving these equations for c, d we find

$$c = a/(a^2 + b^2), \quad d = -b/(a^2 + b^2).$$

Thus $$A^{-1} \text{ is } [a/(a^2 + b^2), \ -b/(a^2 + b^2)].$$

Then if

$$B = (e, f), \ \frac{B}{A} = BA^{-1} = (e, f)\left(\frac{a}{a^2 + b^2}, \ \frac{-b}{a^2 + b^2}\right)$$

$$= \left(\frac{ae + bf}{a^2 + b^2}, \ \frac{-be + af}{a^2 + b^2}\right)$$

We can readily change our "number pair" notation into that used in the previous section.

Write $Z = (x, y) = (x, 0) + (0, y)$, in accordance with the rule for addition.

Now $(x, 0)$ is the real number x. Also $(0, y) = E_1 y = iy$

Thus $$Z = x + iy.$$

Since any complex number Z can be defined by a pair of real numbers (x, y) we can represent it geometrically as a point in a plane in the usual way. This gives the Argand diagram which will be considered in a subsequent section. In dealing with geometrical representation it is sometimes advantageous to use a vector notation and this is a convenient point to give a warning about confusing complex numbers with vectors. Whereas some of the fundamental laws referred to above are satisfied by vectors as well as by complex numbers, thus giving some points of resemblance, vectors do not satisfy the commutative law of multiplication. Thus if A and B are complex numbers then $AB = BA$, but if **A** and **B** are vectors the vector product **AB** is not the same as the vector product **BA**.

8·11. Conjugate Numbers

Let $z = (x, y)$ be a complex number. Then $\bar{z} = (x, -y)$ is defined to be the *conjugate* of z. Observe that $y \neq 0$, for in this case z reduces to the real number $(x, 0) = x$. Represented geometrically

with axes x and y, a complex number and its conjugate are mirror images with respect to the x-axis.

We now prove the following results.

(i) *The sum and product of two conjugate numbers are real.*

(ii) *Conversely, if the sum and product of two complex numbers are both real, then the two numbers are conjugate.*

(iii) *If two complex numbers are equal, these conjugates are equal.*

(i) Let $z = (x, y)$, $\bar{z} = (x, -y)$. Then

$$z + \bar{z} = (x, y) + (x, -y) = (2x, 0) = 2x, \text{ which is real.}$$

$$z\bar{z} = (x, y)(x, -y) = (x^2 + y^2, -xy + xy) = (x^2 + y^2, 0)$$
$$= x^2 + y^2.$$

Hence the product is real and *positive.*

(ii) Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ be the two complex numbers, where $y_1, y_2, \neq 0$. Then

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

$$z_1z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

Since $z_1 + z_2$ and z_1z_2 are both real.

$y_1 + y_2 = 0$, $x_1y_2 + y_1x_2 = 0$. Substituting $y_2 = -y_1$ in the second equation $y_1(-x_1 + x_2) = 0$. Since $y_1 \neq 0$, $x_1 = x_2$. Thus the two numbers have the form $z_1 = (x_1, y_1)$, $z_2 = (x_1, -y_1)$ and are conjugates.

(iii) This result follows immediately from definition. For if $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ then $x_1 = x_2$, $y_1 = y_2$. Hence

$$\bar{z}_1 = (x_1 - y_1) = \bar{z}_2 = (x_2, -y_2).$$

The properties of conjugate numbers can be applied to division as follows:—

$$\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \frac{(x_1, y_1)(x_2, -y_2)}{(x_2, y_2)(x_2, -y_2)} = \frac{(x_1x_2 + y_1y_2, -x_1y_2 + y_1x_2)}{x_2^2 + y_2^2}.$$

Since $x_2^2 + y_2^2$ is real the ratio of two complex numbers has thus been expressed as the product of a real number and a complex number.

The results given above may be expressed immediately in the $x + iy$ notation. Thus if $z = x + iy$, $\bar{z} = x - iy$, $z + \bar{z} = 2x$, $z\bar{z} = x^2 + y^2$.

If $z_1 = x_1 + iy$, $z_2 = x_2 + iy$,

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

$$\frac{i(-x_1y_2 + y_1x_2)}{x_2^2 + y_2^2}$$

wherein the product i^2 has been replaced by -1.

Example.—*Express* $2/(4 + 3i)$ *in the form* $u + iv$.

Multiplying numerator and denominator by $4 - 3i$,

$$\frac{2}{4 + 3i} = \frac{2(4 - 3i)}{4^2 - 3^2i^2} = \frac{8 - 6i}{16 + 9} = \frac{8}{25} - \frac{6}{25}i$$

8·12. Alternative Form for a Complex Number

Consider the complex number $z = x + iy$ and let r, θ be two real quantities defined by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

$$\text{Then } z = r(\cos\theta + i\sin\theta).$$

Squaring and adding the corresponding sides of the equations for x and y,

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2, \text{ i.e. } r = \pm\sqrt{(x^2 + y^2)}.$$

Again by division, $\tan\theta = y/x$; $\therefore\ \theta = \tan^{-1} y/x$.

In order to make r unique it is taken to be the positive square root of $x^2 + y^2$.

It is now necessary to choose θ so that $\cos\theta = x/r$, $\sin\theta = y/r$. If we restrict θ so that $-\pi < \theta \leqslant \pi$, there is *one and only one angle* defined by these equations. This angle is called the principal value. Since $\cos(\theta + 2n\pi) = \cos\theta$, $\sin(\theta + 2n\pi) = \sin\theta$, where n is a positive or negative integer it is clear that there are an infinite number of angles satisfying the equations

$$\cos\theta = x\sqrt{(x^2 + y^2)}, \quad \sin\theta = y\sqrt{(x^2 + y^2)}.$$

The principal value is the simplest value and the other values only differ from it by multiples of 2π.

The real number r is defined to be the modulus; and the number θ the amplitude (or argument or phase) of z. Thus

$$r = |z| = \sqrt{(x^2 + y^2)} \quad \theta = \text{amp. } z \text{ (or arg. } z).$$

In general, when speaking about the amplitude of a complex number, the principal value will be intended unless otherwise stated.

$$\text{Thus, e.g. } |12 + 5i| = \sqrt{(12^2 + 5^2)} = 13.$$

If θ denote the principal value of the amplitude

$$\cos\theta = 12/13, \quad \sin\theta = 5/13.$$

If a denote the acute angle $\sin^{-1} 5/13$, then amp. $(12 + 5i) = a$.

If we do not restrict the angle to be the principal value then amp. $(12 + 5i) = a + 2n\pi$.

8·13. Properties of the Modulus of a Complex Number

(i) Let z be a complex number, $\bar{z}$ its conjugate. Then

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 = |\bar{z}|^2.$$

Hence *a complex number and its conjugate have the same modulus, the value of the modulus being the square root of their product.*

(ii) *If* $z = 0$ *then* $|z| = 0$ *and conversely.* This follows immediately from the fact that $x + iy = 0$ implies $x = 0$, $y = 0$.

(iii) *If* z_1 *and* z_2 *are two complex numbers, then*

$$|z_1 z_2| = |z_1| \times |z_2| \text{ and } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Write $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then
$$\begin{aligned} z_1 z_2 &= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\} \\ &= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}. \end{aligned}$$

$$|z_1| = r_1, \quad |z_2| = r_2, \quad |z_1 z_2| = r_1 r_2.$$

$$\text{Hence } |z_1 z_2| = |z_1| \times |z_2|.$$

Now consider the quotient z_1/z_2.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\left\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right\}.$$

$$\text{Hence } \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}.$$

These results can obviously be extended to the product or quotient of any number of complex numbers.

(iv) *If z and $\bar{z}$ are conjugates, then $z + \bar{z} \leqslant 2 \mid z \mid$, there being strict inequality, unless z is real and positive.*

If $z = x + iy$, then $z + \bar{z} = 2x \leqslant 2\sqrt{(x^2 + y^2)}$.
Since $\mid z \mid = \sqrt{(x^2 + y^2)}$, $\mid z + \bar{z} \mid \leqslant 2 \mid z \mid$ and the inequality is strict unless $x > 0$ and $y = 0$, i.e. z is real and positive.

(v) *If z_1 and z_2 are two complex numbers, then*

$$\mid z_1 + z_2 \mid^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2).$$

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$\mid z_1 + z_2 \mid^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2.$$

Also $(z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$$= \{(x_1 + x_2) + i(y_1 + y_2)\}\{(x_1 + x_2) - i(y_1 + y_2)\}$$
$$= (x_1 + x_2)^2 + (y_1 + y_2)^2,$$

which proves the result.

(vi) *If z_1 and z_2 are any two numbers then*

(*a*) $\mid z_1 + z_2 \mid \leqslant \mid z_1 \mid + \mid z_2 \mid$. (*b*) $\mid z_1 - z_2 \mid \geqslant \mid z_1 \mid - \mid z_2 \mid$.

(*a*) From (v), $\mid z_1 + z_2 \mid^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$$
$$= \mid z_1 \mid^2 + \mid z_2 \mid^2 + w + \bar{w}, \text{ where } w = z_1\bar{z}_2$$

From (iv) $w + \bar{w} \leqslant 2 \mid w$

Now $\mid w \mid = \ z_1\bar{z}_2 \mid = \mid z_1 z_2 \quad (\bar{z}_2 / z_2) \mid = \mid z_1 z_2 \mid = \mid z_1 \mid \mid z_2$

Hence $\mid z_1 + z_2 \mid^2 \leqslant \mid z_1 \mid^2 + \mid z_2 \mid^2 + 2 \mid z_1 \mid \mid z_2 \mid$.

From (iv) the inequality is strict unless $w = z_1\bar{z}_2$ is real and positive. Now $z_1\bar{z}_2 = \left(\dfrac{z_1}{z_2}\right)(z_2\bar{z}_2)$ and $z_2\bar{z}_2$ is real and positive. Hence there is equality only if z_1/z_2 is real and positive.

(*b*) This follows from (*a*) by a simple transformation.

Write $z_1 = Z_1 - Z_2$, $z_2 = Z_2$.

Then (*a*) gives $\mid Z_1 \mid \leqslant \mid Z_1 - Z_2 \mid + \mid Z_2 \mid$, or

$\mid Z_1 - Z_2 \mid \geqslant \mid Z_1 \mid - \mid Z_2 \mid$, which is the required result.

8·14. Properties of the Amplitude

(i) *The amplitude of the product of two complex numbers is equal to the sum of their amplitudes or differs from the sum by 2π.*

From § 8·13 (iii), $z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}$.

Hence if amp. $z_1 = \theta_1$, amp. $z_2 = \theta_2$,

$$\text{amp. } (z_1 z_2) = \theta_1 + \theta_2 \text{ or } \theta_1 + \theta_2 \pm 2\pi.$$

It is assumed in writing down this equation that the principal values are considered in each case. The fact that $\theta_1 + \theta_2$ need not give the principal value of the argument for $z_1 z_2$ is easily seen. Thus if θ_1 and θ_2 are both greater than $\frac{1}{2}\pi$ then $\theta_1 + \theta_2$ is greater than π, so that it will be necessary to subtract 2π in order to obtain an angle between $-\pi$ and π for the new amplitude.

In particular, if

$$z_1 = z_2 = -1 + i\sqrt{3}, \text{ amp. } z_1 = \text{amp. } z_2 = \tfrac{2}{3}\pi.$$

But $z_1 z_2 = -2 - 2i\sqrt{3}$, amp. $(z_1 z_2) = -\frac{2}{3}\pi = \frac{2}{3}\pi + \frac{2}{3}\pi - 2\pi$.

(ii) *The amplitude of the quotient of two complex numbers is equal to the difference of their amplitudes or differs from this by* 2π.

From § 8·13 (iii), $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right\}$.

$$\text{Hence amp. } (z_1/z_2) = \theta_1 - \theta_2 \text{ or } \theta_1 - \theta_2 \pm 2\pi.$$

Thus if $z_1 = -1 + i\sqrt{3}$, $z_2 = -i$,

$$\frac{z_1}{z_2} = \frac{i^2 + i\sqrt{3}}{-i} = -\sqrt{3} - i.$$

Amp. $z_1 = \frac{2}{3}\pi$, amp. $z_2 = -\frac{1}{2}\pi$,

$$\text{amp. } (z_1/z_2) = -\tfrac{5}{6}\pi.$$

Also $-\frac{5}{6}\pi = \frac{2}{3}\pi - (-\frac{1}{2}\pi) - 2\pi$.

8·2. The Argand Plane

Let $z = (x, y) = x + iy$ be a complex number defined by the number pair (x, y). Then we can represent z by a point on a plane where the axes of X and Y are taken arbitrarily. The plane is called the *Argand* plane and is analogous to the Cartesian plane. Take arbitrary axes OX, OY (Fig. 27). Then if P is the point with cartesian coordinates x, y then we can say P corresponds to z and write

$$\vec{OP} = z = (x, y).$$

We can also regard P as a point whose polar coordinates are (r, θ).

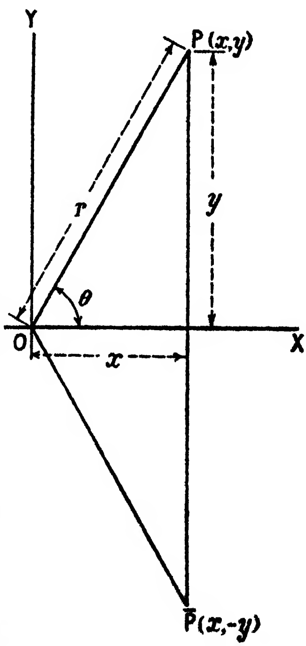


FIG. 27.

Then $|\,OP\,| = |\,z\,| = OP = r = \sqrt{(x^2 + y^2)}$.

If $\overrightarrow{OP}$ has direction θ measured positively as an anticlockwise rotation from the positive sense of the X-axis, then

$$\text{amp. } z = \theta = \tan^{-1}(y/x).$$

When z is real, $y = 0$ and the point z will lie on the X-axis. Thus the line $X'OX$ is usually called the *real axis*.

Again, if z is purely imaginary, i.e. $x = 0$, the point z will lie on the Y-axis, so that this line is usually called the *imaginary axis*.

It should be observed that θ need not be the principal value of amp. z. For in the usual convention for polar coordinates θ is to be the angle $0 \leqslant \theta < 2\pi$ which satisfies

$$x = r \cos \quad y = r \sin \theta,$$
where $r > 0$.

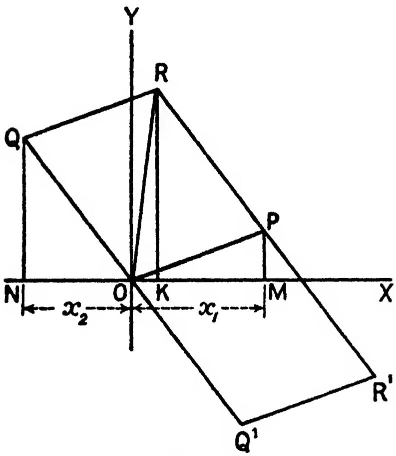


FIG. 28.

In the convention for principal values of the amplitude, θ is to lie between $-\pi$ and π.

In the figure, since θ is an acute angle, θ is the principal value of amp. z. If $\bar{z}$ denote the conjugate of z, then $\bar{z} = x - iy$. Thus the point $\overline{P}$ *corresponding to* $\bar{z}$ *is the mirror image of* P *in the X-axis.*

It will be observed that *if* θ *is the principal value of amp.* z, *then the principal value of amp.* $\bar{z}$ *is* $-\theta$.

8·21. Sum and Difference of two Complex Numbers

Let P, Q (Fig. 28) be points on the Argand plane corresponding to $z_1 = (x_1, y_1) = x_1 + iy_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$ respectively. Then if $z_3 = (x_3, y_3)$ is the sum of (x_1, y_1), (x_2, y_2) then in accordance with the definition of addition,

$$z_3 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Hence $\qquad x_3 = x_1 + x_2,\ y_3 = y_1 + y_2.$

Let M, N be the feet of the perpendiculars from P and Q to the X-axis. The $OM = x_1$, $ON = x_2$. Now complete the parallelogram

whose adjacent sides are OP, OQ obtaining $OPRQ$ and let K be the foot of the perpendicular from R to the real axis.

The coordinates of R are OK and KR and it is easily seen from the figure that $OK = x_1 + x_2$, $KR = y_1 + y_2$. Hence R is the point (x_3, y_3) on the Argand plane corresponding to $z_1 + z_2$. Thus we can write $z_3 = z_1 + z_2$ as

$$\vec{OP} + \vec{OQ} = \vec{OR}.$$

It follows that as far as *addition* is concerned complex numbers behave in a manner similar to vectors. Thus the sum of two complex numbers is found in exactly the same way as we find the resultant of two vectors, *i.e.* by completing the parallelogram, whose adjacent sides represent the two vectors. The resultant is then represented by the diagonal of the parallelogram drawn *from the point of intersection* of the two given vectors. Thus in Fig. 28, OR is the diagonal of the parallelogram, whose adjacent sides are OP and OQ.

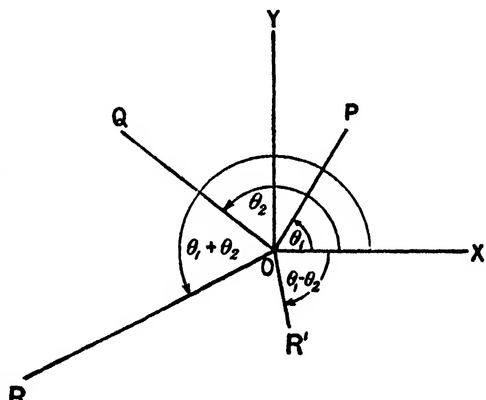


FIG. 29.

Next consider the difference of the two complex numbers z_1, z_2.

Then if $\vec{OQ} = z_2$ then $-z_2 = \vec{OQ'}$ where $|\vec{OQ}| = |\vec{OQ'}|$, i.e. OQ' is obtained by producing QO to Q' making $QO = OQ'$. Then

$z_1 - z_2 = z_1 + (-z_2) = \vec{OP} + \vec{OQ'} = \vec{OR'}$ where OR' is the diagonal of the parallelogram whose adjacent sides and OP are OQ'.

It will be observed that it is not necessary to draw the second parallelogram $OPR'Q'$. For it is clear that the second diagonal QP of $OPRQ$ is equal and parallel to OR'.

The Argand diagram provides geometrical illustrations of two earlier results on modulus, viz. $|z_1 + z_2| \leqslant |z_1| + |z_2|$, and $z_1 - z_2| \geqslant |z_1| -$ The first theorem asserts that any two

sides of a triangle are greater than or equal to the third, while the second asserts that any side of a triangle is greater than or equal to the difference between the other two sides. Cases of equality only arise when the triangle degenerates into a straight line, otherwise there is strict inequality.

8·22. Product and Quotient of two Complex Numbers

Let $\overrightarrow{OP} = z_1 = r \cos \theta_1 + ir \sin \theta$, $\overrightarrow{OQ} = z_2 = r_2 \cos \theta_2 + ir_2 \sin \theta_2$ be two complex numbers represented in the Argand plane (Fig. 29). Since $e^{i\theta} = \cos \theta + i \sin \theta$ we can write $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.

$$\text{Thus} \quad |\overrightarrow{OP}| = r_1, \ |\overrightarrow{OQ}| = r_2, \ \angle XOP = \theta_1, \ \angle XOQ = \theta_2.$$

$$\text{Let} \quad \overrightarrow{OR} = z_1 z_2 = z_2 z_1 = r_1 e^{i\theta_1} . r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Then $|\overrightarrow{OR}| = r_1 r_2$, amp. $(\overrightarrow{OR}) = \theta_1 + \theta_2$. This involves merely the multiplication of two real numbers r_1, r_2 and rotation in the Argand plane of an angle $(\theta_1 + \theta_2)$ counter-clockwise from the positive X-axis. Observe that although θ_1 and θ_2 are the principal values of amp. (OP) and amp. $(\overrightarrow{OQ})$ respectively, $\theta_1 + \theta_2$ need not be the principal value of amp. $(\overrightarrow{OR})$.

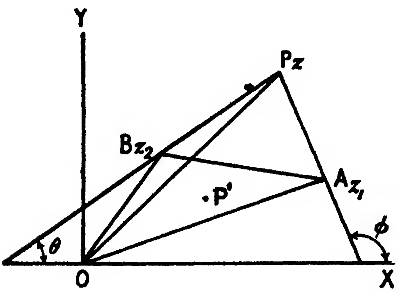


FIG. 30.

Next let $\overrightarrow{OR'} = \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \ \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$ Then $|\overrightarrow{OR'}| = \frac{r_1}{r_2}$

and amp. $(OR') = \theta_1 - \theta_2$. This involves division of one real number by another and a rotation in the Argand plane of θ_1 anti-clockwise, and θ_2 clockwise, each angle being measured from the positive X-axis.

Since we can represent geometrically the sum, difference, product and quotient of two complex numbers it is clear that by a series of geometrical processes we can represent any operation on a complex number, which consists of a combination of the four fundamental operations.

8·23. Geometrical Representation of $(z - z_1)/(z - z_2)$.

Let $\vec{OA} = z_1$, $\vec{OB} = z_2$, $\vec{OP} = z$ (Fig. 30). Then

$$z - z_1 = \vec{OP} - \vec{OA} = \vec{OQ}$$

where OQ is equal to and parallel to AP and in the same directed sense. Thus amp. $(z - z_1) = \phi$, where ϕ is the angle between OX and AP. Again $z - z_2 = \vec{OP} - \vec{OB} = \vec{OR}$ where OR is equal to and parallel to BP and in the same sense. Hence

$$\text{amp. } (z - z_2) = \theta,$$

where θ is the angle between BP and OX. Then

$$\text{amp. } \frac{z - z_1}{z - z_2} = \text{amp. } (z - z_1) - \text{amp. } (z - z_2) = \theta - \phi$$
$$= \angle APB = \psi, \text{ say.}$$

If we regard z_1 *and* z_2 *as fixed,* ψ *as a fixed angle, and* z *variable, then the locus traced out by* P *is an arc of a circle passing through* A *and* B.

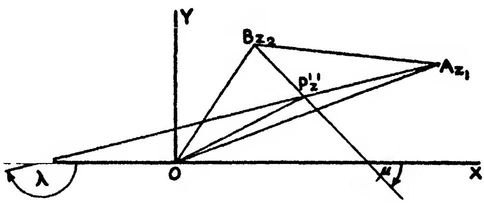


FIG. 31.

Next, let P' be any point on the arc of the circle APB on the side of AB opposite to P and let P' be the point z' (Fig. 31). Then if amp. $(z' - z_1) = \lambda$, amp. $(z' - z_2) = \mu$,

$$\text{amp. } \frac{z' - z_1}{z' - z_2} = \lambda - \mu = -(\pi - \psi), \text{ since } \angle AP'B = \pi - \psi.$$

It should be noted that λ, μ are both negative angles and λ is numerically greater than μ. Thus the second arc of the circle is obtained by replacing ψ in amp. $\dfrac{z - z_1}{z - z_2} = \psi$ by $-\pi + \psi$.

Finally if we regard ψ as a variable parameter the equations

$$\text{amp. } \frac{z - z_1}{z - z_2} = \psi, \text{ amp. } \frac{z - z_1}{z - z_2} = -\pi + \psi$$

define a system of circles passing through A, B.

8·24. Concyclic Points

The following is an important theorem on the representation of four points in a plane. *The necessary and sufficient condition that the points corresponding to the four numbers z_1, z_2, z_3, z_4 be concyclic is that $\{(z_1 - z_3)(z_2 - z_4)/(z_1 - z_4)(z_2 - z_3)\}$ be real.*

If $\{(z_1 - z_3)(z_2 - z_4)/(z_1 - z_4)(z_2 - z_3)\}$ be real the amplitude must be 0, or π. Hence amp. $\{(z_1 - z_3)/(z_1 - z_4)\}$ must differ from amp. $\{(z_2 - z_4)/(z_2 - z_3)\}$ by π or else the two amplitudes are equal. It follows from § 8·23 that the four points are concyclic.

Conversely, if the points are concyclic, then

$$\text{amp. } \{(z_1 - z_3)/(z_1 - z_4)\}$$

must be equal to amp. $\{(z_2 - z_4)/(z_2 - z_3)\}$ or differ from it by π.

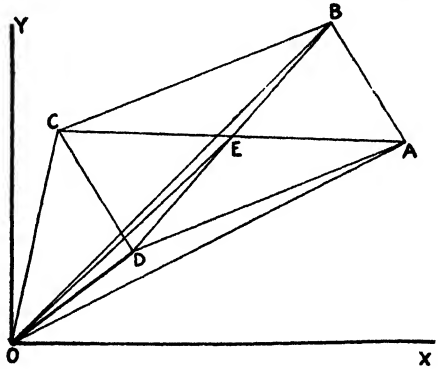


FIG. 32.

Thus $$\text{amp. } \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = 0 \text{ or } \pm\pi,$$

i.e. $(z_1 - z_3)(z_2 - z_4)/(z_1 - z_4)(z_2 - z_3)$ is real

Examples.—(1) *If z_1, z_2, z_3, z_4 are represented by the vertices of a parallelogram, taken in order, writing $z_1 = z - a$, $z_2 = z - b$, $z_3 = z + a$, $z_4 = z + b$, show that*

$$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2 = 4\{|a|^2 + |b|^2\}.$$

What is the geometrical theorem corresponding to this result? [*Lond. B.A.*]

Let $ABCD$ be the parallelogram, with $\overrightarrow{OA} = z_1$, $\overrightarrow{OB} = z_2$, $\overrightarrow{OC} = z_3$, $\overrightarrow{OD} = z_4$, and let E be the point of interesection of the diagonals and $\overrightarrow{OE} = z$. (Fig. 32.)

Write $a = \overrightarrow{OL}$, $b = \overrightarrow{OM}$ where $|a| = |\overrightarrow{OL}| = \frac{1}{2}$ diagonal AC, $|b| = |\overrightarrow{OM}| = \frac{1}{2}$ diagonal BD, $\overrightarrow{OL}$ is parallel to $\overrightarrow{AE}$ and $\overrightarrow{OM}$ to $\overrightarrow{BE}$ in the

sense indicated by the arrows. By consideration of the triangle AEB it is seen that $|a - b| = AB$, and from the triangle BEC, $|a + b| = BC$.

Then $$z_1 = \overrightarrow{OA} = \overrightarrow{OE} - \overrightarrow{OL} = z - a.$$

$$z_2 = \overrightarrow{OB} = \overrightarrow{OE} - \overrightarrow{OM} = z - b.$$

$$z_3 = \overrightarrow{OC} = \overrightarrow{OE} + \overrightarrow{OL} = z + a.$$

$$z_4 = \overrightarrow{OD} = \overrightarrow{OE} + \overrightarrow{OM} = z + b.$$

$z_1 - z_2 = z - a - z + b = b - a$, $|z_1 - z_2| = |a - b| = AB$.

Similarly, $|z_2 - z_3| = |a + b| = BC$, $|z_3 - z_4| = |a - b| = AB$,
$|z_4 - z_1| = |a + b| = BC$.

Hence $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2$
$= 2|a + b|^2 + 2|a - b|^2 = 2BC^2 + 2AB^2$.

Write $a = \overrightarrow{OL} = r(\cos\theta + i\sin\theta)$, $b = \overrightarrow{OM} = \rho(\cos\phi + i\sin\phi)$, where $r = |a|$, $\rho = |b|$.

$|a + b|^2 = |r\cos\theta + \rho\cos\phi + i(r\sin\theta + \rho\sin\phi)|^2$
$= r^2\cos^2\theta + \rho^2\cos^2\phi + 2r\rho\cos\theta\cos\phi + r^2\sin^2\theta + \rho^2\sin^2\phi$
$+ 2r\rho\sin\theta\sin\phi$.
$= r^2 + \rho^2 + 2r\rho\cos(\theta - \phi)$.

Similarly $|a - b|^2 = r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)$.

Hence, $2|a + b|^2 + 2|a - b|^2 = 4r^2 + 4\rho^2 = 4|a|^2 + 4|b|^2$, which proves the result.

Since $2|a|$ and $2|b|$ are the lengths of the diagonals of the given parallelogram the geometrical theorem corresponding to the result is that the sum of the squares on the four sides is equal to the sum of the squares on the diagonals.

*(2) *If the point P on the Argand diagram represents the complex quality z, show how to construct geometrically the point Q corresponding to z^2. If P lies on the circle, centre $(1, 0)$ passing through the origin, show that $|z^2 - z| = |z|$. Show also that amp. $(z - 1)$ = amp. $z^2 = \frac{2}{3}$ amp. $(z^2 - z)$.* [Lond. B.Sc.]

If Q represents z^2, then $OQ = OP^2$ and OQ makes an angle 2θ with OX, where θ is the angle which OP makes with OX. Since the centre of the circle is $(1, 0)$ and the radius is unity, any point on it may be represented in the form

$$z - 1 = \cos\theta + i\sin\theta, \quad -\pi < \theta \leqslant \pi.$$

Now $|z^2 - z| = |z| \times |z - 1| = |z|$,

since $|z - 1| = \sqrt{(\cos^2\theta + \sin^2\theta)} = 1$.

Since $z - 1 = \cos\theta + i\sin\theta$, amp. $(z - 1) = \theta$.

$$\text{amp. } z = \tan^{-1}\frac{\sin\theta}{1 + \cos\theta} = \tan^{-1}\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$
$$= \tan^{-1}(\tan\tfrac{1}{2}\theta) = \tfrac{1}{2}\theta.$$

Hence amp. $z^2 = \theta$.

Again amp. $(z^2 - z)$ = amp. z + amp. $(z - 1) = \frac{1}{2}\theta + \theta$.

Hence $\frac{2}{3}$ amp. $(z^2 - z) = \theta$.

†(3) *α, β, z are three complex numbers which are represented in the Argand diagram by A, B and P respectively. Prove that if k is a constant, A and B fixed points and* $\left|\dfrac{z-\alpha}{z-\beta}\right| = k$, *then z must lie on a fixed circle whose centre lies on AB.*

Now $|z-\alpha| = AP$, $|z-\beta| = BP$. Thus the given relation asserts that $\dfrac{AP}{BP} = k$. Let PT be the tangent to the circle ABP, meeting AB in T. [Fig. 33 (*a*)].

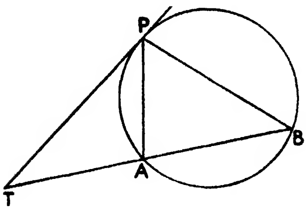


FIG. 33 (*a*).

Then it is easily seen that the triangles APT, PBT are equiangular. Hence

$$\frac{AP}{BP} = \frac{AT}{PT} = \frac{PT}{BT} = k.$$

It follows that $\dfrac{AT}{BT} = \dfrac{PT^2}{BT^2} = k^2$. Thus T is a fixed point. Again $PT^2 = AT.BT$ and since T is a fixed point it follows that PT is of fixed length. Hence P lies on a circle whose centre is T.

† (4) *Prove that if the ratio $(z-i)/(z-1)$ is purely imaginary the point z lies on the circle whose centre is at the point $\frac{1}{2}(1+i)$ and whose radius is $1/\sqrt{2}$.* [*Lond. B.Sc.*]

If $z = x + iy$ then

$$\frac{z-i}{z-1} = \frac{x+i(y-1)}{x-1+iy} = \frac{\{x+i(y-1)\}\{x-1-iy\}}{(x-1)^2+y^2}.$$

Since the number is purely imaginary, the real part is zero. Hence

$$x(x-1) + y(y-1) = 0, \text{ i.e. } (x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2 = \tfrac{1}{2}.$$

This represents a circle whose centre is $(\frac{1}{2}, \frac{1}{2})$ and whose radius is $1/\sqrt{2}$.

(5) *If the amplitude of the complex number $(z-1)/(z+1)$ is $\frac{1}{4}\pi$, show that z lies on a fixed circle whose centre is at the point which represents i.* [*Lond. B.Sc.*]

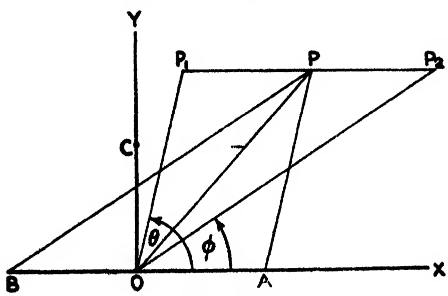


FIG. 33 (*b*)

Let P be the point z, P_1 the point $z-1$, P_2 the point $z+1$, so that $P_1P = PP_2 = 1$ and P_1PP_2 is parallel to OX.

In Fig. 33 (*b*), it is clear that amp. $(z-1) = \theta$, amp. $(z+1) = \phi$. Thus

$$\text{amp. } \{(z-1)/(z+1)\} = \theta - \phi.$$

Hence if A, B are the points $(1, 0)$, $(-1, 0)$ then

$$\theta - \phi = \angle APP_1 - \angle BPP_1 = \angle APB.$$

Thus the angle subtended by AB at P is constant and equal to $\frac{1}{4}\pi$. Hence P lies on a circle through AB.

The centre of the circle lies on the perpendicular bisector of AB, *i.e.* on the imaginary axis. Let Q be the point where the circle cuts this axis, C the centre of the circle.

Then $\quad \angle CQB = \angle CBQ = \frac{1}{8}\pi$.

Hence $\angle BCQ = \frac{3}{4}\pi$, $\angle OCB = \frac{1}{4}\pi$

so that $\quad OC = OB = 1$.

Thus C is the point which represents i in the Argand diagram.

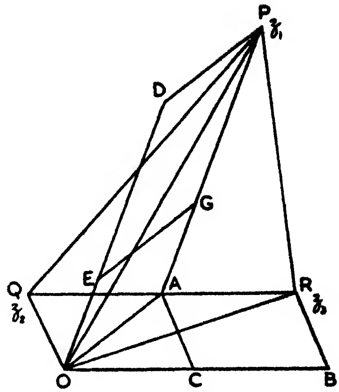


FIG. 33 (*c*).

(6) *If P, Q, R are points in the Argand plane representing z_1, z_2, z_3 respectivly, prove that centroid G of the triangle PQD represents $\frac{1}{3}(z_1 + z_2 + z_3)$.*

In Fig. 33 (*c*), O represents the origin, A the mid point of QR, so that $GA = \frac{1}{3}AP$. Draw the parallelograms $OQRB$, $OQAC$, $OAPD$, $OAGE$.

Then $\quad \vec{OP} = z_1$, $\vec{OQ} = z_2$, $\vec{OR} = z_3$.

$$\vec{OB} = z_3 - z_2,\ \vec{OC} = \tfrac{1}{2}\vec{OB} = \tfrac{1}{2}(z_3 - z_2).$$

$$\vec{OA} = \vec{OQ} + \vec{OC} = z_2 + \tfrac{1}{2}(z_3 - z_2) = \tfrac{1}{2}(z_2 + z_3).$$

$$\vec{OD} = \vec{OP} - \vec{OA} = z_1 - \tfrac{1}{2}(z_2 + z_3).$$

$$\vec{OE} = \tfrac{1}{3}\vec{OD} = \tfrac{1}{3}z_1 - \tfrac{1}{6}(z_2 + z_3).$$

$$\vec{OG} = \vec{OA} + \vec{OE} = \tfrac{1}{2}(z_2 + z_3) + \tfrac{1}{3}z_1 - \tfrac{1}{6}(z_2 + z_3)$$
$$= \tfrac{1}{3}(z_1 + z_2 + z_3).$$

Hence G represents the complex number $\frac{1}{3}(z_1 + z_2 + z_3)$.

Note.—If $P'Q'R'$ is any other triangle in the Argand plane corresponding to z_1', z_2', z_3' then the condition that the centroids of the triangles PQR, $P'Q'R'$ be the same is $z_1 + z_2 + z_3 = z_1' + z_2' + z_3'$.

8·31. The Exponential Form for z

The series for $\cos\theta$ and $\sin\theta$ (Chapter V., § 5·4) converges absolutely for all real values of θ. Thus

$$\cos\theta = \sum_{n=0}^{\infty} (-1)^n \theta^{2n}/(2n)!,\ \sin\theta = \sum_{n=0}^{\infty} (-1)^n \theta^{2n+1}/(2n+1)!$$

Assuming that i can be treated as an ordinary number we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = \cos\theta + i\sin\theta.$$

Now $(-1)^n = i^{2n}$ so that the series may be written in the form

$$\sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (i\theta)^r$$

Now if x is real, $e^x = \sum_{n=0}^{\infty} x^n/n!$

We would expect for z complex that $e^z = \sum_{n=0}^{\infty} z^n/n!$.

Thus if we *define* e^z by this series the definition will agree with the results already known for the particular case in which the imaginary part of z is zero, i.e. z reduces to a real number. Using this result we see that

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

The convergence of the series $\Sigma z^n/n!$ will be discussed in § 8·55. It will be shown that the series converges absolutely for all values of z.

Using this form for $\cos\theta + i\sin\theta$ we can write $z = re^{i\theta}$, and $z^{-1} = r^{-1}e^{-i\theta}$. Also if $\bar{z}$ be the conjugate of z then $\bar{z} = re^{-i\theta}$.

Again, if k is a real positive constant and a is fixed, the equation

$$z - a = ke^{i\theta}, \quad -\pi < \theta \leqslant \pi$$

will represent a circle in the Argand diagram. For

$$|z - a| = k\,|e^{i\theta}| = k\,|\cos\theta + i\sin\theta| = k$$

Also $|z - a|$ is the distance from the point z to the point a. Hence the equation asserts that z always lies at a fixed distance k from the point a, *i.e.* the locus of z is a circle whose centre is the point a and whose radius is k. In particular, if $a = 0$, $k = 1$ the equation $z = e^{i\theta}$ represents the circle whose centre is the origin and whose radius is unity. This circle is usually referred to as *the unit circle*.

Example.—*Express the complex number* $1 + i$ *in the form* $r(\cos\theta + i\sin\theta)$. *Hence, or otherwise, prove that, n being a positive integer,*

$$(1 + i)^n + (1 - i)^n = 2\,(2^{n/2}\cos n\pi/4).$$

If $(1 + x)^n = p_0 + p_1x + p_2x^2 + \ldots + p_nx^n$, *prove that*

$$p_0 - p_2 + p_4 - \ldots = 2^{n/2}\cos n\pi/4$$

and

$$p_1 - p_3 + p_5 - \ldots = 2^{n/2}\sin n\pi/4. \qquad \text{[Lond. B.Sc.]}$$

Write $z = 1 + i$, then $\bar{z} = 1 - i$, where z and $\bar{z}$ are conjugates.

If $1 + i = r(\cos\theta + i\sin\theta)$, $r\cos\theta = 1$, $r\sin\theta = 1$ giving $r = \sqrt{2}$, $\theta = \pi/4$.

Hence $\qquad z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{i\pi/4}$, $\bar{z} = \sqrt{2}e^{-i\pi/4}$.

$$z^n + \bar{z}^n = (\sqrt{2})^n[e^{ni\pi/4} + e^{-ni\pi/4}] = 2(2^{n/2}\cos n\pi/4).$$

also $\qquad z^n - z^{-n} = (\sqrt{2})^n[e^{ni\pi/4} - e^{-ni\pi/4}] = 2i(2^{n/2}\sin n\pi/4)$.

If $(1 + x)^n = p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4 + \ldots$, then changing x into $-x$ we have

$$(1 - x)^n = p_0 - p_1x + p_2x^2 - p_3x^3 + p_4x^4 - \ldots.$$

Hence $\qquad (1 + x)^n + (1 - x)^n = 2(p_0 + p_2x^2 + p_4x^4 + \ldots)$

$$(1 + x)^n - (1 - x)^n = 2x(p_1 + p_3x^2 + p_5x^4 + \ldots).$$

In these equations, write $x = i$. Then $1 + x = z$, $1 - x = \bar{z}$, and

$$z^n + \bar{z}^n = 2(p_0 + p_2i^2 + p_4i^4 + \ldots)$$

$$z^n - \bar{z}^n = 2i(p_1 + p_3i^2 + p_5i^4 + \ldots).$$

Now $i^2 = -1$, $i^4 = (-1)^2 = +1$, $i^6 = i^4 \times i^2 = -1$, $i^8 = +1$, ...

Substituting for powers of i and using the values already found for $z^n \pm \bar{z}^n$ we have

$$p_0 - p_2 + p_4 - \ldots = 2^{n/2}\cos n\pi/4$$

$$p_1 - p_3 + p_4 - \ldots = 2^{n/2}\sin n\pi/4.$$

8·32. Further Properties of Conjugate Numbers

Let α, β be two complex numbers $\bar{\alpha}$, $\bar{\beta}$ their conjugates. Then we have the following results:

(i) $\alpha + \beta$ and $\bar{\alpha} + \bar{\beta}$ are conjugates, *i.e.* $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$.

(ii) $\alpha\bar{\beta}$ and $\bar{\alpha}\beta$ are conjugates, *i.e.* $\overline{\alpha\bar{\beta}} = \bar{\alpha}\beta$.

(iii) If $z = \alpha/\beta$ then $\bar{z} = \bar{\alpha}/\bar{\beta}$.

(i) Write

$$\alpha = re^{i\theta} = r(\cos\theta + i\sin\theta), \quad \beta = \rho e^{i\phi} = \rho(\cos\phi + i\sin\phi).$$

Then $\bar{\alpha} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$, $\bar{\beta} = \rho e^{-i\phi} = \rho(\cos\phi - i\sin\phi)$.

$$\alpha + \beta = r\cos\theta + \rho\cos\phi + i(r\sin\theta + \rho\sin\phi)$$

$$\overline{\alpha + \beta} = r\cos\theta + \rho\cos\phi - i(r\sin\theta + \rho\sin\phi)$$

$$= r(\cos\theta - i\sin\theta) + \rho(\cos\phi - i\sin\phi) = \bar{\alpha} + \bar{\beta}.$$

In particular $|\alpha + \beta| = |\overline{\alpha + \beta}| = |\bar{\alpha} + \bar{\beta}|$.

It is clear that this theorem will extend to any number of complex numbers.

(ii) $\qquad \alpha\bar{\beta} = re^{i\theta}.\rho e^{-i\phi} = r\rho e^{i(\theta - \phi)}$.

$$\overline{\alpha\bar{\beta}} = r\rho e^{-i(\theta - \phi)} = re^{-i\theta}.\rho e^{i\phi} = \bar{\alpha}\beta.$$

In particular if $\alpha = \beta$, $\bar{\alpha}\bar{\bar{\alpha}} = \alpha\bar{\alpha}$, i.e. the conjugate is the number itself. The number $\alpha\bar{\alpha}$ is of course real.

The results (i) and (ii) may be combined in various ways. Thus, *e.g.* if $\alpha = 1$, $\beta = \bar{a}z$ then $\bar{\alpha} = 1$, $\bar{\beta} = a\bar{z}$ so that

$$|1 + \bar{a}z| = |1 + a\bar{z}|.$$

(iii) $$z = \alpha/\beta = re^{i\theta}/\rho e^{i\phi} = \frac{r}{\rho} e^{i(\theta-\phi)}.$$

$$\bar{z} = \frac{r}{\rho} e^{-i(\theta-\phi)} = re^{-i\theta}/\rho e^{-i\phi} = \bar{\alpha}/\bar{\beta}.$$

Examples.—(1) *Prove that if z and $\bar{z}$ be conjugate points in the Argand diagram, then the equation $z\bar{z} - a\bar{z} - \bar{a}z + c = 0$, c being real and $\bar{a}$ the complex conjugate of a, represents a circle. Prove also that*

$$\bar{a}(\bar{z} - z) + a(\bar{z} + z) + c = 0$$

represents two straight lines, and find the angle between them. [*Lond. B.Sc.*]

Write $z = re^{i\theta}$, $a = \rho e^{i\beta}$. Then $\bar{z} = re^{-i\theta}$, $\bar{a} = \rho e^{-i\beta}$.

Thus $z\bar{z} - a\bar{z} - \bar{a}z + c = r^2 - r\rho e^{-i\theta + i\beta} - r\rho e^{i\theta - i\beta} + c$

$$= r^2 - r\rho\{e^{i(\theta-\beta)} + e^{-i(\theta-\beta)}\} + c$$
$$= r^2 - 2r\rho\cos(\theta - \beta) + c.$$

The equation $r^2 - 2r\rho\cos(\theta - \beta) + c = 0$ represents a circle whose centre is (ρ, β) and whose radius is $\sqrt{\rho^2 - c}$, i.e. the centre is the point on the Argand diagram corresponding to a.

The following is an alternative method. The equation

$$z\bar{z} - a\bar{z} - \bar{a}z + c = 0$$

may be written in the form $(z - a)(\bar{z} - \bar{a}) = a\bar{a} - c$. Now $\bar{z} - \bar{a} = \overline{z - a}$. Hence $(z - a)(\bar{z} - \bar{a}) = |z - a|^2$. Thus the equation is $|z - a|^2 = \rho^2 - c$, from which the result follows immediately.

Now consider the equation $\bar{a}(\bar{z} - z) + a(\bar{z} + z) + c = 0$.

$$\bar{a}(\bar{z} - z) + a(\bar{z} + z) + c = \bar{a}\bar{z} + az + c + a\bar{z} - \bar{a}z$$
$$= (\overline{az} + az + c) + (a\bar{z} - \bar{a}z)$$
$$= \{r\rho e^{-i(\theta+\beta)} + r\rho e^{i(\theta+\beta)} + c\}$$
$$+ \{\rho re^{-i(\theta-\beta)} - \rho re^{i(\theta-\beta)}\}$$
$$= \{2r\rho\cos(\theta + \beta) + c - 2i\rho r\sin(\theta - \beta)\}.$$

Since the real and imaginary parts must be zero,

$$2r\rho\cos(\theta + \beta) + c = 0 \quad \text{(i)}$$
$$2r\rho\sin(\theta - \beta) = 0 \quad \text{(ii)}$$

Equation (ii) is equivalent to $\theta = \beta$, i.e. a straight line through the origin and the point which represents a on the Argand diagram.

In (i) write $x = r\cos\theta$, $y = r\sin\theta$. Thus the Cartesian equation of (i) is

$$x\cos\beta - y\sin\beta + c/2\rho = 0,$$

$$\text{i.e. } x\sin(\tfrac{1}{2}\pi - \beta) - y\cos(\tfrac{1}{2}\pi - \beta) + c/2\rho = 0.$$

This line makes an angle of $\frac{1}{2}\pi - \beta$ with the x-axis. Thus the angle between the two lines is $\frac{1}{2}\pi - 2\beta$.

(2) *If* $A + iB = (x + iy)^2 e^{2int}$, find A and B.

$$A + iB = (x + iy)^2 e^{2int}$$
$$= (x^2 + 2ixy - y^2)(\cos 2nt + i \sin 2nt)$$
$$= (x^2 - y^2) \cos 2nt - 2xy \sin 2nt + 2ixy \cos 2nt$$
$$+ i (x^2 - y^2) \sin 2nt.$$

Equating real and imaginary parts.

$$A = (x^2 - y^2) \cos 2nt - 2xy \sin 2nt, \quad B = 2xy \cos 2nt + (x^2 - y^2) \sin 2nt$$

8·4. De Moivre's Theorem

For all real values of n, $\cos n\theta + i \sin n\theta$ is the value, or one of the values of $(\cos \theta + i \sin \theta)^n$.

We give below a proof of the theorem for the case in which n is *rational*. The following cases are considered separately. (i) n a positive integer, (ii) n a negative integer, (iii) n a positive fraction, (iv) n a negative fraction.

(i) We first prove that

$$(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \ldots (\cos \theta_n + i \sin \theta_n)$$
$$= \cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots \theta_n).$$

The proof is by induction. The case of $n = 2$ has been considered in § 8·22. Suppose the result is true for $n = p$,

i.e. $$\prod_{r=1}^{p} (\cos \theta_r + i \sin \theta_r)$$
$$= \cos (\theta_1 + \theta_2 + \ldots + \theta_p) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_p).$$

For brevity write $\theta_1 + \theta_2 + \ldots + \theta_p = \psi$. Then

$$\prod_{r=1}^{p+1} (\cos \theta_r + i \sin \theta_r)$$
$$= (\cos \psi + i \sin \psi)(\cos \theta_{p+1} + i \sin \theta_{p+1})$$
$$= \cos \psi \cos \theta_{p+1} - \sin \psi \sin \theta_{p+1}$$
$$+ i (\sin \psi \cos \theta_{p+1} + \cos \psi \sin \theta_{p+1})$$
$$= \cos (\psi + \theta_{p+1}) + i \sin (\psi + \theta_{p+1})$$
$$= \cos (\theta_1 + \theta_2 + \ldots + \theta_{p+1}) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_{p+1}).$$

Hence if the result is true for $n = p$, it is true for $n = p + 1$. But the result has been proved for $n = 2$ so that it is true in general.

In this result write $\theta_1 = \theta_2 = \ldots = \theta_n$, then the equation asserts that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

and in this case $\cos n\theta + i \sin n\theta$ is *the* value of $(\cos \theta + i \sin \theta)^n$.

(ii) n a negative integer. Write $n = -m$. Then

$$\cos n\theta + i \sin n\theta = \cos(-m\theta) + i \sin(-m\theta)$$
$$= \cos m\theta - i \sin m\theta$$
$$= (\cos m\theta - i \sin m\theta)/(\cos^2 m\theta + \sin^2 m\theta)$$
$$= 1/(\cos m\theta + i \sin m\theta)$$
$$= 1/(\cos \theta + i \sin \theta)^m \quad \text{from case (i)}$$
$$= (\cos \theta + i \sin \theta)^{-m} = (\cos \theta + i \sin \theta)^n.$$

Hence as in the case of n a positive integer $\cos n\theta + i \sin n\theta$ is *the* value $(\cos \theta + i \sin \theta)^n$.

(iii) $n = p/q$, a positive fraction in its lowest terms where p and q are positive integers. In this case we show that

$$(\cos \frac{p}{q} \theta + i \sin \frac{p}{q} \theta)$$

is *one of the values of* $(\cos \theta + i \sin \theta)^{\frac{p}{q}}$. For this it is sufficient to show that

$$\left(\cos \frac{p}{q} \theta + i \sin \frac{p}{q} \theta\right)^q = (\cos \theta + i \sin \theta)^p.$$

From case (i)

$$\left(\cos \frac{p}{q} \theta + i \sin \frac{p}{q} \theta\right)^q = \cos p\theta + i \sin p\theta = (\cos \theta + i \sin \theta)^p.$$

(iv) $n = -p/q$, p and q as in case (iii). In this case it is sufficient to show that

$$\left\{\cos\left(-\frac{p}{q}\theta\right) + i \sin\left(-\frac{p}{q}\theta\right)\right\}^q = (\cos \theta + i \sin \theta)^{-p}.$$

Now $\left\{\cos\left(-\frac{p}{q}\theta\right) + i \sin\left(-\frac{p}{q}\theta\right)\right\}^q$

$$= \cos(-p\theta) + i \sin(-p\theta), \quad \text{case (i)},$$
$$= (\cos \theta + i \sin \theta)^{-p} \quad \text{case (ii)}.$$

In this case, as in (iii), $\cos n\theta + i \sin n\theta$ is only one of the values of $(\cos \theta + i \sin \theta)^n$.

The other values of $(\cos \theta + i \sin \theta)^n$ when n is fractional.

In this case $n = \pm p/q$ so that it is sufficient to consider $(\cos \theta + i \sin \theta)^{p/q}$, where p may now denote either a positive or negative integer. Let r denote an integer positive or negative.

$$\text{Now } \left\{\cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)\right\}^q$$

$$=\cos(p\theta+2r\pi)+i\sin(p\theta+2r\pi)$$

$$=\cos p\theta+i\sin p\theta=(\cos\theta+i\sin\theta)^p.$$

Hence $\cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)$ is one of the values of $(\cos\theta+i\sin\theta)^{p/q}$.

Now consider the values of the angles $\frac{p\theta}{q}+\frac{2r\pi}{q}$ where $r=0, 1, 2, \ldots (q-1)$. As r ranges over these values $2r\pi/q$ ranges from 0 to $2(q-1)\pi/q<2\pi$. Thus the angles are all different and no two have at the same time equal cosines and equal sines.

Hence the values of $\cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)$ are different for each value of r, i.e. we obtain q distinct values for this expression by taking the values $r=0, 1, 2, \ldots\ q-1$. If we put r equal to any other integral value we repeat one of the values of the expression already found.

Thus we obtain the following important result. *The q different values of the expression $(\cos\theta+i\sin\theta)^{p/q}$ are obtained by writing $r=0, 1, \ldots, q-1$ in the expression*

$$\cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right).$$

In particular *the qth roots of* $\cos\theta+i\sin\theta$ *are given by*

$$\cos\frac{\theta+2r\pi}{q}+i\sin\frac{\theta+2r\pi}{q}$$

where $r=0\ 1, 2, \ldots q-1$.

8·41. The *n*th Roots of Unity

It follows from § 8·4 that if z is a real or complex number and $z=\rho(\cos\theta+i\sin\theta)$ then the nth roots of z, i.e. the n values of $z^{\frac{1}{n}}$ are

$$\rho^{\frac{1}{n}}\left(\cos\frac{\theta+2r\pi}{n}+i\sin\frac{\theta+2r\pi}{n}\right),$$

where $r = 0, 1, 2, \ldots (n-1)$. If z is real, then we can write $\theta = 0$ or π and the n values are

$$\rho^{\frac{1}{n}}\left(\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right) \text{ or } \rho^{\frac{1}{n}}\left(\cos\frac{(2r+1)\pi}{n} + i\sin\frac{(2r+1)\pi}{n}\right),$$

$r = 0, 1, 2, \ldots, (n-1)$. In particular, if $z = 1$, the nth roots are given by

$$\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}.$$

Hence the roots of the equation $z^n = 1$ are obtained by writing $r = 0, 1, 2, \ldots, (n-1)$ in

$$\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n},$$

and thus the values are

$$1, \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots,$$
$$\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}.$$

It is easy to see that all the roots are different. For suppose that

$$\cos\frac{2s\pi}{n} + i\sin\frac{2s\pi}{n} = \cos\frac{2t\pi}{n} + i\sin\frac{2t\pi}{n},$$

where s, t lie between 0 and $n-1$ and $s \neq t$. Then equating real and imaginary parts

$$\cos\frac{2s\pi}{n} = \cos\frac{2t\pi}{n}, \quad \sin\frac{2s\pi}{n} = \sin\frac{2t\pi}{n},$$

and thus the angles $2s\pi/n$ and $2t\pi/n$ must be coterminal. Since both the angles lie between 0 and 2π this is only possible if $s = t$.

If n is even and equal to $2p$ (say) the roots may be written in the form

$$1, \cos\frac{\pi}{p} + i\sin\frac{\pi}{p}, \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}, \ldots,$$
$$\cos\frac{(p-1)\pi}{p} + i\sin\frac{(p-1)\pi}{p},$$
$$\cos\pi + i\sin\pi, \cos\frac{(p+1)\pi}{p} + i\sin\frac{(p+1)\pi}{p}, \ldots$$
$$\cos\frac{(2p-1)\pi}{p} + i\sin\frac{(2p-1)\pi}{p}.$$

Now $\cos\pi = -1$, $\sin\pi = 0$, $\cos\frac{(2p-r)\pi}{p} = \cos\frac{r\pi}{p}$,

$$\text{and } \sin\frac{(2p-r)\pi}{p} = -\sin\frac{r\pi}{p}.$$

Thus the $2p$ roots may be grouped in pairs:

$$\pm 1,\ \cos\frac{\pi}{p} \pm i\sin\frac{\pi}{p},\ \cos\frac{2\pi}{p} \pm i\sin\frac{2\pi}{p}, \ldots,$$
$$\cos\frac{(p-1)\pi}{p} \pm i\sin\frac{(p-1)\pi}{p}.$$

Next suppose that n is odd and equal to $2p+1$ (say). Then the $2p+1$ roots are

$$1,\ \cos\frac{2r\pi}{2p+1} + i\sin\frac{2r\pi}{2p+1},$$

where $r=0, 1, 2, \ldots, 2p$. Then as before it is easily seen that the $2p+1$ roots are

$$1,\ \cos\frac{2\pi}{2p+1} \pm i\sin\frac{2\pi}{2p+1}, \ldots,\ \cos\frac{2p\pi}{2p+1} \pm \sin\frac{2p\pi}{2p+1}.$$

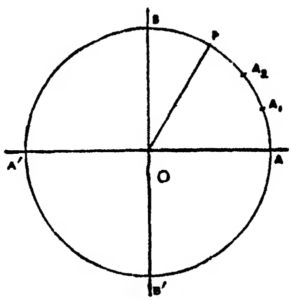


FIG. 34.

8·42. Geometrical Representation of the nth Roots of Unity

Take a circle $ABA'B'$ of unit radius having its centre at the origin, then a point P on this circle represents the quantity

$$\cos\theta + i\sin\theta$$

where θ is the $\angle AOP$ measured as usual, and the point A represents unity.

Now divide the circumference into n equal parts, starting from A, and let $A_1A_2\ldots A_{n-1}$ be the remaining points of section. Then

$$\angle AOA_1 = \frac{2\pi}{n},\ AOA_2 = \frac{4\pi}{n}, \text{ etc.}$$

Hence the points $A, A_1, A_2 \ . \ . \ . \ A_{n-1}$ represent the nth roots of unity. Thus *e.g.* in the figure the points $ABA'B'$ represent the fourth roots of unity.

8·43. The Factors of $z^n - 1$ and $z^n + 1$

Consider $z^n - 1 = 0$ and suppose first that $n = 2p$. Then corresponding to the roots

$$\cos\frac{r\pi}{p} \pm i\sin\frac{r\pi}{p}, \quad r = 1, \ . \ . \ . \ , (p-1)$$

of $z^n - 1 = 0$ we obtain the factors

$$\left(z - \cos\frac{r\pi}{p} - i\sin\frac{r\pi}{p}\right)\left(z - \cos\frac{r\pi}{p} + i\sin\frac{r\pi}{p}\right)$$

$$= \left(z - \cos\frac{r\pi}{p}\right)^2 + \sin^2\frac{r\pi}{p} = z^2 - 2z\cos(r\pi/p) + 1$$

of $z^n - 1$. Hence

$$z^{2p} - 1 = (z^2 - 1)\prod_{r=1}^{p-1}\{z^2 - 2z\cos(r\pi/p) + 1\}.$$

Next suppose that $n = 2p + 1$. Then we obtain in a similar way

$$z^{2p+1} - 1 = (z - 1)\prod_{r=1}^{p}[z^2 - 2z\cos\{2r\pi/(2p+1)\} + 1].$$

These results may be represented in the following forms:

If n is even, $$z^n - 1 = (z^2 - 1)\prod_{r=1}^{\frac{1}{2}n-1}\left\{z^2 - 2z\cos\frac{2r\pi}{n} + 1\right\}.$$

If n is odd, $$z^n - 1 = (z - 1)\prod_{r=1}^{\frac{1}{2}(n-1)}\left\{z^2 - 2z\cos\frac{2r\pi}{n} + 1\right\}.$$

In a similar way the following results may be proved.

If n is even, $$z^n + 1 = \prod_{r=0}^{\frac{1}{2}n-1}[z^2 - 2z\cos\{(2r+1)\pi/n\} + 1].$$

If n is odd, $$z^n + 1 = (z + 1)\prod_{r=0}^{\frac{1}{2}(n-3)}[z^2 - 2z\cos\{(2r+1)\pi/n\} + 1].$$

Examples.—(1) *Find the square root of* $2 + 6\sqrt{(-7)}$, *and show that* $1 - \sqrt{(-3)}$ *is a sixth root of* 64. *What are the other five sixth roots of* 64?

Write $2 + 6i\sqrt{7} = r(\cos\theta + i\sin\theta)$. Then

$$r\cos\theta = 2, \quad r\sin\theta = 6\sqrt{7}.$$

Hence $r^2 = 2^2 + (6\sqrt{7})^2 = 256$, $r = 16$. $\cos\theta = 1/8$, $\sin\theta = 3\sqrt{7}/8$.
Thus we may take θ to be acute angle $\cos^{-1} 1/8$.

The two square roots are

$$r^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta),\ r^{\frac{1}{2}}\{\cos\tfrac{1}{2}(2\pi+\theta) + i\sin\tfrac{1}{2}(2\pi+\theta)\},$$

$$\text{i.e. } \pm r^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta).$$

Now $\cos^2\frac{1}{2}\theta = \frac{1}{2}(1+\cos\theta) = \frac{1}{2}\cdot\frac{9}{8} = \frac{9}{16}$, $\cos\frac{1}{2}\theta = \pm\frac{3}{4}$.

Again, $\sin^2\frac{1}{2}\theta = \frac{1}{2}(1-\cos\theta) = \frac{1}{2}\cdot\frac{7}{8}$, $\sin\frac{1}{2}\theta = \pm\frac{\sqrt{7}}{4}$.

Since $\frac{1}{2}\theta$ is an acute angle, the positive square root must be chosen in each case.

Thus the two square roots of $2 + 6\sqrt{(-7)}$ are $\pm 4\left(\frac{3}{4} + i\frac{\sqrt{7}}{4}\right)$, i.e. $\pm(3 + i\sqrt{7})$.

Next consider the equation $z^6 = 64 = 64(\cos 0 + i\sin 0)$. The six sixth roots are given by

$$64^{\frac{1}{6}}\left\{\cos\frac{2r\pi}{6} + i\sin\frac{2r\pi}{6}\right\},\ \text{i.e. } 2\left\{\cos\frac{r\pi}{3} + i\sin\frac{r\pi}{3}\right\},\ r = 0, 1, 2, 3, 4, 5.$$

The values $r = 0$, $r = 3$ give $z = \pm 2$.

Grouping the other four values in pairs we obtain

$$2\{\cos\tfrac{1}{3}\pi \pm \sin\tfrac{1}{3}\pi\},\ 2\{\cos\tfrac{2}{3}\pi \pm i\sin\tfrac{2}{3}\pi\}$$

$$\text{i.e. } 1 \pm i\sqrt{3},\ -1 \pm i\sqrt{3}.$$

(2) *Prove that all the roots of* $x^7 - 1 = 0$ *may be expressed as powers of*

$$a = \cos\tfrac{2}{7}\pi + i\sin\tfrac{2}{7}\pi.$$

Prove also that $a + a^2 + a^4$ and $a^3 + a^5 + a^6$ *are roots of the equation* $x^2 + x + 2 = 0$. [*Lond. B.Sc.*]

$$\text{Now } x^7 = 1 = \cos 2r\pi + i\sin 2r\pi.$$

$$\text{Hence } x = \cos\frac{2r\pi}{7} + i\sin\frac{2r\pi}{7},\ r = 0, 1, 2, \quad , 6.$$

Thus $x = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^r = a^r$, $r = 1, 2, \ldots, 6$. When $r = 0, x = 1$.

$$\text{Let } \beta = a + a^2 + a^4,\ \gamma = a^3 + a^5 + a^6.$$

The quadratic equation which has β, γ for its roots is

$$x^2 - (\beta + \gamma)x + \beta\gamma = 0.$$

$$\text{Now } \beta + \gamma = a + a^2 + a^3 + a^4 + a^5 + a^6.$$

Again $x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$, so that a is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

$$\text{Hence } a^6 + a^5 + a^4 + a^3 + a^2 + a = -1.$$

$$\begin{aligned}\text{Again, } \beta\gamma &= a^4(1 + a + a^3)(1 + a^2 + a^3)\\ &= a^4(1 + a + a^2 + a^3 + a^4 + a^5 + a^6) + 2a^7\\ &= 2a^7,\ \text{since } a \text{ is a root of } \sum_{r=1}^{6} x^r + 1 = 0,\end{aligned}$$

Hence the quadratic equation is $x^2 + x + 2 = 0$.

(3) *Find the three roots of the equation* $z^3 = 8\,(1 - z)^3$, *expressing the roots in the simplest form.* [*N. Sc.*]

The equation may be written in the form

$$\left(\frac{z}{1-z}\right)^3 \qquad 8 = 2^3\,(\cos 0 + i \sin 0).$$

Hence the three values of $z/(1 - z)$ are

$$2, \quad 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), \quad 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$\text{i.e. } 2, \quad -1 \pm i\sqrt{3}.$$

The corresponding values of z are

$$\tfrac{2}{3}, \quad \frac{-1+i\sqrt{3}}{i\sqrt{3}}, \quad \frac{-1-i\sqrt{3}}{-i\sqrt{3}},$$

$$\text{i.e. } \tfrac{2}{3}, \quad 1 + \tfrac{1}{3}i\sqrt{3}, \quad 1 - \tfrac{1}{3}i\sqrt{3}.$$

(4) *Solve the equation* $(1 + z)^{2n} - (1 - z^2)^n + (1 - z)^{2n} = 0$, *where n is a positive integer.* [*Madras, B.A.*]

Write $(1 + z)^n = u$, $(1 - z)^n = v$ and the equation becomes

$$u^2 - uv + v^2 = 0.$$

Now $u^2 - uv + v^2 = (u^3 + v^3)/(u + v)$. Hence the roots of the original equation will be the roots of $u^3 + v^3 = 0$, excluding those which are also roots of $u + v = 0$. Now the roots of $u^3 + v^3 = 0$ are given by

$$\left(\frac{1+z}{1-z}\right)^{3n} = -1.$$

$$\text{Hence } \left(\frac{1+z}{1-z}\right)^{3n} = \cos(2r+1)\pi + i\sin(2r+1)\pi, \text{ and}$$

$$\frac{1+z}{1-z} = \cos\frac{(2r+1)\pi}{3n} + i\sin\frac{(2r+1)\pi}{3n},$$

where $r = 0, 1, 2, 3, \ldots, 3n - 1$.

The roots of $u + v = 0$ are given by

$$\left(\frac{1+z}{1-z}\right)^{n} = -1 = \cos(2r+1)\pi + i\sin(2r+1)\pi.$$

$$\text{Thus } \frac{1+z}{1-z} = \cos\frac{(2r+1)\pi}{n} + i\sin\frac{(2r+1)\pi}{n},$$

$$r = 0, 1, 2, 3, \ldots (n-1).$$

Hence the roots of the required equation are

$$\frac{1+z}{1-z} = \cos\frac{(2r+1)\pi}{3n} + i\sin\frac{(2r+1)\pi}{3n},$$

where r has any of the values 0 to $3n - 1$, except those for which $2r + 1$ is a multiple of 3. Writing $(2r + 1)\pi/3n = \lambda_r$, the values of z are

$$(\cos\lambda_r - 1 + i\sin\lambda_r)/(1 + \cos\lambda_r + i\sin\lambda_r).$$

(5) *Show that all the roots of the equation* $(1 + x)^{2n+1} = (1 - x)^{2n+1}$ *are given by* $\pm i \tan\{r\pi/(2n + 1)\}$, *where r has the values* $0, 1, 2, \ldots, n$. *By putting* $n = 2$, *or otherwise, show that*

$$\tan^2\frac{\pi}{5}.\tan^2\frac{2\pi}{5} = 5. \qquad \text{[Lond. B.Sc.]}$$

Now $\{(1+x)/(1-x)\}^{2n+1} = 1$. Hence the $2n+1$ values of $(1+x)/(1-x)$ are given by

$$\cos\frac{2r\pi}{2n+1} + i\sin\frac{2r\pi}{2n+1}, \quad r = 0, 1, 2, \ldots, 2n.$$

Write $2r\pi/(2n+1) = t$. Then

$$\frac{1+x}{1-x} = \cos t + i\sin t.$$

$$\frac{(1+x)-(1-x)}{(1+x)+(1-x)} \quad \frac{\cos t + i\sin t - 1}{\cos t + i\sin t + 1}$$

$$\text{i.e. } x = \frac{2i\sin\frac{1}{2}t\cos\frac{1}{2}t - 2\sin^2\frac{1}{2}t}{2i\sin\frac{1}{2}t\cos\frac{1}{2}t + 2\cos^2\frac{1}{2}t} = \frac{i\tan\frac{1}{2}t - \tan^2\frac{1}{2}t}{i\tan\frac{1}{2}t + 1}$$

$$= \frac{i\tan\frac{1}{2}t\,(1 + i\tan\frac{1}{2}t)}{i\tan\frac{1}{2}t + 1} = i\tan\tfrac{1}{2}t.$$

Thus the roots are 0, $i\tan\{r\pi/(2n+1)\}$, $r = 1, 2, \ldots 2n$.

Now if s denote a positive integer, $0 < s < n-1$,

$$\tan\{(2n-s)\,\pi/(2n+1)\} = \tan\{\pi - (s+1)\,\pi/(2n+1)\}$$
$$= -\tan\{(s+1)\,\pi/(2n+1)\}.$$

Hence the $2n$ roots corresponding to $r = 1, 2, \ldots, 2n$ may be grouped in pairs of the form $\pm\, i\tan r\pi/(2n+1)$, i.e. the $2n$ roots are

$$\pm\, i\tan\{r\pi/(2n+1)\}, \quad r = 0, 1, 2, \ldots n.$$

If $n = 2$, then the roots of the equation $(1+x)^5 = (1-x)^5$ are

$$0, \ \pm\, i\tan\tfrac{1}{5}\pi, \ \pm\, i\tan\tfrac{2}{5}\pi.$$

The last equation may be written in the form $x\,(x^4 + 10x^2 + 5) = 0$. The roots of $x^4 + 10x^2 + 5 = 0$ are $\pm\, i\tan\frac{1}{5}\pi$, $\pm\, i\tan\frac{2}{5}\pi$.

Putting $x^2 = y$ it is clear that the roots of $y^2 + 10y + 5 = 0$ are $-\tan^2\frac{1}{5}\pi$, $-\tan^2\frac{2}{5}\pi$.

The product of the two roots is 5. Hence

$$\tan^2\tfrac{1}{5}\pi . \tan^2\tfrac{2}{5}\pi = 5.$$

(6) *If n is a positive integer prove that*

$$\left\{\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right\}^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right).$$

Write $\theta = \frac{1}{2}\pi - \phi$. Then

$$\left\{\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right\}^n = \left\{\frac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi}\right\}^n$$

$$= \left\{\frac{2\cos^2\frac{1}{2}\phi + 2\,i\sin\frac{1}{2}\phi.\cos\frac{1}{2}\phi}{2\cos^2\frac{1}{2}\phi - 2\,i\sin\frac{1}{2}\phi.\cos\frac{1}{2}\phi}\right\}^n$$

$$= \left\{\frac{\cos\frac{1}{2}\phi + i\sin\frac{1}{2}\phi}{\cos\frac{1}{2}\phi - i\sin\frac{1}{2}\phi}\right\}^n$$

$$= \left\{\frac{(\cos\frac{1}{2}\phi + i\sin\frac{1}{2}\phi)^2}{\cos^2\frac{1}{2}\phi + \sin^2\frac{1}{2}\phi}\right\}^n$$

$$= \{\cos\tfrac{1}{2}\phi + i\sin\tfrac{1}{2}\phi\}^{2n} = \cos n\phi + i\sin n\phi$$

$$= \cos(\tfrac{1}{2}n\pi - n\theta) + i\sin(\tfrac{1}{2}n\pi - n\theta).$$

(7) *Prove that* $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$. *Hence show that* $x = 2\cos\frac{1}{7}\pi$ *is a root of the equation*

$$x^3 - x^2 - 2x + 1 = 0$$

and find the other roots.

From de Moivre's theorem

$$(\cos 7\theta + i\sin 7\theta) = (\cos\theta + i\sin\theta)^7$$

$$= \cos^7\theta + 7\cos^6\theta . i\sin\theta + 21\cos^5\theta . i^2\sin^2\theta + 35\cos^4\theta . i^3\sin^3\theta$$
$$+ 35\cos^3\theta . i^4\sin^4\theta + 21\cos^2\theta . i^5\sin^5\theta + 7\cos\theta . i^6\sin^6\theta + i^7\sin^7\theta.$$

Equating real parts,

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta . \sin^2\theta + 35\cos^3\theta . \sin^4\theta - 7\cos\theta . \sin^6\theta.$$

In this equation write $\sin^2\theta = (1 - \cos^2\theta)$ and we have the required form.

In $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$, write $x = 2\cos\theta$.

$$\text{Then } 2\cos 7\theta = x^7 - 7x^5 + 14x^3 - 7x.$$

In this equation write $\cos 7\theta = -1$. Then

$$\theta = \tfrac{1}{7}\pi, \tfrac{3}{7}\pi, \tfrac{5}{7}\pi, \pi, \tfrac{9}{7}\pi, \tfrac{11}{7}\pi, \tfrac{13}{7}\pi, \ldots \text{ satisfy } \cos 7\theta = -1.$$

It follows that $2\cos\frac{1}{7}\pi$, $2\cos\frac{3}{7}\pi$, ... are roots of

$$-2 = x^7 - 7x^5 + 14x^3 - 7x.$$

Now one root is $x = 2\cos\pi = -2$. Hence $x + 2$ must be a factor of $x^7 - 7x^5 + 14x^3 - 7x + 2$.

It is easily verified that

$$x^7 - 7x^5 + 14x^3 - 7x + 2 = (x + 2)(x^6 - 2x^5 - 3x^4 + 6x^3 + 2x^2 - 4x + 1),$$

Again, $\cos\frac{1}{7}\pi = \cos\frac{13}{7}\pi$, $\cos\frac{3}{7}\pi = \cos\frac{11}{7}\pi$, $\cos\frac{5}{7}\pi = \cos\frac{9}{7}\pi$.

Further, if we take more values of θ the new angles will be coterminal with one of those already considered. Thus, e.g. $\theta = \frac{15}{7}\pi = \frac{1}{7}\pi + 2\pi$ which is coterminal with $\frac{1}{7}\pi$. It follows that we can obtain no further values for $\cos\theta$. Thus the only distinct roots of

$$x^6 - 2x^5 - 3x^4 + 6x^3 + 2x^2 - 4x + 1 = 0$$

are $x = 2\cos\frac{1}{7}\pi$, $2\cos\frac{3}{7}\pi$, $2\cos\frac{5}{7}\pi$. It follows that the equation must have repeated roots, and it is easily verified that

$$x^6 - 2x^5 - 3x^4 + 6x^3 + 2x^2 - 4x + 1 = (x^3 - x^2 - 2x + 1)^2.$$

Hence the roots of $x^3 - x^2 - 2x + 1 = 0$ are $x = 2\cos\frac{1}{7}\pi$, $2\cos\frac{3}{7}\pi$ and $2\cos\frac{5}{7}\pi$.

8·44. Complex Sequences

Let $\{x_n\}$, $\{y_n\}$ denote two sequences of real numbers. Then the sequence $\{z_n\} \equiv \{x_n + iy_n\}$ is a sequence whose terms are complex.

A complex sequence $\{z_n\}$ is *bounded* if there exists a positive number k independent of n such that $|z_n| \leqslant k$, for all values of n. Since $|z_n| = \sqrt{(x_n^2 + y_n^2)}$ it follows that the sequences $\{x_n\}$, $\{y_n\}$ must also be bounded.

A sequence $\{z_n\}$ is said to be *convergent* or tend to a limit as $n \to \infty$ if $\{x_n\}$ and $\{y_n\}$ separately converge as $n \to \infty$. Thus if $x_n \to x$, $y_n \to y$ then $z_n \to z = x + iy$.

The *general principle of convergence* for complex sequences may be stated as follows: Let ϵ be an arbitrary positive number. Then *the necessary and sufficient condition that the sequence* $\{z_n\}$ *converge is that there exists a positive number* n_0 *such that for all* $n \geqslant n_0$ *and for every positive integer* p,

$$| z_{n+p} - z_n | < \epsilon.$$

The result may be proved from the properties of real numbers.

(i) The condition is *necessary.*

Suppose z_n converges so that by definition x_n and y_n converge. Hence corresponding to $\frac{1}{2}\epsilon$ there exist positive integers n_1, n_2 such that

$$\text{(i)} \qquad | x_{n+p} - x_n | < \tfrac{1}{2}\epsilon,\ n \geqslant n_1.$$

$$\text{(ii)} \qquad | y_{n+p} - y_n | < \tfrac{1}{2}\epsilon,\ n \geqslant n_2.$$

If n_0 denote the greater of n_1, n_2 then

$$\begin{aligned} | z_{n+p} - z_n | &= | x_{n+p} - x_n + i\,(y_{n+p} - y_n) | \\ &\leqslant | x_{n+p} - x_n | + | y_{n+p} - y_n | \\ &< \epsilon,\ n \geqslant n_0. \end{aligned}$$

Thus the condition is necessary.

(ii) The condition is *sufficient.*

We know that corresponding to ϵ, there exists n_0 such that

$$| z_{n+p} \quad z_n | < \epsilon,\ n \geqslant n_0,\ p \text{ a positive integer}$$

Since $| x_{n+n} - x. \quad \leqslant | z_{n+p} - z_n |$ it follows that

$$| x_{n+p} - x_n | < \epsilon,\ n \geqslant n_0,\ p \text{ a positive integer.}$$

Hence $x_n \to$ a limit. Similarly $y_n \to$ a limit and so $z_n \to$ a limit also.

If $\{z_n\}$ is bounded and does not tend to a limit as $n \to \infty$ then the sequence is said to oscillate finitely.

If $\{z_n\}$ is unbounded then at least one of $\{x_n\}$, $\{y_n\}$ is unbounded, and we can regard the sequence as oscillating infinitely.

If $x_n \to \alpha$, $y_n \to \infty$ we can write $z_n \to \alpha + i\,\infty$ and say that z_n tends to infinity in the direction of the y-axis. If $x_n \to \infty$, $y_n \to \beta$, so that $z_n \to \infty + i\beta$, we can say that $z_n \to \infty$ in the direction of the x-axis.

In general, if $y_n/x_n = \tan \gamma_n$ and $\tan \gamma_n \to \tan \gamma$ we can say that $z_n \to \infty$ in the direction γ. Observe that γ_n is the amplitude of z_n, so that what we are asserting is that amp. z_n tends to a limit.

As an example, consider the sequence $\{z_n\} = z^n$ where $z = re^{i\theta}$, r being the modulus, θ the amplitude of z. Then $x_n = r^n \cos n\theta$, $y_n = r^n \sin n\theta$, $z_n = r^n (\cos n\theta + i \sin n\theta)$. Since $|\cos n\theta| \leqslant 1$, $|\sin n\theta| \leqslant 1$ for all real values of θ, $\{x_n\}$ and $\{y_n\}$ are bounded for $r \leqslant 1$, and unbounded for $r > 1$.

For $0 < r < 1$, $\lim\limits_{n \to \infty} r^n = 0$ so that $z_n \to 0$ $0 < r < 1$ for all values of θ. Thus the sequence converges for all points inside the circle $|z| = 1$ in the Argand plane. When $r = 1$, $x_n = \cos n\theta$, $y_n = \sin n\theta$.

If $\theta = 0$ or $2p\pi$ where p is a positive or negative integer $x_n = 1$, $y_n = 0$, giving $z_n = 1$. For other values of θ, x_n and y_n oscillate between $+1$ and -1.

Hence, on the circle $|z| = 1$ the sequence oscillates finitely except at $z = 1$ where it converges to unity. When $r > 1$, $r^n \to \infty$, and the sequence oscillates infinitely.

8·5. Series with Complex Terms

Let $\{a_n\} \equiv \{u_n + iv_n\}$ denote a complex sequence. Consider the sum

$$z_n = \sum_{r=1}^{n} a_n = \sum_{r=1}^{n} (u_n + iv_n) = x_n + iy_n \text{ where}$$

$$x_n = \sum_{r=1}^{n} u_n, \; y_n = \sum_{r=1}^{n} v_n.$$

If the series $\sum\limits_{r=1}^{\infty} u_n$, $\sum\limits_{r=1}^{\infty} v_n$ are both convergent, *i.e.* both x_n and y_n tend to limits as $n \to \infty$ then the complex series Σa_n is said to *converge*. Thus if $x_n \to x$, $y_n \to y$, then the sum of the series is $z = x + iy$. If *either* x_n or y_n diverges the series Σa_n is said to diverge.

If $\{z_n\}$ is bounded and z_n does not tend to a limit the series Σa_n is said to oscillate finitely.

If $\{z_n\}$ is unbounded then at least one of the sequences $\{x_n\}$, $\{y_n\}$ must be unbounded and we can say, in general, that the series oscillates infinitely.

8·51. Absolute Convergence of a Complex Series

If $\Sigma |a_n| = \Sigma |u_n + iv_n| = \Sigma \sqrt{(u_n^2 + v_n^2)}$ converges the series is said to be absolutely convergent. It follows that since

$|u_n| \leqslant |a_n|$, $|v_n| \leqslant |a_n|$ the series Σu_n, Σv_n are absolutely convergent series of real terms.

Thus Σu_n, Σv_n may have the order of their terms deranged without affecting the sum of the series and the same property is true for Σa_n if $\Sigma |a_n|$ converges.

We now consider a ratio test for absolute convergence. Since $|a_n|$ is a *positive real number* it is clear that we can apply any suitable test from among those considered in connection with real numbers. Probably the most important test is the extension of Gauss' rule. Suppose that the ratio a_n/a_{n+1} is expressed in the form

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right), \quad \left\{\begin{matrix} \mu = \lambda + i\nu \\ p > 1 \end{matrix}\right\}.$$

In order to test for absolute convergence, consider

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right)\right| = \left|1 + \frac{\lambda}{n} + \frac{i\nu}{n} + O\left(\frac{1}{n^p}\right)\right|$$

$$= \left[\left\{1 + \frac{\lambda}{n} + O\left(\frac{1}{n^p}\right)\right\}^2 + \left\{\frac{\nu}{n} + O\left(\frac{1}{n^p}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= \left[1 + \frac{2\lambda}{n} + O\left(\frac{1}{n^p}\right) + O\left(\frac{1}{n^2}\right)\right]^{\frac{1}{2}}$$

$$= 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^q}\right), \; q > 1.$$

Hence the series converges absolutely if $\lambda > 1$. Thus *if*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right), \; p > 1$$

the series converges absolutely, provided $R(\mu) > 1$, *where* $R(\mu)$ *denotes the real part of* μ.

Example.—*Consider the hypergeometric series*

$$1 + \frac{\alpha.\beta}{1.\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\gamma(\gamma+1)} z^2 + \ldots$$

[*Chapter VII.*, § 7·51.]

where α, β, γ. z *are supposed to be complex.*

Now $\dfrac{a_n}{a_{n+1}} \quad \dfrac{n(\gamma+n-1)}{(\alpha+n-1)(\beta+n-1)} \cdot \dfrac{\,}{z}, \quad \lim_{n\to\infty} \quad \dfrac{1}{|z|}$.

Hence the series converges absolutely if $|z| < 1$. Now suppose $|z| = 1$.

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{1 + \dfrac{\gamma - 1}{n}}{\left(1 + \dfrac{\alpha - 1}{n}\right)\left(1 + \dfrac{\beta - 1}{n}\right)}\right| = \left|1 + \frac{\gamma + 1 - \alpha - \beta}{n} + O\left(\frac{1}{n^2}\right)\right|$$

Hence if $|z| = 1$ the series will converge absolutely if the real part of $\gamma - \alpha - \beta$ is positive.

8·52. Comparison Test for Absolute Convergence*

If Σa_n is an absolutely convergent series of complex terms and λ_n is a complex sequence such that $|\lambda_n| < k$ for all n, where k denotes a fixed positive number, then the series $\Sigma a_n \lambda_n$ is absolutely convergent.

Since Σa_n is absolutely convergent, there exists a number n_0 corresponding to the arbitrary positive ϵ such that

$$\sum_{r=n+1}^{n+p} |a_r| < \epsilon/k,\ n \geqslant n_0.$$

for every positive integer p.

$$\text{Now}\quad \sum_{r=n+1}^{n+p} |a_r\lambda_r| = \sum_{r=n+1}^{n+p} \{|a_r| \times |\lambda_r|\} < \sum_{r=n+1}^{n+p} k\,|a_r|$$

$$\text{which is} < \epsilon,\ n \geqslant n_0.$$

Hence the series $\Sigma a_r \lambda_r$ converges absolutely.

8·53. Power Series

A power series $\Sigma a_n z^n$ may be convergent for all values of z, for a certain region of values of z only, or for no values of z except $z = 0$. Examples of the first two types are given in § 8·55. Thus the binomial series $\sum_{n=0}^{\infty} \binom{n}{r} z^n$ converges for $|z| < 1$, while the exponential series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all values of z. An example of a power series which converges only for $z = 0$ is $\Sigma a_n z^n$ where $a_n = n!$. For $\frac{a_{n+1}z^{n+1}}{a_n z^n} = (n+1) \quad \infty$ as $n \quad \infty$.

Let $\Sigma a_n z^n$ denote a power series where z, a_n may be real or complex and write $\lim_{n \to \infty} |a_n|^{\frac{1}{n}} = 1/l$. The series will converge

* For a further discussion on convergence of series of complex terms, and for the treatment of infinite products and uniform convergence when the numbers are complex, the reader may refer to Bromwich, *Infinite Series*.

absolutely if

$$\lim_{n \to \infty} | a_n z^n |^{\frac{1}{n}} < 1$$

and cannot converge if $\lim\limits_{n \to \infty} \; | a_n z^n \qquad 1$, for in this case it is clear that $a_n z^n$ cannot tend to zero as $n \to \infty$.

It follows that $\Sigma a_n z^n$ converges absolutely if $| z | < l$ and cannot converge if $| z | > l$.

For a complex series, l is called the radius of convergence. We can see the reason for this by considering the interpretation in the Argand diagram. Taking OX and OY as the real and imaginary axes as before, draw a circle whose centre is O and whose radius is l (Fig. 35). Then if P is any point z_1 inside the circle, we have $| z_1 | = OP < l$, while if Q is any point z_2 outside the circle, $| z_2 | = OQ > l$. Hence the power series converges absolutely if z is a point inside the circle and does not converge if z lies outside the circle. At points on the boundary of the circle $| z | = l$ and at such points the series may or may not converge. Thus the circle is called **the circle of convergence.**

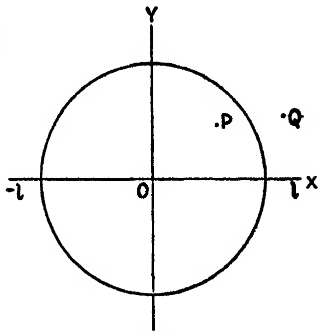


FIG. 35.

We now show that if $\lim\limits_{n \to \infty} \; a_n$ exists, then this limit is l, the radius of convergence of the power series. The result follows immediately from the following theorem on limits. *Let $\{b_n\}$ denote a sequence of positive terms such that lim. $\{b_{n+1}/b_n\}$ exists. Then*

$\lim\limits_{n \to \infty} (b_n)^{\frac{1}{n}}$ also exists and the two limits are equal.

Write $\lim\limits_{n \to \infty} \{b_{n+1}/b_n\} = \rho$. Then corresponding to ϵ there exists a positive integer ν such that for $n \geqslant \nu$,

$$\rho (1 - \epsilon) < b_{n+1}/b_n < \rho (1 + \epsilon).$$

$$\text{Thus } \rho (1 - \epsilon) < b_{\nu+1}/b_\nu < \rho (1 + \epsilon),$$

$$\rho (1 - \epsilon) < b_{\nu+2}/b_{\nu+1} < \rho (1 + \epsilon)$$

. . . .

$$\rho(1-\epsilon) < b_n/b_{n-1} < \rho(1+\epsilon).$$

Multiplying the inequalities together,

$$\rho^{n-\nu}(1-\epsilon)^{n-\nu} < b_n/b_\nu < \rho^{n-\nu}(1+\epsilon)^{n-\nu}$$

$$\text{i.e. } \frac{b_\nu}{\rho^\nu}(1-\epsilon)^{n-\nu} < \frac{b_n}{\rho^n} < \frac{b_\nu}{\rho^\nu}(1+\epsilon)^{n-\nu},$$

$$\text{i.e. } \frac{b_\nu}{\rho^\nu}(1-\epsilon)^{n} < \frac{b_n}{\rho^n} < \frac{b_\nu}{\rho^\nu}(1+\epsilon)^{n}.$$

Taking the nth root, $\left(\frac{b_\nu}{\rho^\nu}\right)^{\frac{1}{n}}(1-\epsilon) < \frac{b_n^{\frac{1}{n}}}{\rho} < \left(\frac{b_\nu}{\rho^\nu}\right)^{\frac{1}{n}}(1+\epsilon)$.

Now if a is any fixed positive number, $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1$. Hence there exists a number $\nu_1 \geqslant \nu$ such that

$$1-\epsilon < \left(\frac{b_\nu}{\rho^\nu}\right)^{\frac{1}{n}} < 1+\epsilon,\ n \geqslant \nu_1.$$

$$\text{Thus } (1-\epsilon)^2 < \frac{b_n^{\frac{1}{n}}}{\rho} < (1+\epsilon)^2,\ n \geqslant \nu_1$$

$$\text{i.e. } -\epsilon(2-\epsilon) < \frac{b_n^{\frac{1}{n}}}{\rho} - 1 < \epsilon(2+\epsilon).$$

$$\text{Hence } \lim_{n\to\infty}(b_n^{\frac{1}{n}}/\rho) = 1, \text{ i.e. } \lim_{n\to\infty} b_n^{\frac{1}{n}} = \rho.$$

This proves the theorem. Now write

$$b_n = |a_n|,\ b_{n+1} = |a_{n+1}|,\ \rho = 1/l.$$

Then $\lim\limits_{n\to\infty} |a_n/a_{n+1}| = 1/\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = l$.

We now prove the following theorem.

If the power series $\Sigma a_n z^n$ converges for $z = z_1$, it is absolutely convergent for $|z| < |z_1|$.

Write $f(z) = \Sigma a_n z^n = a_0 + a_1 z + \ldots + a_n z^n + R_n(z)$, where

$$R_n(z) = a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots \text{ to } \infty$$

$$= a_{n+1}z_1^{n+1}\left(\frac{z}{z_1}\right)^{n+1} + a_{n+2}z_1^{n+2}\left(\frac{z}{z_1}\right)^{n+2} + \ldots$$

Since the series converges for $z = z_1$, there exists a positive constant k such that $|a_r z_1^r| < k$, for all r. Hence

$$|R_n(z)| \leqslant \left|a_{n+1}z_1^{n+1}\left(\frac{z}{z_1}\right)^{n+1}\right| + \left|a_{n+2}z_1^{n+2}\left(\frac{z}{z_1}\right)^{n+2}\right| + \dots$$

$$< k\left\{\left|\frac{z}{z_1}\right|^{n+1} + \left|\frac{z}{z_1}\right|^{n+2} + \dots\right\}$$

$$= k\left|\frac{z}{z_1}\right|^{n+1}\left\{1 + \left|\frac{z}{z_1}\right| + \dots\right\}$$

$$= k\left|\frac{z}{z_1}\right|^{n+1}\Big/\left\{1 - \left|\frac{z}{z_1}\right|\right\}.$$

Since $k\Big/\left(1 - \left|\frac{z}{z_1}\right|\right)$ is finite and $\left|\frac{z}{z_1}\right| < 1$ it follows that $\lim\limits_{n\to\infty} |R_n(z)| = 0$. It follows that the series must converge absolutely.

Interpreted in the Argand plane this theorem asserts that *if the power series converges at a point P then it converges absolutely at all points nearer to the origin than P.*

It follows that any given power series will possess one of the three following properties.

Either (i) *it converges absolutely for all values of* z; or

(ii) *it converges absolutely for all values of* z *within a certain circle and does not converge for any value of* z *outside the circle*; or

(iii) *it converges for* $z = 0$ *and for no other value of* z.

8·54. Theorem on Identical Equality between Power Series

The case for real series has been considered in Chapter I., § 1·8.

Let $\Sigma a_n z^n$, $\Sigma b_n z^n$ *be two power series such that each converges to the same value for all values of* z *such that* $|z| < l$, $l > 0$. *Then* $a_n = b_n$ *for every value of* n.

Using the notation of § 8·53:

$$a_0 + a_1 z + a_2 z^n + \dots + a_n z^n + R_n(z)$$
$$= b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n + R^1{}_n(z), \quad \dots\text{(i)}$$

for $|z| < l$, where $R_n(z) \to 0$, $R^1{}_n(z) \to 0$ as $n \to \infty$, for each value of z.

Now let $|z| \to 0$. Then it is clear that for $|z|$ sufficiently small

$$|a_0 - b_0| < \epsilon$$

ϵ is arbitrary. It follows that $a_0 = b_0$.

Again, since the equality (i) holds for $|z| > 0$, we can divide throughout by z. Thus

$$a_1 + a_2z + a_2z^2 + \ldots = b_1 + b_2z + b_2z^2 + \ldots$$

Proceeding as before it follows that $a_1 = b_1$. Continuing the process, $a_n = b_n$ for every n.

This theorem asserts the *uniqueness* of a power series, i.e. *the same function $f(z)$ cannot be represented by two different power series.*

8·55. Special Series

(*a*) THE BINOMIAL SERIES.—*If n is real and z complex the series*

$$1 + nz + \frac{n(n-1)}{2!}z^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}z^r + \ldots$$

$$= \sum_{n=0}^{\infty} \binom{n}{r} z^r$$

converges absolutely if $|z| < 1$. Further, if $n > 0$, the series converges absolutely at all points on the circle of convergence, while for $0 > n > -1$ the series converges at all points of the circle except at $z = -1$, but the convergence is not absolute.

The radius l of the circle of convergence of the power series is

$$\lim_{r\to\infty} \left|\frac{a_r}{a_{r+1}}\right| = \lim_{r\to\infty} \left|\frac{r}{n-r+1}\right| = 1.$$

Thus the series converges absolutely if $|z| < 1$ and is not convergent for $|z| > 1$.

At points on the circle of convergence $|z| = 1$ and for such points we may write $z = \cos\theta + i\sin\theta$. Thus the series becomes $\Sigma a_r(\cos r\theta + i\sin r\theta)$ and will be convergent provided $\Sigma a_r \cos r\theta$ and $\Sigma a_r \sin r\theta$ are convergent.

From Chapter I., § 1·52 it follows that Σa_r converges if $n + 1 > 0$, and $\Sigma|a_r|$ converges for $n > 0$. Thus Σa_r is absolutely convergent for $n > 0$. Hence $\Sigma a_r(\cos r\theta + i\sin r\theta)$ is absolutely convergent for $n > 0$.

Next consider $-1 < n < 0$. In this case as the terms are alternately positive and negative the series may be written in the form

$$\Sigma (-1)^r \mid a_r \mid (\cos r\theta + i \sin r\theta)$$
$$= \Sigma \mid a_r \mid \{\cos r(\theta + \pi) + i \sin r(\theta + \pi)\}.$$

Now $\left|\dfrac{a_r}{a_{r+1}}\right| = \dfrac{r}{r - n - 1} > 1$, i.e. $\mid a_r \mid > \mid a_{r+1} \mid$. Also $\lim\limits_{r \to \infty} \mid a_r \mid = 0$.

Hence $\Sigma \mid a_r \mid \{\cos r(\theta + \pi) + i \sin r(\theta + \pi)\}$ converges provided $\theta + \pi$ is not zero or an even multiple of π, *i.e.* θ is not an odd multiple of π. It will be observed that the convergence is not absolute, since $\Sigma \mid a_r \mid$ diverges.

In the exceptional case the series $\Sigma \mid a^r \mid \cos r(\theta + \pi)$ diverges, i.e. $\Sigma a_r \cos r\theta$ diverges. Now when θ is an odd multiple of π, $z = -1$. Hence we conclude that the series $\Sigma \dbinom{n}{r} z^r$ converges at all points of the circle of convergence if $0 > n > -1$, except at $z = -1$.

Note.—If we replace n by a *complex number* ν, it follows from Gauss' test, § 8·51, that the binomial series will converge absolutely on the circle of convergence if the real part of ν is positive.

(*b*) THE EXPONENTIAL SERIES.—Consider the series

$$E(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots = \sum_{=0} \frac{z^n}{n!}.$$

The radius of convergence of the power series is given by $\lim\limits_{n \to \infty} \dfrac{(n+1)!}{n!} \to \infty$. Hence if R be any real number as large as we like, the series $z^n/n!$ converges absolutely for $\mid z \mid \leqslant R$.

In Chapter V., § 5·1, it is proved that if ν, y, x are real, and $\lim\limits_{\nu \to \infty} (\nu y) = x$, then

$$\lim_{\nu \to \infty} (1 + y)^\nu = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

It is easy to extend the argument to the case in which y, x are complex, ν still being real. Thus if ζ, z be any complex numbers such that $\lim\limits_{\nu \to \infty} (\nu\zeta) = z$, then

$$\lim_{\nu \to \infty} (1 + \zeta)^\nu = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

It may then be shown that the function $E(z)$ defined by this series satisfies the fundamental equation

$$E(z_1 + z_2) = E(z_1) \times E(z_2).$$

It is usual to denote the function $E(z)$ by e^z when z is complex, the fundamental equation just stated being then written in the form

$$e^{z_1+z_2} = e^{z_1} \times e^{z_2}.$$

(c) THE LOGARITHMIC SERIES.—This is the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} z^n$.

The radius of convergence is $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$. Hence the series converges absolutely for $|z| < 1$ and does not converge for $|z| > 1$. When z is on the circle of convergence we can write $z = e^{i\theta}$, $z^n = e^{ni\theta}$ and the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{ni\theta} = - \sum_{n=1}^{\infty} \frac{e^{n\pi i}}{n} . e^{ni\theta} = - \sum_{n=1}^{\infty} \frac{e^{ni\phi}}{n},$$

where $\phi = \theta + \pi$, $= - \sum\limits_{n=1}^{\infty} \dfrac{\cos n\phi}{n} - i \sum\limits_{n=1}^{\infty} \dfrac{\sin n\phi}{n}$.

If s denotes a positive integer,

$$\sum_{n=1}^{s} \cos n\phi + i \sum_{n=1}^{s} \sin n\phi = \frac{1 - e^{si\theta}}{1 - e^{i\theta}}.$$

Now $|1 - e^{si\phi}| \leqslant 2$ and $|1 - e^{i\theta}| = \sqrt{[(1 - \cos\phi)^2 + \sin^2\phi]}$
$= 2 \sin \tfrac{1}{2}\phi.$

Hence $\left| \sum\limits_{n=1}^{s} \cos n\phi + i \sum\limits_{n=1}^{s} \sin n\phi \right| \leqslant \operatorname{cosec} \tfrac{1}{2}\phi$, giving

$$\sum_{n=1}^{s} \cos n\phi \leqslant \operatorname{cosec} \tfrac{1}{2}\phi, \quad \text{and} \quad \sum_{n=1}^{s} \sin n\phi \leqslant \operatorname{cosec} \tfrac{1}{2}\phi.$$

Now $\operatorname{cosec} \tfrac{1}{2}\phi$ is finite provided $\phi \neq 0$ or an even multiple of π, *i.e.* provided $\theta = \phi - \pi$ is not an odd multiple of π.

It follows from Dirichlet's test (§ 7·7) that each of the series

$$- \sum_{n=1}^{\infty} \frac{\cos n\phi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos n\theta, \quad - \sum_{n=1}^{\infty} \frac{\sin n\phi}{n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin n\theta$$

converges provided $\theta \neq$ an odd multiple of π.

If θ is an odd multiple of π, say $(2p + 1)$, then

$$z = e^{i(2p+1)\pi} = -1$$

and the series becomes $-\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges to $-\infty$. Hence the logarithmic series converges at all points of its circle of convergence except at $z = -1$.

8·56. Behaviour of a Power Series on the Circle of Convergence

I. In the previous section the behaviour of the logarithmic series on its circle of convergence has been investigated. The method can be extended so as to apply to certain types of power series with *real* coefficients and with finite radii convergence. First we observe that consideration of a series with radius of convergence l, $0 < l < \infty$, can be reduced to one in which $l = 1$, *i.e.* unit circle. Thus if the series $\Sigma a_n z^n$ has a radius of convergence l, then the substitution $z = l\zeta$ will change $\Sigma a_n z^n$ into $\Sigma b_n \zeta^n$, where $b_n = a_n l^n$, the new power series in ζ having unity for its radius of convergence.

Consider, then, a series $\Sigma a_n z^n$ which has the circle $|z| = 1$ as its circle of convergence. On this circle we can write

$$z = \cos\theta + i\sin\theta$$

so that $\Sigma a_n z^n$ becomes $\Sigma a_n \cos n\theta + i\Sigma a_n \sin n\theta$. We distinguish three cases.

(i) $\Sigma |a_n|$ converges, *i.e.* Σa_n is absolutely convergent. On this assumption $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$ converge for all values of θ and the original power series converges at all points of the circle of convergence.

Thus the series $\Sigma a_n z^n$ where $a_n = 1/n^p$, $p > 1$ converges at all points of its circle of convergence $|z| = 1$.

(ii) $a_n > 0$, $a_n \to 0$ steadily. From Dirichlet's test (§ 7·7) the series $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$ are convergent provided $\theta \neq 2m\pi$, where m denotes a positive or negative integer or zero. For under this condition $\Sigma \cos n\theta$ and $\Sigma \sin n\theta$ are both bounded. If $\theta = 2m\pi$, $\sin n\theta = 0$, $\cos n\theta = 1$. Hence the sine series converges to zero while the cosine series converges only if Σa_n converges.

Since $a_n > 0$ this implies that Σa_n would be absolutely convergent.

Thus the series $\Sigma a_n z^n$, where $a_n = 1/n^p$, $0 < p \leqslant 1$, converges at all points of its circle of convergence $|z| = 1$ except at $z = 1$.

(iii) The terms of the sequence $\{a_n\}$ are alternately positive and negative. In this case we can write $a_n = \pm(-1)^n b_n$ so that

$b_n > 0$ and the series becomes

$$\pm \Sigma (-1)^n b_n z^n = \pm \Sigma b_n e^{n\pi i} z^n.$$

It is sufficient to consider the $+$ sign only. On $|z| = 1$ the series becomes $\Sigma b_n e^{ni(\pi+\theta)} = \Sigma b_n \cos n\phi + i \Sigma b_n \sin n\phi$, where $\phi = \theta + \pi$. As the coefficients b_n are now all positive we can proceed as in (ii). In particular, there will be convergence at all points on $|z| = 1$ except at $z = -1$.

Observe that if the sequence $\{a_n\}$ is such that a_n/a_{n+1} can be expressed in the form

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right), \; p > 1$$

then $a_n \to 0$ steadily for $\mu > 0$. If $\mu > 1$, the series Σa_n is absolutely convergent.

The Geometric Series.—$1/(1 + z) = \sum_{n=0}^{\infty} (-1)^n z^n.$

Since $|a_n| = 1$, the circle of convergence is the unit circle $|z| = 1$.

Since $|a_n| \not\to 0$ the series cannot converge at any point on the circle. This result can also be seen by writing the series in the form

$$\sum_{n=0}^{\infty} (-1)^n \cos n\theta + i \sum_{n=0}^{\infty} (-1)^n \sin n\theta$$
$$= \sum_{n=0}^{\infty} \cos n(\theta + \pi) + i \sum_{n=0}^{\infty} \sin n(\theta + \pi).$$

II. Now suppose that a_n is complex and that $\lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right| = 1$, so that the circle of convergence is $|z| = 1$.

Then if it can be shown that a_n/a_{n+1} is of the form

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + \frac{\lambda_n}{n^p}, \; p > 1, \; |\lambda_n| \text{ bounded},$$

the series converges if $0 < R(\mu) \leqslant 1$, except at $z = 1$.

If $R(\mu) > 1$ we know that the series converges absolutely (§ 8·51) and then the series converges at all points on the circle.

8·57. Functions of a Complex Variable

Consider a region A in the Argand plane. If the boundary of the region is included it can be described as closed, if the boundary

is excluded the region is open. Following the definition for a real variable we can say that w is a function of z and write $w = f(z)$, if to every point z in A there correspond one or more values of w. Where there is one value only w will be single valued.

If we write $z = x + iy$, w can be expressed in the form

$$w = u(x, y) + iv(x, y)$$

u, v being functions of the real variables x, y. On this definition then we obtain simply a complex function of two real variables and no new fundamental idea emerges. The class of complex function defined in the way is too general to be of use and it is necessary to select a sub-class.

Again, following real variable theory we might choose the sub-class consisting of *continuous functions*. Thus if z_0 is a point of A, $f(z)$ is continuous at z_0 if corresponding to the arbitrary positive number ϵ there exists a positive number η such that for all z satisfying $|z - z_0| < \eta$, $|f(z) - f(z_0)| < \epsilon$.

This asserts that if we take any point z inside the circle $|z - z_0| = \eta$ then $f(z) \to f(z_0)$ as $z \to z_0$ by whatever path we choose so long as it lies inside the circle. From the definition it follows that $u(x, y)$ are continuous functions of x, y and conversely if $u(x, y)$ and $v(x, y)$ are continuous functions of x, y, so also is $f(z)$ a continuous function of z. Again, this does not suggest any new important conception and the class of continuous functions is restricted still further. From it is chosen a sub-class consisting of *differentiable functions*.

A complex function z will be differentiable at $z = z_0$ if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists as a unique limit. The function will be differentiable in A if it is differentiable at each point z_0 of A. Denoting the differential coefficient by $f'(z_0)$ the property is expressed precisely in the following way. Corresponding to the arbitrary positive ϵ there exists a positive number η such that for all z satisfying $|z - z_0| < \eta$.

$$\left| f'(z_0) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon.$$

This asserts that if z is any point inside the circle $|z - z_0| = \eta$, then *no matter what path we choose inside the circle*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

always exists and is equal to the unique number denoted by $f'(z_0)$. Clearly the condition is a much more stringent one than that of continuity and so may be expected to have more far-reaching effects. Complex functions which are differentiable are said to be *analytic* and it is with such functions that the theory of the complex variable deals. The discussion of the conditions that a function shall be analytic and of the properties of analytic functions belongs then to complex variable theory. A function which is analytic in a region has differential coefficients *of all orders* at all points of the region and so can be expanded in a power series similar to Taylor's theorem. It is also of interest to note that a power series represents an analytic function inside its circle of convergence.

8·58. Uniform Convergence for Complex Series

The fundamental ideas on uniformity for real series, considered in Chapters IV. and VII. extend readily to complex series. The complex series is denoted by $\Sigma u_n(z)$, the remainder after n terms by $R_n(z)$, and ϵ is an arbitrary positive number. *The series $\Sigma u_n(z)$ is said to converge uniformly in a given region (or domain) provided there exists a number n_0* independent *of z such that $|R_n(z)| < \epsilon$, for all $n \geqslant n_0$.* The number n_0 will depend on ϵ but must be the same for all values of z in the domain.

One essential difference is to be noted. In the case of real series the interval of uniform convergence was a straight line; for complex series the values of z lie in a region or domain in the Argand plane so that the region of uniform convergence will be an *area*. We can, if we wish, restrict z to lie on a curve in the Argand plane and then speak about uniform convergence along a curve.

As in § 4·32 the region of uniform convergence must be *closed*. Thus if a series converges uniformly at all points inside a region then the series converges uniformly at all points on the boundary of the region.

For complex series the most important test for uniform convergence is Weierstrass's M-test which may be stated as follows:—

Let $\{M_n\}$ denote a sequence of positive constants such that ΣM_n converges. Then *if the series $\Sigma u_n(z)$ has the property that $|u_n(z)| \leqslant M_n$ for all values of n, at all points of a region A in the Argand plane, then the series $\Sigma u_n(z)$ is absolutely and uniformly convergent in A.* The proof follows the same lines as that for real series (§ 4·33).

Abel's theorem, given in § 4·7 extends to complex series. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = z_0$ where z_0 is a point on the circle of convergence of the given series. Then $\lim_{z \to z_0} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z_0^n$ *where* $z \to z_0$ *along a radius of the circle of convergence.* We may assume without loss of generality that $|z_0| = 1$. Write amp. $z_0 = \alpha$ so that $z_0 = \cos\alpha + i\sin\alpha$. If $z \to z_0$ along the radius of amplitude α, then $z = r(\cos\alpha + i\sin\alpha)$ where $r \to 1$.

Then $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n (\cos n\alpha + i \sin n\alpha)$, $0 \leqslant r \leqslant 1$.

Since both the series $\sum_{n=0}^{\infty} a_n \cos n\alpha$, $\sum_{n=0}^{\infty} b_n \sin n\alpha$ are convergent it follows from Abel's theorem that

$$\lim_{r \to 1} \sum_{n=0}^{\infty} a_n r^n \cos n\alpha = \sum_{n=0}^{\infty} a_n \cos n\alpha, \quad \lim_{r \to 1} \sum_{n=0}^{\infty} a_n r^n \sin n\alpha$$
$$= \sum_{n=0}^{\infty} a_n \sin n\alpha.$$

Hence $\lim_{r \to 1} \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} a_n (\cos n\alpha + i \sin n\alpha) = \sum_{n=0}^{\infty} a_n z_0^n$.

Complex power series *may be multiplied* in the same way as for real series (§ 1·6). Consider $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence l_1, and $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence l_2. If l is the smaller of l_1, l_2, and $0 \leqslant k < l$ both series are absolutely convergent for $0 \leqslant |z| \leqslant k$. The proof of § 1·6 then extends immediately and if they have the following result.

The product $\left(\sum_{n=0}^{\infty} a_n z^n\right) \times \left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n$, *where*

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0, \quad 0 \leqslant |z| \leqslant k.$$

8·61. Relation between e^z and sin z and cos z

In § 8·31 the equation $e^{ix} = \cos x + i \sin x$, where x and y are real, has been considered. Assuming that ix can be treated as an ordinary index we have

$$e^{-ix} = 1/(\cos x + i \sin x) = (\cos^2 x + \sin^2 x)/(\cos x + i \sin x)$$
$$= \cos x - i \sin x.$$

Adding and subtracting the two equations,

$$e^{ix} + e^{-ix} = 2 \cos x, \quad e^{ix} - e^{-ix} = 2i \sin x.$$

Thus when x is real,

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

It is convenient to define $\cos z$ and $\sin z$ by these equations when z is complex. Hence when z is real or complex,

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)!,$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)!.$$

Similar forms may be used to define the hyperbolic functions for complex values of the variable (Chapter V., § 5·42). Thus

$$\cosh z = \tfrac{1}{2}(e^{z} + e^{-z}) = \sum_{n=0}^{\infty} z^{2n}/(2n)!,$$

$$\sinh z = \tfrac{1}{2}(e^{z} - e^{-z}) = \sum_{n=0}^{\infty} z^{2n+1}/(2n+1)!.$$

The usual properties of the circular and the hyperbolic functions may be developed from the definitions given above. Some indication of the methods which may be adopted, is given in Chap. V., § 5·4.

8·62. Inequalities for $\sin z$ and $\cos z$

When x is real, $|\sin x| \leqslant |x|$, $|\cos x| \leqslant 1$. These inequalities are no longer true when the variable is complex. Now

$$|\sin z| = |\Sigma(-1)^n z^{2n+1}/(2n+1)!|$$
$$\leqslant \Sigma |z|^{2n+1}/(2n+1)!.$$

If $|z| \leqslant 1$, $|\sin z| \leqslant |z| \Sigma 1/(2n+1)!$

$$= |z| \left\{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots\right\}$$
$$< |z| \{1 + \tfrac{1}{6} + (\tfrac{1}{6})^2 + (\tfrac{1}{6})^3 + \dots\}$$
$$= \tfrac{6}{5}|z|,$$

i.e. $|\sin z| < \tfrac{6}{5}|z|$.

It is clear that $\frac{6}{5}$ is not the best possible factor. Again,

$$|\cos z| \leqslant \Sigma |z|^{2n}/(2n)!$$

$$< 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots, \text{ if } |z| \leqslant 1$$

$$< 1 + \tfrac{1}{2} \sum_{n=0}^{\infty} (\tfrac{1}{2})^{3n} = 1\tfrac{4}{7}.$$

Hence $|\cos z| < 1\frac{4}{7}$, if $|z| \leqslant 1$.

8·63. Note on Conjugate Functions

Suppose we have a function $f(z)$ of a complex variable $z = x + iy$. Then $f(z) = f(x + iy)$ being itself a complex number can be represented in the form

$$f(x + iy) = u + iv, \text{ where } u \text{ and } v \text{ are real.}$$

From the properties of conjugate numbers it follows that

$$f(x - iy) = u - iv.$$

This principle is a very useful one and examples of its use will be given below.

Examples.—(1) *Prove the following properties of the hyperbolic functions.*

(i) $\sinh iz = i \sin z$, (ii) $\cosh iz = \cos z$.

(iii) $\cosh u + \cosh v = 2 \cosh \frac{1}{2}(u + v) . \cosh \frac{1}{2}(u - v)$.

(iv) $\cosh u - \cosh v = 2 \sinh \frac{1}{2}(u + v) . \sinh \frac{1}{2}(u - v)$.

(v) $\cosh^2 z - \sinh^2 z = 1$.

(i) $\sinh iz = \frac{1}{2}\left(e^{iz} - e^{-iz}\right) = i . \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = i \sin z$.

(ii) $\cosh iz = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$.

(iii) $4 \cosh \frac{1}{2}(u + v) . \cosh \frac{1}{2}(u - v)$

$$= \left\{e^{\frac{1}{2}(u+v)} + e^{-\frac{1}{2}(u+v)}\right\}\left\{e^{\frac{1}{2}(u-v)} + e^{-\frac{1}{2}(u-v)}\right\}$$
$$= e^u + e^{-u} + e^v + e^{-v}$$
$$= 2 \cosh u + 2 \cosh v.$$

(iv) $4 \sinh \frac{1}{2}(u + v) . \sinh \frac{1}{2}(u - v)$

$$= \left\{e^{\frac{1}{2}(u+v)} - e^{-\frac{1}{2}(u+v)}\right\}\left\{e^{\frac{1}{2}(u-v)} - e^{-\frac{1}{2}(u-v)}\right\}$$
$$= e^u + e^{-u} - (e^v + e^{-v})$$
$$= 2 \cosh u - 2 \cosh v.$$

(v) $\cosh^2 z - \sinh^2 z = \frac{1}{4}(e^z + e^{-z})^2 - \frac{1}{4}(e^z - e^{-z})^2$

$$= \tfrac{1}{4}\{e^{2z} + 2 + e^{-2z} - e^{-2z} + 2 - e^{-2z}\} = 1.$$

(2) *Two complex numbers* $z = x + iy$, $w = u + iv$, *are connected by an equation* $z = c \cosh w$. *Prove that*

$$\frac{x^2}{c^2 \cosh^2 u} + \frac{y^2}{c^2 \sinh^2 u} = \frac{x^2}{c^2 \cos^2 v} - \frac{y^2}{c^2 \sin^2 v} = 1$$

and find the values of x and y that correspond to $u = v = 1$.

[*Lond. B.Sc., Eng.*]

$$x + iy = c \cosh (u + iv); \quad x - iy = c \cosh (u - iv).$$

Hence $\dfrac{2x}{c} = \cosh (u + iv) + \cosh (u - iv)$ ....(i)

$\dfrac{2iy}{c} = \cosh (u + iv) - \cosh (u - iv)$ ....(ii)

From (i) $\dfrac{x}{c} = \cosh u \cosh iv = \cosh u \cos v$ ....(iii)

From (ii) $\dfrac{iy}{c} = \sinh u . \sinh iv = i \sinh u . \sin v$ ....(iv)

From (iii) and (iv), $\dfrac{x}{c \cosh u} = \cos v$, $\dfrac{y}{c \sinh u} = \sin v$.

Squaring and adding $\dfrac{x^2}{c^2 \cosh^2 u} + \dfrac{y^2}{c^2 \sinh^2 u} = \cos^2 v + \sin^2 v = 1$.

Again, $\dfrac{x}{c \cos v} = \cosh u$, $\dfrac{y}{c \sin v} = \sinh u$.

Squaring and subtracting $\dfrac{x^2}{c^2 \cos^2 v} - \dfrac{y^2}{c^2 \sin^2 v} = \cosh^2 u - \sinh^2 u = 1$.

When $u = v = 1$, $x = c \cosh 1 . \cos 1 = c . \frac{1}{2} (e + e^{-1}) \cos 57° 18'$, since 1 radian $= 57° 18'$ approx. Substituting $e = 2{\cdot}718 \ldots$ the approximate numerical value of x can be calculated.

Similarly, $y = c \sinh 1 . \sin 1 = c . \frac{1}{2} (e - e^{-1}) \sin 57° 18'$, and y may be found in the same way.

(3) *Given that* $x + iy = \coth \frac{1}{2} (\xi + i\eta)$, *express x and y separately in in real form in terms of ξ and η and show that, when* $\xi < 0$, *y can be expressed in a series*

$$- 2 (e^{\xi} \sin \eta + e^{2\xi} \sin 2\eta + e^{3\xi} \sin 3\eta + \ldots).$$

What is the corresponding series when $\xi > 0$? [*Camb. Sch.*]

Since $x + iy = \coth \frac{1}{2} (\xi + i\eta)$,

$x - iy = \coth \frac{1}{2} (\xi - i\eta)$

Thus
$$\begin{aligned} 2x &= \coth \tfrac{1}{2} (\xi + i\eta) + \coth \tfrac{1}{2} (\xi - i\eta) \\ &= \frac{\cosh \frac{1}{2} (\xi + i\eta)}{\sinh \frac{1}{2} (\xi + i\eta)} + \frac{\cosh \frac{1}{2} (\xi - i\eta)}{\sinh \frac{1}{2} (\xi - i\eta)} \\ &= \frac{e^{\frac{1}{2} (\xi + i\eta)} + e^{- \frac{1}{2} (\xi + i\eta)}}{e^{\frac{1}{2} (\xi + i\eta)} - e^{- \frac{1}{2} (\xi + i\eta)}} \\ &\qquad + \frac{e^{\frac{1}{2} (\xi - i\eta)} + e^{- \frac{1}{2} (\xi - i\eta)}}{e^{\frac{1}{2} (\xi - i\eta)} - e^{- \frac{1}{2} (\xi - i\eta)}} \end{aligned}$$

$$= \frac{e^{\xi + i\eta} + 1}{e^{\xi + i\eta} - 1} + \frac{e^{\xi - i\eta} + 1}{e^{\xi - i\eta} - 1}$$

$$= \frac{2\,(e^{2\xi} - 1)}{e^{2\xi} - e^{\xi + i\eta} - e^{\xi - i\eta} + 1}$$

$$= \frac{2\,(e^{2\xi} - 1)}{e^{2\xi} - 2e^{\xi}\cos\eta + 1}.$$

Hence $x = \dfrac{e^{2\xi} - 1}{e^{2\xi} - 2e^{\xi}\cos\eta + 1}$. Similarly $y = \dfrac{-\,e^{\xi}\sin\eta}{e^{2\xi} - 2e^{\xi}\cos\eta + 1}$.

Now y may be written in the form

$$-\,e^{\xi}\sin\eta\,\{1 - e^{\xi + i\eta}\}^{-1}\{1 - e^{\xi - i\eta}\}^{-1}$$

If $\xi < 0$, $\left| e^{\xi \pm i\eta} \right| = e^{\xi} < 1$. Thus we may expand by the binomial theorem, the series obtained being absolutely convergent. Thus

$$y = -\,e^{\xi}\sin\eta \left\{ \sum_{n=0}^{\infty} e^{n\xi + ni\eta} \right\} \left\{ \sum_{n=0}^{\infty} e^{n\xi - ni\eta} \right\}.$$

In the product of these infinite series the coefficient of $e^{\eta\xi}$ is

$$e^{ni\eta} + e^{(n-2)\,i\eta} + e^{(n-4)\,i\eta} + \ldots + e^{-(n-2)i\eta} + e^{-ni\eta}.$$

This is a geometrical progression of $(n + 1)$ terms, the first term being $e^{ni\eta}$ and common ratio $e^{-2i\eta}$. Thus its sum is

$$e^{ni\eta}\,\frac{1 - e^{-2i\,(n+1)\eta}}{1 - e^{-2i\eta}} = \frac{e^{(n+1)\,i\eta} - e^{-(n+1)i\eta}}{e^{i\eta} - e^{-i\eta}} = \frac{\sin\,(n+1)\eta}{\sin\eta}.$$

$$\text{Hence } y = -\,e^{\xi} \sum_{n=0}^{\infty} e^{n\xi}\sin\,(n+1)\eta,$$

which is the required result.

If $\xi > 0$ we can write y in the form

$$y = -\,\{e^{-\xi}\sin\eta\}/(1 - .2e^{-\xi}\cos\eta + e^{-2\xi})$$

which is the same as the original form with $-\,\xi$ instead of ξ. Hence if $\xi > 0$

$$y = -\,e^{-\xi} \sum_{n=0}^{\infty} e^{-\eta\xi}\sin\,(n+1)\,\eta.$$

(4) *The real quantities x, y, u, v are connected by the equation*

$$\cosh\,(x + iy) = \cot\,(u + iv).$$

Prove that

$$\frac{\sinh 2v}{\sin 2u} = -\,\tanh x\,.\,\tan y,$$

and that $\coth 2v = -\,(\cosh 2x + \cos 2y + 2)/4\sinh x\,.\,\sin y.$ [*Camb. Sch.*]

Now $\cosh\,(x + iy) = \cot\,(u + iv)$ .... (i)

$\cosh\,(x - iy) = \cot\,(u - iv)$ .... (ii)

Adding (i) and (ii),

$$\cosh(x+iy)+\cosh(x-iy)=\frac{\cos(u+iv)}{\sin(u+iv)}+\frac{\cos(u-iv)}{\sin(u-iv)}$$

$$=\frac{\cos(u+iv)\sin(u-iv)+\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}.$$

Hence $\cosh x\cosh iy=\sin 2u/\{\cos 2iv-\cos 2u\}$.

i.e. $\cosh x\cos y=\sin 2u/(\cosh 2v-\cos 2u)$ .... (iii)

Again, subtracting (ii) from (i)

$$\cosh(x+iy)-\cosh(x-iy)$$

$$=\frac{\cos(u+iv)\sin(u-iv)-\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}$$

i.e. $\sinh x\sinh iy=\sin(-2iv)/\{\cos 2iv-\cos 2u\}$

i.e. $i\sinh x\sin y=-i\sinh 2v/(\cosh 2v-\cos 2u)$

Hence $\sinh x\sin y=-\sinh 2v/(\cosh 2v-\cos 2u)$ ...... (iv)

From (iii) and (iv), $\dfrac{\sinh 2v}{\sin 2u}=-\tanh x.\tan y.$

Again $u+iv=\cot^{-1}\{\cosh(x+iy)\}$

$u-iv=\cot^{-1}\{\cosh(x-iy)\}$

$2iv=\cot^{-1}\{\cosh(x+iy)\}-\cot^{-1}\{\cosh(x-iy)\}$

i.e. $$\cot 2iv=\frac{\cosh(x+iy)\cosh(x-iy)+1}{\cosh(x+iy)-\cosh(x-iy)}$$

$$=\frac{\frac{1}{2}\cosh 2x+\frac{1}{2}\cosh 2iy+1}{2\sinh x\sinh iy}$$

Hence $$4i\coth 2v=\frac{\cosh 2x+\cos 2y+2}{i\sinh x\sin y},$$

i.e. $\coth 2v=-(\cosh 2x+\cos 2y+2)/4\sinh x\sin y.$

8·7. Logarithm of a Complex Number

MULTIPLE-VALUED FUNCTIONS.—Let $f(z)$ denote a function depending on z. Then if to each value of z there corresponds *more than one* value $f(z)$, $f(z)$ is said to be multiple-valued. Examples of this type of function have occurred already. Thus $z^{\frac{1}{n}}$ is a multiple-valued function of z, for to each value of z there correspond n values of $z^{\frac{1}{n}}$. Each of these n values is a function of z, and we may speak of each one of them as a *branch* of the multiple-valued function $z^{\frac{1}{n}}$ Thus $z^{\frac{1}{n}}$ has n branches. In particular $z^{\frac{1}{2}}$ has two branches.

As another example consider amp. z. This is a multiple-valued function, but it is made *single-valued* by considering the *principal value* of the angle. This is equivalent to restricting ourselves to the consideration of one particular branch.

THE LOGARITHM AND ITS PRINCIPAL BRANCH.—Now if x, y are real, $x > 0$, then $\log x$ is defined as the number y satisfying the equation $x = e^y$. In Chapter V., § 5·13, it has been shown that this number is *unique*, i.e. given $x > 0$, there exists *one and only one* real number y such that $y = \log x$.

Now consider the equation $z = e^w$ where w, z are no longer restricted to be real. If v is any real angle,

$$e^{iv} = \cos v + i \sin v.$$

Writing $v = 2n\pi$, where n is a positive or negative integer, it follows that

$$e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1.$$

Again, if $w = u + iv$, u and v being real,

$$e^w = e^{u+iv} = e^u . e^{iv} = e^u (\cos v + i \sin v).$$

Let us consider in particular the solution of the equation $e^w = 1$.

$$\text{Then } e^u \cos v = 1, \quad e^u \sin v = 0.$$

From the second equation, since $e^u \neq 0$ it follows that $\sin v = 0$, i.e. $v = n\pi$.

But if $v = n\pi$, $\cos v = \pm 1$. Since e^u is never negative it follows that $\cos v$ cannot be negative so that $\cos v = 1$. Thus $v = 2n\pi$, *i.e.* only even multiples of π. Also $e^u = 1$, i.e. $u = 0$.

Thus the solutions of the equation $e^w = 1$ are

$$u = 0, \quad v = 2n\pi, \quad \text{i.e. } w = 2n\pi i.$$

Hence the solution of the equation $e^w = z$ is no longer unique, for in the particular case considered there are an infinite number of solutions, each differing from the other by a multiple of 2π. This result is to be expected since

$$e^w = e^w \times 1 = e^w \times e^{2n\pi i} = e^{w + 2n\pi i}.$$

Now let z be any complex number and write it in the form

$$z = r (\cos \theta + i \sin \theta) = re^{i\theta} = e^{\log r + i\theta},$$

where $\log r$ is the logarithm of the real number defined as in Chapter V. It follows that we can take one value of $\log z$ to be $\log r + i\theta$ and the general solution to be

$$w = \log z = \log r + i (\theta + 2n\pi),$$

where n is zero, or a positive or negative integer.

The logarithmic function is defined as consisting of all these branches. It is convenient to specify *a single-valued branch of the logarithm* in the following way.

Consider the representation of z in the Argand diagram and suppose that a cut is made along the negative part of the real axis OX' as indicated in Fig. 36. Suppose further, the point P describes any path in the plane, but is never allowed to cross the cut. Then it is clear that under these conditions that if P returns to its original position its *amplitude will be the same as its original value.*

The only other possibility is that the new amplitude should differ from the old by a multiple of 2π. This is impossible since P has described a path which has *not passed around the origin.*

Thus if we define the principal branch of Log z to be $\log z = \log r + i\theta$,

$$-\pi < \theta \leqslant \pi$$

then no matter what path z describes under the restriction just stated the amplitude will always lie between $-\pi$ and π.

The reader should note the distinction drawn between *Log z* and *log z*.

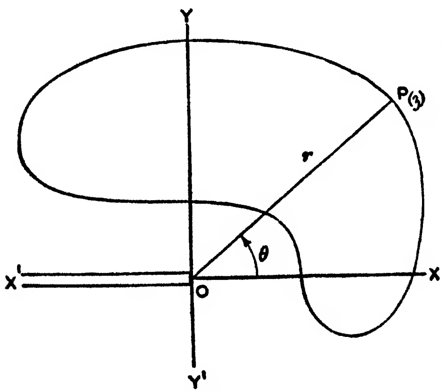


FIG. 36.

Note.—Care is necessary in using such formula as

$$\log (z_1 z_2) = \log z_1 + \log z_2.$$

For each logarithm has its principal value, which means that the value of θ corresponding to $z_1 z_2$ must lie between $-\pi$ and π. But it is easily seen that when we add the angles corresponding to z_1 and z_2 the resulting angle may not lie on the range and it will be necessary to add or subtract $2\pi i$. Thus in general

$$\log (z_1 z_2) = \log z_1 + \log z_2 + 2k\pi i,$$

where $k = 0$, 1 or -1.

In any particular case it is easy to see the value of k, which must be taken. The last point has already arisen in connection with the amplitude of the product or quotient of two complex numbers (§ 8·22).

8·71. The Logarithmic Series

It has been proved in Chapter V., § 5·5, that if x is real,

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x_4}{4} + \ldots, \ |x| < 1.$$

The algebraic method adopted there may be extended directly to complex numbers, provided the series involved converges absolutely.

The radius of convergence of the series $\sum_{n=1}^{\infty} (-1)^{n-1}z^n/n$ has been shown in § 8·55 to be unity.

It follows that the series is one of the solutions of the equation $e^w = 1 + z$, provided $|z| < 1$. Thus $\sum_{n=1}^{\infty} (-1)^{n-1}z^n/n$ is one of the values of Log $(1 + z)$.

Now Log $(1 + z) = \log (1 + z) \pm 2m\pi i$, where m is a positive integer or zero. Hence

$$\log (1 + z) \pm 2m\pi i = \sum_{n=1}^{\infty} (-1)^{n-1}z^n/n.$$

Now write $z = re^{i\theta}$. Then

$$\log (1 + r\cos\theta + ir\sin\theta) \pm 2m\pi i = \sum_{n=1} (-1)^n r^n e^{ni\theta}/n$$

When $\theta = 0$, i.e. z is real, the equation becomes

$$\log (1 + r) \pm 2m\pi i = \sum_{n=1}^{\infty} (-1)^r r^n/n.$$

It follows that $m = 0$, since the other terms are real. Hence

$$\log (1 + z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \ldots, \ |z| < 1,$$

where $\log (1 + z)$ is the principal branch of Log $(1 + z)$. When z lies on the circle of convergence $|z| = 1$ it has been proved in § 8·55 that *the series converges for all such values of z except at $z = -1$.*

Examples.—(1) *If x, y, θ, ϕ are real quantities and if $\log_e \sin (\theta + i\phi) = x + iy$, prove that*

$$2e^{2x} = \cosh 2\phi - \cos 2\theta \qquad \text{[Camb. Sch.]}$$

Since $x + iy = \log \sin (\theta + i\phi)$,

$e^{x+iy} = \sin (\theta + i\phi)$, and $e^{x-iy} = \sin (\theta - i\phi)$.

Multiplying the two equations,

$$e^{2x} = \sin(\theta + i\phi)\sin(\theta - i\phi) = \tfrac{1}{2}\{\cos 2i\phi - \cos 2\theta\};$$

$$\therefore\ 2e^{2x} = \cosh 2\phi - \cos 2\theta.$$

In this example the question of the principal value of the logarithm does not arise since the exponential form is used.

(2) *Express the fourth root of* (-2) *and its logarithm in the form* $u + iv$. [*Lond., B.Sc., Eng.*]

$$\text{Now}\quad -2 = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

$$\text{Thus}\quad r\cos\theta = -2,\ r\sin\theta = 0,\ r = 2,\ \theta = \pi.$$

Hence $-2 = 2e^{i\pi} = 2e^{i(\pi + 2n\pi)} = 2e^{(2n+1)i\pi}$ where n is zero or an integer.

$$\text{Thus}\quad (-2)^{\frac{1}{4}} = 2^{\frac{1}{4}}\, e^{\frac{1}{4}(2n+1)\pi i}.$$

The four values may be obtained by writing $n = 0, 1, 2, 3$, and are

$$2^{\frac{1}{4}}e^{\frac{1}{4}\pi i},\ 2^{\frac{1}{4}}e^{\frac{3}{4}\pi i},\ 2^{\frac{1}{4}}e^{\frac{5}{4}\pi i},\ 2^{\frac{1}{4}}e^{\frac{7}{4}\pi i},$$

i.e, $2^{\frac{1}{4}}(\cos\tfrac{1}{4}\pi + i\sin\tfrac{1}{4}\pi)$, $2^{\frac{1}{4}}(\cos\tfrac{3}{4}\pi + i\sin\tfrac{3}{4}\pi)$, $2^{\frac{1}{4}}(\cos\tfrac{5}{4}\pi + i\sin\tfrac{5}{4}\pi)$,

$$2^{\frac{1}{4}}(\cos\tfrac{7}{4}\pi + i\sin\tfrac{7}{4}\pi),$$

$$\textit{i.e.}\ 2^{\frac{1}{4}}\left(\pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right) = 2^{-\frac{1}{4}}(\pm 1 \pm i).$$

Taking the principal values the logarithms are

$$\tfrac{1}{4}\log 2 + \tfrac{1}{4}\pi i,\ \tfrac{1}{4}\log 2 + \tfrac{3}{4}\pi i,\ \tfrac{1}{4}\log 2 - \tfrac{3}{4}\pi i,\ \tfrac{1}{4}\log 2 - \tfrac{1}{4}\pi i.$$

(3) *Considering only the principal value, prove that the real part of*

$$(1 + i\sqrt{3})^{1 + i\sqrt{3}} \text{ is } 2e^{-\pi/\sqrt{3}}\cos(\tfrac{1}{3}\pi + \sqrt{3}\log 2).$$

[*Lond. B.Sc.*]

$$\text{Write}\quad (1 + i\sqrt{3})^{1 + i\sqrt{3}} = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

$$\text{Taking logarithms,}\quad (1 + i\sqrt{3})\log(1 + i\sqrt{3}) = \log r + i\theta \quad \ldots\ldots \text{(i)}$$

Now $\log(1 + i\sqrt{3}) = \log 2 + i\tan^{-1}\sqrt{3} = \log 2 + i\tfrac{1}{3}\pi$, the principal value being taken.

$$\text{Then}\quad (1 + i\sqrt{3})(\log 2 + i.\tfrac{1}{3}\pi) = (\log 2 - \pi/\sqrt{3}) + i(\tfrac{1}{3}\pi + \sqrt{3}\log 2)\ldots \text{(ii)}$$

$$\text{From (i) and (ii),}\ \log r = \log 2 - \pi/\sqrt{3},\ \theta = \tfrac{1}{3}\pi + \sqrt{3}\log 2.$$

$$\text{Hence}\ r = 2e^{-\pi/\sqrt{3}},\ \cos\theta = \cos(\tfrac{1}{3}\pi + \sqrt{3}\log 2).$$

$$\text{The real part of } re^{i\theta} = r\cos\theta = 2e^{-\pi/\sqrt{3}}\cos(\tfrac{1}{3}\pi + \sqrt{3}\log 2).$$

(4) *If* $u + iv = \log\dfrac{z - a}{z + a}$, *where* $z = x + iy$ *and* a *is a real constant, find the values of* u *and* v *in terms of* x *and* y *and show that* $u =$ *const. and* $v =$ *const. represent in the z-plane two sets of circles which cut each other orthogonally.* [*Lond. B.Sc., Eng.*]

$$\frac{z - a}{z + a} = e^{u+iv},\ \text{i.e.}\ \frac{x - a + iy}{x + a + iy} = e^{u+iv} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The conjugate functions give $\dfrac{x-a-iy}{x+a-iy} = e^{u-iv}$ .... (ii)

Multiplying (i) and (ii), $e^{2u} = \{(x-a)^2 + y^2\}/\{(x+a)^2 + y^2\}$.

Thus $u = \frac{1}{2}\log\,[\{(x-a)^2 + y^2\}/\{(x+a)^2 + y^2\}]$.

Dividing (i) by (ii) $e^{2iv} = \dfrac{x-(a-iy)}{x+(a+iy)}\cdot\dfrac{x+(a-iy)}{x-(a+iy)}$

$$e^{2iv} = \frac{x^2-(a-iy)^2}{x^2-(a+iy)^2} = \frac{x^2-a^2+y^2+2aiy}{x^2-a^2+y^2-2aiy}.$$

Hence $\dfrac{e^{2iv}-1}{e^{2iv}+1} = \dfrac{x^2-a^2+y^2+2aiy-x^2+a^2-y^2+2aiy}{x^2-a^2+y^2+2aiy+x^2-a^2+y^2-2aiy}$

$$= \frac{2aiy}{x^2+y^2-a^2},$$

i.e. $\dfrac{e^{iv}-e^{-iv}}{e^{iv}+e^{-iv}} = \dfrac{2aiy}{x^2+y^2-a^2}$.

Hence $\tan v = 2ay/(x^2+y^2-a^2)$.

The curves for which $u =$ const., $v =$ const. are given by

$$(x-a)^2 + y^2 = k\,\{(x+a)^2 + y^2\},\ x^2+y^2-a^2 = 2\,aly$$

where k and l are constants. These equations may be written in the form

$$(x-am)^2 + y^2 = a^2\,(m^2-1);\ x^2+(y-al)^2 = a^2(1+l^2),$$

where $m = (1+k)/(1-k)$. These equations represent circles; the first has centre $(am, 0)$ and radius $a\sqrt{(m^2-1)}$, the second has centre $(0, al)$ and radius $a\sqrt{(1+l^2)}$. The distance between the centres is $\sqrt{(a^2m^2+a^2l^2)}$.

The condition that the circle should cut orthogonally is that the sum of the squares of the radii be equal to the square of the distance between the centres, *i.e.* $a^2m^2 + a^2l^2 = a^2(m^2-1) + a^2(1+l^2)$.
Thus the condition is satisfied.

8·8. Transformations

Let $w = u + iv$ be a complex number which is related to the variable $z = x + iy$ by some relation $w = f(z)$.

Consider the geometrical interpretation of w and z. The plane in which z is represented may be called the xoy plane or the z-plane, and the plane of w by the uOv plane or the w-plane, the origins being o and O. Suppose now that z describes a curve in the xoy plane. Then w will describe some curve in the uOv plane. Thus to any figure in the first plane corresponds one in the second plane. The passage from a figure in the xoy plane to the corresponding figure in the uOv plane may be called a *transformation*, and *the function* $w = f(z)$ *is said to transform* z *into* w.

If $w = az + b$, where a and b are constants, the transformation is said to be *linear*. If $w = (az + b)/(cz + d)$ the transformation is said to be *bilinear*.

Again, if $f(z)$, $g(z)$ are polynomials in z the transformation defined by $w = f(z)/g(z)$ is said to be rational. Thus the linear and bilinear transformations are particular cases of rational transformations.

8·81. The Linear Transformation

We consider the transformation $w = az + b$ in two stages as follows. Write $Z = az$, $w = Z + b$ where $Z = X + iY$. Then Z represents an intermediate transformation and the complete

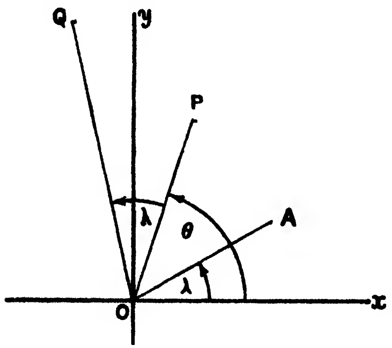


FIG. 37.

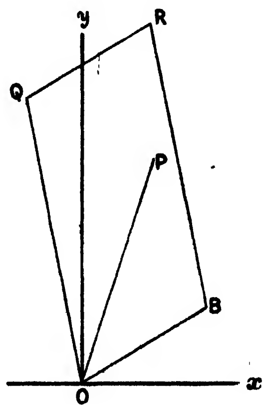


FIG. 38.

transformation is obtained by carrying out the two other transformations in succession.

Now if $Z = az$, $|Z| = |a||z|$, amp. Z = amp. a + amp. z. In the last equation amp. Z need not be the principal value of the amplitude when amp. a and amp. z are principal values, but may differ from it by 2π.

Corresponding points may be conveniently represented in the one figure. Thus if P and A are the points z and a respectively, Q the point Z, $\angle xoA = \lambda$, $\angle xoP = \theta$ then $\angle xoQ = \theta + \lambda$ and $oQ = oA \cdot oP$ (Fig. 37). Thus the radius vector oQ is obtained from the radius vector oP by turning it through an angle λ = amp. a and multiplying its magnitude by the length of $oA = |a|$.

It follows that if P describes a curve on the xoy plane, the corresponding curve described by Q will be similar.

Next consider $w = Z + b$. Let B represent the point b, Q the point Z (Fig. 38). Then if R is the other vertex of the parallelogram whose adjacent sides are oB, oQ then R is the point w. Thus the effect of the second intermediate transformation is to *translate* every point Z through a distance oB in the direction of oB.

The complete transformation $w = az + b$ involves then three separate transformations: (i) a *rotation*, θ changed into $\theta + \lambda$; (ii) a *magnification*, oP changed into $oP \times |a|$; (iii) a *translation* equivalent in magnitude and direction to oB.

8·82. The Bilinear Transformation $w = (az + b)/(cz + d)$.

As in the case of the linear transformation we may consider the transformation by means of a series of intermediate transformations. Thus, provided $a/b \neq c/d$,

$$w = \frac{(bc - ad)/c^2}{z + d/c} + \frac{a}{c}.$$

If $\dfrac{a}{b} = \dfrac{c}{d}$, it is clear that w is a constant. Writing

$$(bc - ad)/c^2 = m$$

we may express the complete transformation as follows:

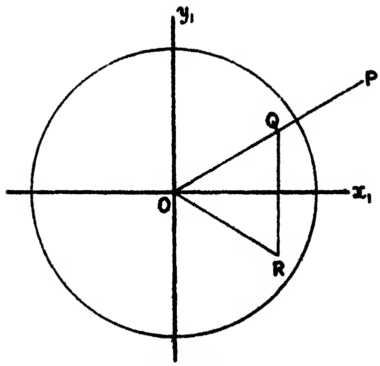


FIG. 39.

(i) $z_1 = z + d/c$, (ii) $z_2 = 1/z_1$,

(iii) $z_3 = mz_2$, (iv) $w = z_3 + a/c$.

The first, third and fourth transformations have already been considered as they are all linear transformations. There remains $z_2 = 1/z_1$.

Let P be the point z_1 (Fig. 39). Then with centre O and radius unity draw a circle in the x_1Oy_1 plane. This is the unit circle in the z_1-plane.

Since $z_2 = 1/z_1$, $|z_2| = 1/|z_1|$, amp. $z_2 = -$ amp. z_1.

Let Q be a point on OP such that $OP.OQ = 1$ Thus $OQ = |z_2|$. Then in geometrical notation Q is the *inverse* of P with respect to the unit circle.

Next let R be the *reflexion* of Q in the real axis. Then since $\angle x_1OP = \angle x_1OR$ numerically, amp. $z_2 = x_1OR$ and since $OR = OQ$ it follows that R is the point z_2.

Expressed in geometrical language we pass from the point z_1 to the point z_2 by first inverting the point with respect to the unit circle and then constructing the image of the new point in the real axis.

We now show that *the bilinear transformation transforms circles into circles or straight lines.*

In order to prove this it is sufficient to show that the transformation $w = 1/z$ transforms circles into circles or straight lines. For as regards the intermediate transformations (i), (iii), (iv) it is clear that circles will transform into circles and straight lines into straight lines.

Any circle in the z-plane may be represented in the form $|z - a| = \rho$, where a is the centre, ρ the radius of the circle. Thus

$$z - a = \rho(\cos\theta + i\sin\theta), \quad -\pi < \theta \leqslant \pi.$$

Write $a = \alpha + i\beta,\ w = u + iv,$

$$1/w = 1/(u + iv) = (u - iv)/(u^2 + v^2).$$

Then $(u - iv)/(u^2 + v^2) = \alpha + i\beta + \rho(\cos\theta + i\sin\theta).$

Equating real and imaginary parts,

$$u/(u^2 + v^2) = \alpha + \rho\cos\theta; \quad -v/(u^2 + v^2) = \beta + \rho\sin\theta.$$

Hence
$$u - \alpha(u^2 + v^2) = \rho(u^2 + v^2)\cos\theta,$$
$$-v - \beta(u^2 + v^2) = \rho(u^2 + v^2)\sin\theta.$$

Squaring and adding we obtain

$$(u^2 + v^2)\{(\alpha^2 + \beta^2 - \rho^2)(u^2 + v^2) - 2\alpha u + 2\beta v + 1\} = 0.$$

Since $\rho > 0$, $w \neq 0$ and $u^2 + v^2 > 0$. Thus the equation gives

$$(\alpha^2 + \beta^2 - \rho^2)(u^2 + v^2) - 2\alpha u + 2\beta v + 1 = 0.$$

Then if $\alpha^2 + \beta^2 - \rho^2 \neq 0$ this represents a circle in the u, v plane whose centre is the point

$$\{\alpha/(\alpha^2 + \beta^2 - \rho^2),\ -\beta/(\alpha^2 + \beta^2 - \rho^2)\}.$$

If $\alpha^2 + \beta^2 - \rho^2 = 0$, i.e. the circle $|z - a| = \rho$ passes through the origin, the equation reduces to

$$-2\alpha u + 2\beta v + 1 = 0.$$

8·83. Circle in the Argand Diagram

We now show that *the equation of any circle may be written in the form*

$$z\bar{z} + az + \bar{a}\bar{z} + b = 0,$$

where z, $\bar{z}$ denote conjugate numbers and b is real. Conversely if z satisfies an equation of the above form, then z must lie on a circle. [Compare alternative proof in § 8·32, Ex. 1.]

The equation of the circle whose centre is λ and whose radius is ρ may be written in the form $|z - \lambda| = \rho$.

Now the conjugate of $z - \lambda$ is $\bar{z} - \bar{\lambda}$, and since the product of two conjugate numbers is equal to the square of the modulus of the original, we may write the equation of the circle in the form

$$(z - \lambda)(\bar{z} - \bar{\lambda}) = \rho^2, \text{ i.e. } z\bar{z} - \bar{\lambda}z - \lambda\bar{z} + \lambda\bar{\lambda} - \rho^2 = 0.$$

Writing $a = -\bar{\lambda}$ so that $\bar{a} = -\lambda$, and $\lambda\bar{\lambda} - \rho^2 = b$ the equation takes the form

$$z\bar{z} + az + \bar{a}\bar{z} + b = 0.$$

If $\lambda = \mu + i\nu$, $\bar{\lambda} = \mu - i\nu$ then $a = -\mu + i\nu$, $\bar{a} = -\mu - i\nu$.

$$\text{Hence } \mu = -\tfrac{1}{2}(a + \bar{a}), \quad \nu = \tfrac{1}{2}i(\bar{a} - a).$$

Again, $\rho^2 = \bar{\lambda}\lambda - b = a\bar{a} - b$.

Thus the equation $z\bar{z} + az + \bar{a}\bar{z} + b = 0$ represents a circle whose centre is $\{-\tfrac{1}{2}(a + \bar{a}) - \tfrac{1}{2}(\bar{a} - a)\}$ and whose radius is $(a\bar{a} - b)^{\frac{1}{2}}$.

This result may be used to give an alternative proof of the result that the bilinear transformation transforms circles into circles, with straight lines as particular cases. As before it is sufficient to prove the result for the transformation $w = 1/z$.

Let the equation of the given circle be

$$z\bar{z} + az + \bar{a}\bar{z} + b = 0.$$

Substituting $z = 1/w$, $\bar{z} = 1/\bar{w}$ the equation becomes

$$1 + a\bar{w} + \bar{a}w + bw\bar{w} = 0.$$

Dividing throughout by $b \neq 0$,

$$w\bar{w} + \frac{\bar{a}}{b}w + \frac{a}{b}\bar{w} + \frac{1}{b} = 0.$$

Since b is real, the conjugate of $\bar{a}/b$ is a/b. Thus this equation has the same form as the original and hence represents a circle in the w-plane. A special form occurs when $b = 0$ in which case the equation for w becomes

$$1 + a\bar{w} + \bar{a}w = 0.$$

Writing $w = u + iv$, $\bar{w} = u - iv$, $a = b + ic$, $\bar{a} = b - ic$, this equation becomes

$$1 + (b + ic)(u - iv) + (b - ic)(u + iv) = 0,$$

i.e. $1 + 2bu + 2cv = 0$. This represents a straight line.

8·9. One-one Correspondence

Let C_1 and C_2 be two curves in the Argand diagram. Then if there exists some relation such that by means of it we can define a correspondence between points on the two curves in such a way that *to each point of* C_1 *there corresponds one and only one point of* C_2, and conversely *to a point on* C_2 *there corresponds one and only one point of* C_1, the correspondence is said to be one-one.

In a similar way we may define one-one correspondence between two areas in the planes. To each point of the first area will correspond one and only one point of the second area, and conversely.

The functional relation which defines the one-one correspondence may be said to give a **one-one transformation.**

The general bilinear transformation $w = (az + b)/(cz + d)$ is the most general type of transformation which possesses the property that to each value of z there corresponds one and only one value of w and conversely. For the equation may also be written in the form

$$z = -\frac{dw - b}{cw - a},$$

which clearly defines one and only one value of z corresponding to each value of w.

Examples.—(1) *Prove that the transformation* $w = (2z + i)/(2 - iz)$ *transforms the interior of unit circle in the* (x, y) *plane into the interior of the unit circle in the* (u, v) *plane, where* $z = x + iy$, $w = u + iv$.

At any point on the boundary of unit circle in the (x, y) plane, $|z| = 1$, $z = (\cos\theta + i\sin\theta)$, $-\pi < \theta \leqslant \pi$. To show that this circle is transformed into unit circle in the (u, v) plane it is sufficient to show that when

$$z = \cos\theta + i\sin\theta, \ |w| = 1.$$

$$|w| = \left|\frac{2z + i}{2 - iz}\right| = \left|\frac{2\cos\theta + i(2\sin\theta + 1)}{2 + \sin\theta - i\cos\theta}\right|$$

$$= \left\{\frac{4\cos^2\theta + (2\sin\theta + 1)^2}{(2 + \sin\theta)^2 + \cos^2\theta}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{4 + 4\sin\theta + 1}{4 + 4\sin\theta + 1}\right\}^{\frac{1}{2}} = 1.$$

To show that the interior of unit circle in the (x, y) plane transforms into the interior of unit circle in the (u, v) plane it is sufficient to show that if z_0 be any particular point such that $|z_0| < 1$, then $|w_0| < 1$, where $w_0 = (2z_0 + i)/(2 - iz_0)$. To see this let z_0, z_1 be two points such that $|z_0| < 1$, $|z_1| < 1$ and suppose that $|w_0| < 1$. Then must $|w_1| < 1$.

Suppose the contrary, i.e. $|w_1| > 1$. In the (u, v) plane join w_0, w_1 by any curve. Then since w_0 lies inside and w_1 outside, this curve must cut unit circle at a point w_2 (say).

Denote the curve by C_2 and let C_1 be the corresponding curve in the (x, y) plane. C_1 is uniquely determined since the transformation is one-one. Further the point z_2 corresponding to w_2 must lie on the boundary of unit circle in the (x, y) plane. Hence the curve C_1 must cut the circle.

This is true for any curve joining w_0 and w_1 in the (u, v) plane. It follows that every curve joining z_0, z_1 must cut the unit circle in the (x, y) plane. This is obviously false since z_0, z_1 both lie inside the circle. Thus we have a contradiction and hence $|w_1| < 1$.

As the particular point take $z_0 = 0$. Then $w_0 = i/2$, $|w_0| = \frac{1}{2} < 1$. Thus the result is proved.

(2) *Prove that the transformation* $w = (z - a)/(1 - \bar{a}z)$, *where* a *is any complex number whose modulus is not equal to unity,* $\bar{a}$ *is the conjugate complex number, transforms the inside of the unit circle into the inside or the outside of the unit circle in the plane of* w *and distinguish the two cases.*

We first show that if $|z| = 1$ then $|w| = 1$.

$$|w| = |z - a|/|1 - \bar{a}z| = |z - a|/|1 - a\bar{z}|, \quad [\S\ 8\cdot32, \text{(ii)}).$$

$$\text{Now write } z = e^{i\theta}, \quad a = \rho e^{i\phi}, \quad \rho \neq 1$$

$$|w| = |e^{i\theta} - \rho e^{i\phi}|/|1 - \rho c^{i\phi} e^{-i\theta}| = |e^{i\theta}| = 1.$$

Thus unit circle in z-plane transforms into unit circle in the w-plane.

Consider the point $z = 0$ and let w_0 be the corresponding value of w. Then $w_0 = -a$. Thus the point is inside unit circle in the w-plane if $|a| < 1$ and outside if $|a| > 1$.

(3) *In an Argand diagram the points P and Q represent the complex quantities* z *and* z_1 *respectively, where* $z_1 = (z - 1)/(z + 1)$. *Find the locus of Q if P describes a line through the origin inclined at an angle* α *to the x-axis and show that if Q describes a circle of the coaxal system whose limiting points are* $(1, 0)$, $(-1, 0)$ *then P describes a circle whose centre is the origin.* [*Lond. B.Sc.*]

Write $z = x + iy$, $z_1 = x_1 + iy_1$.

If P lies on the line through the origin inclined at angle α to the x-axis, amp. $z = \alpha$, i.e. $y/x = \tan\alpha$.

Now the relation $z_1 = (z - 1)/(z + 1)$ may be written in the form

$$\begin{aligned} z &= (z_1 + 1)/(1 - z_1) = (x_1 + 1 + iy_1)/(1 - x_1 - iy_1) \\ &= (x_1 + 1 + iy_1)(1 - x_1 + iy_1)/\{(1 - x_1)^2 + y_1^2\} \\ &= (1 + 2iy_1 - x_1^2 - y_1^2)/\{(1 - x_1)^2 + y_1^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)} \end{aligned}$$

$$\text{Hence } \tan \alpha = 2y_1/(1 - x_1^2 - y_1^2),$$
$$\text{i.e. } x_1^2 + (y + \cot \alpha)^2 = 1 + \cot^2 \alpha = \operatorname{cosec}^2 \alpha.$$

Thus Q describes a circle whose centre is $(0, -\cot \alpha)$ and whose radius is $\operatorname{cosec} \alpha$.

A system of coaxal circles may be represented by the equation
$$x^2 + y^2 - 2gx + c = 0$$
the y-axis being the common radical axis, g being a variable parameter, the limiting points being $(\pm \sqrt{c}, 0)$. Hence Q lies on a circle of the system
$$x_1^2 + y_1^2 - 2gx_1 + 1 = 0.$$

From (i) it follows that $z = (2 - 2gx_1 + 2iy_1)/2x_1(g - 1)$,
$$\text{i.e. } z = (1 - gx_1 + iy_1)/x_1(g - 1).$$
$$\begin{aligned}\text{Hence } |z|^2 &= \{(1 - gx_1)^2 + y_1^2\}/x_1^2(g-1)^2 \\ &= (1 - 2gx_1 + g^2x_1^2 + y_1^2)/x_1^2(g-1)^2 \\ &= (g^2x_1^2 - x_1^2)/x_1^2(g-1)^2 = (g+1)/(g-1).\end{aligned}$$

Hence z lies on a circle whose centre is the origin and whose radius is $(g+1)^{\frac{1}{2}}/(g-1)^{\frac{1}{2}}$.

(4) *If* $\left(\dfrac{z+c}{z-c}\right)^2 = \dfrac{z_1 + 2c}{z_1 - 2c}$, *where* $z = x + iy$ *and* x, y, c *are real, prove that, when* $z = ce^{i\theta}$, $z_1 = 2c \cos \theta$. *Hence, or otherwise, prove that if the point representing* z *describes the circle* $x^2 + y^2 = c^2$, *the point representing* z_1 *describes the segment of the x-axis from* $+2c$ *to* $-2c$ *twice, once in either direction.*

[*Lond. B.Sc., Eng.*]

$$\text{Now } \frac{(z+c)^2}{(z-c)^2} = \frac{z_1 + 2c}{z_1 - 2c}.$$

Applying componendo and dividendo,
$$\frac{(z+c)^2 + (z-c)^2}{(z+c)^2 - (z-c)^2} = \frac{z_1 + 2c + z_1 - 2c}{z_1 + 2c - z_1 + 2c}, \text{ i.e. } \frac{z^2 + c^2}{2cz} = \frac{z_1}{2c}.$$
$$\text{Write } z = ce^{i\theta}. \text{ Then } \frac{e^{2i\theta} + 1}{2e^{i\theta}} = \frac{z_1}{2c}.$$

Since $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, $z_1 = 2c \cos \theta$.

Take the initial amplitude of z to be zero and consider the variation of z_1 as amp. z increases from 0 to 2π, when $|z| = c$. As θ increases from 0 to $\frac{1}{2}\pi$, z_1 moves from the point $(2c, 0)$ on the real axis, along the real axis to the origin. As θ increases from $\frac{1}{2}\pi$ to π, z_1 moves along the real axis from the origin to the point $(-2c, 0)$.

Again, as θ increases from π to 2π, z_1 passes along the real axis from $(-2c, 0)$ to $(2c, 0)$.

Thus as z describes the circle $|z| = c$, z_1 describes the x-axis from $+2c$ to $-2c$ twice, once in either direction.

(5) *If* $zz_1 = k^2$, *where* z *and* z_1 *are complex quantities and* k *is real, and if the point which represents* z *in an Argand diagram describes a circle whose centre is* (a, b) *and whose radius is* r, *show that the point which represents* z_1 *describes a circle of radius* $k^2r/(a^2 + b^2 - r^2)$. [*Lond. B.Sc.*]

Now $zz_1 = k^2$. Hence $|z_1| = k^2/|z|$. Also amp. $z_1 = -$ amp. z.

The curve described by z_1 as z describes the given circle is obtained by first considering the curve for which $|z_1| = k^2/|z|$ and then reflecting the curve in the x-axis. The second step only changes a circle into a circle of equal radius. Thus it is sufficient to prove that the curve described by z_1 where $|z|\ |z_1| = k^2$ and arg. $z =$ arg. z_1, is a circle of radius $k^2r/(a^2 + b^2 - r^2)$.

Let P be any point z on circle $(x - a)^2 + (y - b)^2 = r^2$, Q the corresponding point z_1, O the origin. Then $OP = |z|$, $OQ = |z_1|$, and hence $OP.OQ = k^2$.

Let amp. z be θ so that the equation of OP is $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta} = \rho$, where ρ is the distance from the origin to the point (x, y). If $OP = \rho$ then $OQ = k^2/\rho$.

Write $z_1 = u + iv$. Then arg. $z_1 = \theta$ and $\sin\theta = v/(u^2 + v^2)^{\frac{1}{2}}$, $\cos\theta = u/(u^2 + v^2)^{\frac{1}{2}}$. Since P lies on the circle $(x - a)^2 + (y - b)^2 = r^2$,

$$(\rho\cos\theta - a)^2 + (\rho\sin\theta - b)^2 = r^2 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also, since Q lies on the line OP, $\dfrac{u}{\cos\theta} = \dfrac{v}{\sin\theta} = \dfrac{k^2}{\rho}$. Thus

$$\cos\theta = u\rho/k^2, \quad \sin\theta = v\rho/k^2.$$

Squaring, adding and taking the square root, we obtain $\rho = k^2/(u^2 + v^2)^{\frac{1}{2}}$. Thus $\rho\cos\theta = uk^2/(u^2 + v^2)$, $\rho\sin\theta = vk^2/(u^2 + v^2)$. Substituting in (i) and simplifying we obtain

$$\left(u - \frac{ak^2}{a^2 + b^2 - r^2}\right)^2 + \left(v - \frac{bk^2}{a^2 + b^2 - r^2}\right)^2 = \frac{k^4r^2}{(a^2 + b^2 - r^2)^2}.$$

Thus Q describes a circle of radius $k^2r/(a^2 + b^2 - r^2)$.

EXERCISES VIII

1. Find the value of $(i - 1)^3$.

2. Write down the square, cube and fourth powers of $a + ib$, and show that they are of the same form.

3. Simplify $(3 + i)^4 + (3 - i)^4$.

4. Find the fourth power of $-\sqrt{\{-3\sqrt{(-4)}\}}$.

5. Resolve $x^4 - y^4$ into four factors of the first degree.

6. Show that $\dfrac{1}{(1 - i)^2} - \dfrac{1}{(1 + i)^2} = i$.

7. Simplify $\dfrac{a + bi}{a - bi} - \dfrac{a - bi}{a + bi}$.

8. Express $\dfrac{\{1 + \sqrt{(-1)}\}^2}{1 - \sqrt{(-1)}}$ in the the form of $a + ib$.

9. Find the square roots of (i) $-1 + 2\sqrt{(-2)}$, (ii) $-1 + 4\sqrt{(-5)}$.

10. Prove that $\sqrt{(4 + 3i\sqrt{20})} + \sqrt{(4 - 3i\sqrt{20})} = 6$.

11. Show that $\frac{1}{\sqrt{(3+4i)}} + \frac{1}{\sqrt{(3-4i)}} = \frac{4}{5}$.

12. Find the moduli of the following expressions:

(i) $(1+i)^2$; (ii) $(40+9i)(24+7i)$; (iii) $\frac{3+4i}{12+5i}$; (iv $\frac{3-4i}{3+4i} + \frac{12-5i}{12+5i}$.

13. Reduce the expression $\{(4+3i)\sqrt{3+4i}\}/(3+i)$ to the form $\frac{5}{2}(1+i)$. [*N.Sc.*]

14. Suppose that $x + iy = Ae^{int} + Be^{-int}$ where

$$A = a_1 + ia_2, \quad B = b_1 + ib_2.$$

Assuming that $x, y, a_1, a_2, b_1, b_2, n, t$ are all real, find the value of x and y. [*Lond. B.Sc., Eng.*]

15. If $(a_1 + ib_1)(a_2 + ib_2) = A + iB$, show that

$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} \cdot \tan^{-1}\frac{B}{A},$$

$$\text{and } (a_1^2 + b_1^2)(a_2^2 + b_2^2) = A^2 + B^2.$$

16. Explain the representation of complex numbers by means of points in a plane. Mark on a diagram the points which represent the complex numbers $2+3i$, $1/(2+3i)$, $(1+i)/(1-i)$, $\{(1+i)/(1-i)\}^2$. [*Lond. B.Sc.*]

17. Give a geometrical construction for the product of two complex numbers. Find the modulus and amplitude of (i) $(2+3i)/(1-i)$; (ii) $(\cos\theta + i\sin\theta - 1)/(\cos\theta + i\sin\theta + 1)$. [*Lond. B.A.*]

18. Draw a rough diagram showing the positions of the two points $z_1 = \sqrt{3} + i$, $z_2 = \sqrt{2} + i\sqrt{2}$ on the Argand diagram. What are the moduli and amplitudes of these two complex numbers? Illustrate on the same diagram the numbers $z_1 z_2$ and z_1/z_2.

Three complex numbers are such that $z_1/z_2 = z_2/z_3$; what geometrical significance has this in relation to the corresponding points in the Argand diagram?

19. Represent in a diagram the complex numbers $5+2i$, $(9-7i)/(i-2)$. If $(z_2 - z_3)/(z_3 - z_1) = (z_3 - z_1)/(z_1 - z_2)$, prove that the points which represent the complex numbers z_1, z_2, z_3 form an equilateral triangle. [*Lond. B.A.*]

20. Explain what is meant by the *modulus* and *argument* of a complex number, and how they are represented in the Argand diagram. Prove that if $(1+z)(z'-1)/(z-z')$ is real, the four points, $1, -1, z, z'$ are concyclic. [*Lond. B.Sc.*]

21. A, B, C are three points in the Argand diagram, D is the foot of the perpendicular from A upon BC and AE is a diameter of the circle through ABC. Interpret geometrically the equation (which need not be proved)

$$(b-a)(c-a) = (d-a)(e-a),$$

where a, b, c, d, e are the complex numbers represented in the diagram by A, B, C, D, E. [*Lond. B.A.*]

22. Prove that

$$(\cos \alpha_1 + i \sin \alpha_1)(\cos \alpha_2 + i \sin \alpha_2) \ldots (\cos \alpha_n + i \sin \alpha_n)$$
$$= \cos (\alpha_1 + \alpha_2 + \ldots + \alpha_n) + i \sin (\alpha_1 + \alpha_2 + \ldots + \alpha_2).$$

Factorize $bc(b - c) + ca(c - a) + ab(a - b)$ and by means of the substitution

$$a = \cos 2\alpha + i \sin 2\alpha,\ b = \cos 2\beta + i \sin 2\beta,\ c = \cos 2\gamma + i \sin 2\gamma,$$

prove that

$$\cos (\beta + \gamma - 2\alpha) \sin (\beta - \gamma) + \cos (\gamma + \alpha - 2\beta) \sin (\gamma - \alpha) + \cos (\alpha + \beta - 2\gamma) \sin (\alpha - \beta)$$
$$= 4 \sin (\beta - \gamma) \sin (\gamma - \alpha) \sin (\alpha - \beta).$$

[*Lond. B.Sc.*]

23. State, and prove, De Moivre's theorem for a positive fractional exponent. Find all the values of $(i - 1)^{\frac{1}{6}}$. [*Lond. B.Sc.*]

24. Explain carefully how the Argand diagram may be used to determine the nth roots of unity. Find correct to three places of decimals one of the sixth roots of $35 + 120i$. [*Lond. B.A.*]

25. Express in the form $a + ib$, where a and b are real and in their simplest form

$$\text{(i) } \frac{3 + 4i}{2 - 5i};\quad \text{(ii) } (1 + i\sqrt{3})^8;\quad \text{(iii) } \cos\left(\frac{\pi}{3} + i \log 2\right).$$

[*Lond B.Sc.*]

26. Find u and v, the real and imaginary parts of

$$u + iv = (z - 1)e^{-i\alpha} + \frac{e^{i\alpha}}{z - 1},$$

where $z = x + iy$ and α is real.

Prove that the locus of the points on the Argand diagram representing the complex number z such that $v = 0$ is a circle of unit radius with centre at the point $(1, 0)$ and a straight line through the centre of the circle.

[*Lond. B.Sc.*]

27. If $z = x + iy$ is a complex number, show that

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

[*Lond. B.Sc.*]

28. Represent on an Argand diagram the number $3 + 4i$, its square and its square roots.

Apply de Moivre's theorem to express $\cos 5\theta$ as a function of $\cos \theta$.

[*Lond. B.Sc., Eng.*]

29. Calculate the sums

$$x = e^{\frac{2}{7}\pi i} + e^{\frac{4}{7}\pi i} + e^{\frac{8}{7}\pi i},$$
$$y = e^{\frac{6}{7}\pi i} + e^{\frac{10}{7}\pi i} + e^{\frac{12}{7}\pi i}.$$

(Hint: show that $x + y = -1$, $xy = 2$.)

30. Find the solutions of the equation $x^n = 1$, n being a positive integer. Find all the roots of the equation $(1 + x)^n = x^n$ and show that the real part of each root is $-\frac{1}{2}$. [*Lond. B.A.*]

31. Resolve $x^{2n} + 1$ into factors. Plot the roots of $x^8 + 1 = 0$ in an Argand diagram. [*Lond. B.Sc.*]

32. Prove that if $s = \sigma + it$, where σ and t are real, then the series Σn^{-s} converges absolutely if $\sigma > 1 + \delta$, where δ is any positive number.

33. Prove that the series $\sum_{n=0}^{\infty} (-1)^n/(z + n)$ is conditionally convergent if z is not a negative integer.

34. Discuss the convergence of the series $\Sigma n^a e^{cni + b\sqrt{n}}$ for all real values of a, b, c.

35. $a_1, a_2, a_3, \ldots$ is a sequence of complex numbers such that $\Sigma |a_n|^{-k-1}$ is convergent.

Prove that if $u_n(x) = \dfrac{1}{a_n - x} - \dfrac{1}{a_n} - \dfrac{x}{a_n^2} \cdots - \dfrac{x^{k-1}}{a_n^k}$, the series $\Sigma u_n(x)$ is absolutely convergent for every value of x different from all the a_n.

36. If $x + iy = a \cos(u + iv) + ib \sin(u + iv)$, where x, y, u, v, a and b are real quantities and i denotes $\sqrt{-1}$, show that

$$\frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = a^2 - b^2.$$ [*Camb. Sch.*]

37. Prove that, if x and y are real,

$$|\cot(x + iy)| \leqslant |\coth y|, \quad |\tan(x + iy)| \leqslant |\coth y|.$$ [*Camb. Sch.*]

38. Define $\cosh u$ and $\sinh u$ in terms of exponentials, and express them as trigonometrical functions of imaginary numbers.

If $x + iy = c \sec(\xi + i\eta)$, and if $x = r\cos\theta$, $y = r\sin\theta$ show that

$$4c^2 \sin(\xi - \theta)\sin(\theta + \xi) = r^2 \sin^2 2\xi.$$ [*Lond. B.Sc.*]

39. If $\log \sin(\theta + i\phi) = \alpha + i\beta$, prove that

$$2e^{2\alpha} = \cosh 2\phi - \cos 2\theta.$$ [*Camb. Sch.*]

40. If the points representing z_1, z_2, z_3 on Argand's diagram form a triangle of constant species, prove that $(z_3 - z_1)/(z_2 - z_1)$ is constant.

If z_1, z_2 move uniformly on fixed straight lines, prove that z_3 also moves uniformly on a fixed straight line. [*Lond. B.A.*]

41. Express $\log(-2)$ and $\sin^{-1}(2)$ in the form $a + ib$, where a, b are real.

If $u = \dfrac{\quad + az + b}{z^2 + cz + \quad}$ where a, b, c, d are real, show that the values of z, for which u is real, lie in the Argand diagram either on the real axis or on a circle whose centre is on the real axis. [*Camb. Sch.*]

42. Find the modulus and amplitude of $(1 + i)(1 + 2i)/(1 + 3i)$. The point x moves along the real axis, and the point z moves so that z/x is a (complex) constant; what is the locus of z? [*Lond. B.A.*]

43. Find all the values of e^z, where $z^n = a + ib$, and a and b are real. Consider, in particular, the case where $n = 2$, $a = 1$, $b = 1$.

[*Lond. B.Sc., Eng.*]

44. Define e^x and state the connection between this function and the trigonometric functions $\sin x$ and $\cos x$.

Express in the form $a + ib$ the numbers $\cos^{-1} 2$, 10^{2+3i}, $\log_e (1 - 4)$.

[*Lond. B.Sc., Eng.*]

45. A circle in the z-plane, whose centre is $ai \sin \beta$ and whose radius is a, is transformed into a curve in the w-plane by the transformation

$$w = z + a^2 (\cos^2 \beta)/z.$$

Show that the transformed curve is portion of a circle.

46. If z_1, z_2, z_3 are complex numbers, interpret geometrically the complex numbers $z_1 - z_3$, $(z_1 - z_2)/(z_3 - z_2)$.

Triangles BCX, CAY, ABZ are described on the sides of a triangle ABC. If the points A, B, C, X, Y, Z, in the Argand diagram represent the complex numbers a, b, c, x, y, z respectively, and

$$\frac{x - c}{b - c} = \frac{y - a}{c - a} = \frac{z - b}{a - b}$$

show that the triangles BCX, CAY, ABZ are similar. Prove also that the centroids of ABC, XYZ are coincident. [*Lond. B.Sc.*]

47. If the complex numbers z_1 and z_2 are represented in the Argand diagram by the points P and Q respectively, interpret geometrically the modulus and amplitude of $z_2 - z_1$.

If a third complex number z_3 is represented by the point R, and the angles of the triangle PQR at Q and R are each $\frac{1}{2} (\pi - \alpha)$, prove that

$$(z_3 - z_2)^2 = 4 (z_3 - z_1) (z_1 - z_2) \sin^2 \alpha/2. \qquad \text{[Lond. B.Sc.]}$$

48. Prove that the radius of convergence of the series

$$\frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \dots.$$

is equal to unity and that the series converges absolutely and uniformly at all points on its circle of convergence.

CHAPTER IX

PARTIAL FRACTIONS, RECURRING SERIES, DIFFERENCE EQUATIONS

We first consider some properties of polynomials in a single variable x. The symbols $Y, Y_1, Y_2, \ldots, X_2, X_3, \ldots,$ represent polynomials, which in particular cases may be constants.

9·1. Theorems on Polynomials

Theorem I.—If $Y = Y_1X_2 + Y_2$ then the common factors of Y and Y_1 are the same as those of Y_1 and Y_2.

For every common factor of Y_1 and Y_2 is a factor of Y, since $Y = Y_1X_2 + Y_2$. Again, since $Y_2 = Y - Y_1X_2$, every common factor of Y and Y_1 is a factor of Y_2.

Theorem II.—If the degree of Y_1 does not exceed that of Y then there exist pairs of polynomials (X_2, Y_2), (X_3, Y_3), ..., which may in particular cases be constants, such that

$$Y = Y_1X_2 + Y_2,\ \ Y_1 = Y_2X_3 + Y_3,\ \ Y_2 = Y_3X_4 + Y_4, \ldots$$

and the sequence of polynomials

$$Y_1, Y_2, Y_3, Y_4 \ldots$$

has the property that the degree of any member is less than that of the preceding member.

That this process is possible may be seen quite readily from the existence of the first pair X_2, Y_2 Thus $\dfrac{Y}{Y_1} = X_2 + \dfrac{Y_2}{Y_1}$ so that X_2 is the quotient, Y_2 the remainder when Y is divided by Y_1.

If Y and Y_1 are of the same degree, X_2 will be a constant, *i.e.* a polynomial of degree zero; if Y_1 is of lower degree than Y then X_2 will be a polynomial whose degree is the difference between the degrees of the polynomials Y, Y_1. Further, since Y_2 is the remainder Y_2 will be of lower degree than Y_1.

Arguing as before, it follows that since the degree of Y_2 is less than that of Y_1, Y_1 can be written in the form $Y_2X_3 + Y_3$ where Y_3 is of degree less than Y_2, and so on.

It is clear that the process must come to an end after a finite number of steps. There are only two possibilities. Either the remainder Y_n is zero or is reduced to a constant different from zero, for some value of n. In either case the process stops.

9·11. Application to the Determination of H.C.F. of Two Polynomials

From Theorem I. it follows that at any stage of the process the common factors of Y_n and Y_{n+1} are the same as those of Y_{n+1} and Y_{n+2}. It follows that if Y and Y_1 have any common factor, then this is a factor of each member of the sequence $Y, Y_1, Y_2, Y_3, \ldots$

If there exists a value such that $Y_p = 0$, $Y_{p-1} \neq 0$ then Y_{p-1} is a factor of Y_{p-2} since $Y_{p-2} = Y_{p-1} X_p$. It follows that Y_{p-1} must be the H.C.F. of Y and Y_1.

On the other hand if there exists a value of q such that $Y_q = k$, a constant different from zero, it follows that the original polynomials Y and Y_1 have no algebraic H.C.F. For the H.C.F. must be a factor of Y_q which has no algebraic factors. It should be observed that any *numerical factor only* would not be regarded as forming an algebraic H.C.F.

Thus in the process of finding the H.C.F. in particular cases, we may multiply the polynomials involved by constants, without affecting the result. This is frequently convenient in order to simplify numerical working and avoid fractions, Further, the work may sometimes be shortened by observing that any factor of one of the polynomials $Y, Y_1, Y_2, \ldots$ which is not a factor of the preceding members may be omitted.

The method of dealing with particular cases is illustrated in the following examples.

Examples.—(1) *Find the highest common factor of* $2x^4 + x^3 - 6x^2 - 2x + 3$ *and* $2x^4 - 3x^3 + 2x - 3$.

Here we can take

$$Y = 2x^4 + x^3 - 6x^2 - 2x + 3, \quad Y_1 = 2x^4 - 3x^3 + 2x - 3.$$

$$\begin{array}{r|l} 2x^4 - 3x^3 + 2x - 3 \,) & 2x^4 + x^3 - 6x^2 - 2x + 3 \,(\, 1 \\ & \underline{2x^4 - 3x^3 \qquad\quad + 2x - 3} \\ & \quad 4x^3 - 6x^2 - 4x + 6. \end{array}$$

At this stage the remainder may be divided by 2. For the next step,

$$\begin{array}{l} 2x^3 - 3x^2 - 2x + 3 \,)\, 2x^4 - 3x^3 + 2x - 3 \,(\, x \\ \qquad\qquad\qquad\qquad\quad \underline{2x^4 - 3x^3 - 2x^2 + 3x} \\ \qquad\qquad\qquad\qquad\qquad\quad 2x^2 - x - 3 \,)\, 2x^3 - 3x^2 - 2x + 3 \,(\, x - 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{2x^3 - x^2 - 3x} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x^2 + x + 3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x^2 + x + 3 \end{array}$$

Hence the H.C.F. is $2x^2 - x - 3$.

(2) *Find the greatest common factor of*

$30x^5 + 37x^4 + 110x^3 + 110x^2 + 103x + 30$ and $30x^3 + 17x^2 + 77x + 30$. [*Camb. Sch.*]

$$
\begin{array}{r}
30x^3 + 17x^2 + 77x + 30\ )\ 30x^5 + 37x^4 + 110x^3 + 110x^2 + 103x + 30\ (\ x^2 \\
\underline{30x^5 + 17x^4 + 77x^3 + 30x^2} \\
20x^4 + 33x^3 + 80x^2 + 103x + 30 \quad \times 3 \\
60x^4 + 99x^3 + 240x^2 + 309x + 90\ (\ 2x \\
\underline{60x^4 + 34x^3 + 154x^2 + 60x} \\
65x^3 + 86x^2 + 249x + 90\ (\ 2 \\
\underline{60x^3 + 34x^2 + 154x + 60} \\
5x^3 + 52x^2 + 95x + 30
\end{array}
$$

In this last step the degree of the polynomial is not reduced by the subtraction of twice the devisor. The point of this subtraction is to obtain a *simpler* polynomial of the third degree.

$$
\begin{array}{r}
5x^3 + 52x^2 + 95x + 30\ )\ 30x^3 + 17x^2 + 77x + 30\ (\ 6 \\
\underline{30x^3 + 312x^2 + 570x + 180} \\
-\ 295x^2 - 493x - 150.
\end{array}
$$

Instead of proceeding further with the general method we observe that

$$295x^2 + 493x + 150 = (5x + 2)(59x + 75).$$

Clearly $59x + 75$ is not a factor of either of the given polynomials. Hence if there is a common factor apart from unity this factor must be $5x + 2$.

In order to verify that this is the case it is only necessary to show that $5x + 2$ is a factor of $5x^3 + 52x^2 + 95x + 30$. Substituting $x = -2/5$ in this expression,

$$-5\cdot\tfrac{8}{125} + 52\cdot\tfrac{4}{25} - 95\cdot\tfrac{2}{5} + 30 = 0.$$

Hence the greatest common factor is $5x + 2$.

9·12. Prime Polynomials

Two polynomials are said to be *prime to each other* or *relatively prime* if they have no common factor, apart from numerical factors. Thus, *e.g.* $2(x + 1)$, $6(x^2 + x + 1)$ are prime to each other.

THEOREM III.—**If P, Q, R are three polynomials such that R is a factor of PQ while Q is prime to R, then R must be a factor of P.** This result will be almost obvious to the student, but we may prove it precisely as follows.

If R is prime to Q, then R and Q have no common factor. Hence the highest common factor of RP and PQ is P. But R is a factor of PQ and hence is a common factor of RP and PQ. Thus R must equal P or be a factor of it.

THEOREM IV.—**If A and B are two polynomials, then there exist polynomials X and Y with no common factor, such that**

$$AX + BY = P \text{ or } AX + BY = 1$$

according as A and B have or have not common factors, P denoting a polynomial.

Let $X_1, X_2, X_3, \ldots$ be the quotients, $Y_1, Y_2, Y_3, \ldots$ the remainders in the process of determining highest common factor.

Then $A = BX_1 + Y_1,\ B = Y_1X_2 + Y_2,\ Y_1 = Y_2X_3 + Y_3, \ldots$

$$\therefore\ Y_1 = A - BX_1,$$

$$Y_2 = B - Y_1X_2 = B - X_2(A - BX_1) = -AX_2 + B(1 + X_1X_2),$$

$$Y_3 = Y_1 - Y_2X_3 = A - BX_1 - X_3(-AX_2 + B + BX_1X_2)$$

$$= A(1 + X_2X_3) - B(X_1 + X_3 + X_1X_2X_3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence each member of the sequence $Y_1, Y_2, Y_3, \ldots$ is of the form $AX + BY$, where X and Y are polynomials.

Continuing the process of finding the H.C.F. we obtain either $Y_q = k$, a constant different from zero or $Y_{p-1} = P$, the highest common factor of A and B. Thus either

$$k = AX + BY \quad \text{or} \quad P = AX + BY.$$

It remains to show that X and Y are relatively prime. In the case of $k = AX + BY$ a common factor of X and Y would be a factor of k since A and B are prime to each other. Thus the only possible common factor is a constant. Hence X is prime to Y. Again, the second equation may be written in the form

$$\frac{A}{P}\cdot X + \frac{B}{P}\cdot Y = 1.$$

Since P is the H.C.F. of A and B, A/P and B/P have no common factors, except possibly a constant. Hence X is prime to Y in this case also.

9·13. Rational Fractions

We now pass to the consideration of functions of the form P/Q where P and Q are polynomials. Such an expression is called a **rational fraction.** If the degree of P is less than that of Q the fraction may be called a **proper fraction,** while if the degree of P is greater than or equal to that of Q, the fraction may be called **improper.**

By division in the usual way an improper fraction may be expressed as the sum of a polynomial and a proper fraction. If the degrees of P and Q are equal this polynomial will be a constant.

Thus, e.g. $\dfrac{2x^3 + x^2 + 3x + 2}{x^2 + 1} = \dfrac{(2x + 1)(x^2 + 1) + x + 1}{x^2 + 1}$

$$= 2x + 1 + \frac{x + 1}{x^2 + 1}.$$

THEOREM V.—EQUALITY OF TWO RATIONAL FUNCTIONS.—Let X, X', P, P', Q, Q' be polynomials such that $X + P/Q \equiv X' + P'/Q'$, where P/Q and P'/Q' are proper fractions. Then $X \equiv X'$, $P/Q \equiv P'/Q'$.

$$\text{Now } X - X' = \frac{P'}{Q'} - \frac{P}{Q} = \frac{P'Q - PQ'}{QQ'}.$$

Let the degrees of P, P', Q, Q' be p, p', q, q' respectively. Then since P/Q, P'/Q' are proper fractions, $p < q$, $p' < q'$. The degree of $QQ' = q + q'$, while that of $P'Q - PQ'$ is the greater of $p' + q, p + q'$.

Since $p' + q < q + q'$, $p + q' < q + q'$ it follows that $(P'Q - PQ')/QQ'$ is a proper fraction. But $X - X'$ is a polynomial since it is the difference of two polynomials.

Hence the only possibility is $X = X'$ and $P/Q = P'/Q'$.

THEOREM VI.—If P/QR is a proper fraction, where Q and R are polynomials with no common factor, then there exist proper fractions X/Q, Y/R such that

$$\frac{P}{QR} = \frac{X}{Q} + \frac{Y}{R}.$$

From Theorem IV. it follows there exist polynomials X', Y' which have no common factor and such that $QY' + RX' = 1$,

$$\text{i.e. } P/QR = PX'/Q + PY'/R.$$

If PX'/Q, PY'/R are both proper fractions then we have the required form.

If they are not both proper fractions then either one or both must be improper fractions.

CASE I. One improper fraction, say PX'/Q, while the other PY'/R is a proper fraction of the form Y/R.

We can write $PX'/Q = X'' + X/Q$, where X'' is a polynomial and X/Q is a proper fraction. Then

$$\frac{P}{QR} = X'' + \frac{X}{Q} + \frac{Y}{R} = X'' + \frac{XR + YQ}{QR}.$$

Since X and Y are of lower degree than Q and R respectively $(XR + YQ)/QR$ is a proper fraction. Since P/QR is a proper fraction, $X'' \equiv 0$ and PX'/Q must be a proper fraction.

CASE II. Both PX'/Q and PY'/R are improper fractions.

We can write

$$PX'/Q = X'' + X/Q', \; PY'/R = Y'' + Y/R$$

where X'', Y'' are polynomials and X/Q, Y/R are proper fractions.

Thus
$$\frac{P}{QR} = X'' + Y'' + \frac{X}{Q} + \frac{Y}{R}$$
$$= X'' + Y'' + \frac{XR + YQ}{QR}.$$

Since X and Y are of lower degree than Q and R respectively, it follows that $(XR + YQ)/QR$ is a proper fraction.

Since P/QR is a proper fraction, it follows that $X'' + Y'' \equiv 0$.

Thus
$$\frac{P}{QR} = \frac{X}{Q} + \frac{Y}{R}$$

and P/QR has been expressed in the required form.

9·14. Factors of Polynomials

THEOREM VII.—**Every polynomial whose coefficients are real can be resolved into real factors of the first and second degree.**

Let Q denote a polynomial of the nth degree in x. Then

$$Q \equiv k\,(x - a_1)\,(x - a_2) \ldots (x - a_n),$$

where k is a constant and $a_1, a_2, \ldots, a_n$ are the n roots of $Q = 0$.

Also if $\alpha + i\beta$ is a root of the equation, the conjugate number $\alpha - i\beta$ is also a root. The corresponding factors are

$$(x - \alpha - i\beta)\,(x - \alpha + i\beta) = (x - \alpha)^2 + \beta^2.$$

Thus the two roots give rise to a real quadratic factor.

If the factor $(x - \alpha - i\beta)$ is repeated r times then it follows that factor $(x - \alpha + i\beta)$ must also be repeated r times. These factors give rise to the real factor $\{(x - \alpha)^2 + \beta^2)^r$.

Thus the factors of Q must be represented in one or other of the following forms, where a, p, q are real.

(i) A non-repeated simple factor of the form $x - a$;
(ii) A repeated simple factor of the form $(x - a)^r$;
(iii) A non-repeated quadratic factor of the form $x^2 + px + q$;
(iv) A repeated quadratic factor of the form $(x^2 + px + q)^r$.

THEOREM VIII.—**If $f(x)$ is any polynomial of degree n, h any number, then $f(x)$ may be expressed uniquely in the form.**

$$f(x) \equiv a_0\,(x - h)^n + a_1\,(x - h)^{n-1} + a_2\,(x - h)^{n-2} + \ldots + a_n.$$

Let $f(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n$.

Expanding the terms of $\sum_{r=0}^{n} a_r (x-h)^{n-r}$ by the binomial theorem and collecting terms

$$\sum_{r=0}^{n} a_r (x-h)^{n-r} = a_0 x^n + \{a_1 - {}_nC_1 h a_0\}x^{n-1} + \ldots$$
$$+ \{a_n - ha_{n-1} + \ldots + (-1)^n h^n a_0\}.$$

Hence $b_0 = a_0$,

$b_1 = a_1 - {}_nC_1 h a_0$,

. . . . . .

$b_n = a_n - ha_{n-1} + \ldots + (-1)^n h^n a_0$.

These $(n + 1)$ equations determine uniquely the $n + 1$ unknowns $a_0, a_1, \ldots a_n$.

It will be observed that a_n is the remainder when $f(x)$ is divided by $x - h$. If the quotient be Q then a_{n-1} is the remainder when Q is divided by $x - h$. Thus the coefficients may be calculated in succession.

The simplest way of setting out the numerical working in a particular case may be seen by a study of the following examples.

Examples.—(1) *Express* $x^3 - 6x^2 + 10x - 3$ *as a polynomial in* $x - 2$.

The first step is to find the quotient and remainder when $x^3 - 6x^2 + 10x - 3$ is divided by $x - 2$. First expressing the working in full we obtain

$$\begin{array}{rl} x - 2)\, x^3 - 6x^2 + 10x - 3 & (x^2 - 4x + 2. \\ \underline{x^3 - 2x^2} \qquad\qquad\quad & \\ -4x^2 + 10x \qquad & \\ \underline{-4x^2 + \ 8x} \qquad & \\ 2x - 3 & \\ \underline{2x - 4} & \\ 1 & \end{array}$$

Thus the quotient is $x^2 - 4x + 2$ and the remainder 1.

The result may be set out as follows. First write down the coefficients *inserting* 0 *if any power of* x *is missing.*

$$\begin{array}{cccc} 1 & -6 & +10 & -3 \\ & \underline{\ \ 2} & \underline{-\ 8} & \underline{\ \ 4} \\ & -4 & 2 & 1 \end{array}$$

Multiply the first coefficient by 2 and add it to the second giving -4. Multiply the second number by 2 and add to the third coefficient giving 2. Multiply the last result by 2 and add to the fourth coefficient giving 1. The interpretation of the working as set out is that $x^2 - 4x + 2$ is the quotient and 1 the remainder.

The next step may be carried out in the same way. Thus

$$\begin{array}{ccc} 1 & -4 & 2 \\ & \underline{\ \ 2} & \underline{-4} \\ & -2 & -2 \end{array}$$

Hence the new quotient is $x - 2$ and the remainder -2.

The third remainder is clearly zero. Thus beginning from the last the remainders are 1, − 2, 0. Hence

$$x^3 - 6x^2 + 10x - 3 = (x-2)^3 + 0\,(x-2)^2 - 2\,(x-2) + 1$$
$$= (x-2)^3 - 2\,(x-2) + 1.$$

The complete working may be conveniently set out as follows:

1	− 6	10	− 3
	2	− 8	4
	− 4	2	1
	2	− 4	
	− 2	− 2	
	2		
	0		

(2) *Express* $x^5 - 5x^4 + 20x^2 + x - 1$ *as a polynomial in* $x - 3$.

The working may be set out as follows:

1	− 5	0	+ 20	+ 1	− 1
	3	− 6	− 18	6	21
	− 2	− 6	2	7	20
	3	3	− 9	− 21	
	1	− 3	− 7	− 14	
	3	12	27		
	4	9	20		
	3	21			
	7	30			
	3				
	10				

Hence $x^5 - 5x^4 + 20x^2 + x - 1$

$$= (x-3)^5 + 10\,(x-3)^4 + 30\,(x-3)^3 + 20\,(x-3)^2 - 14\,(x-3) + 20.$$

9·2. Resolution of P/Q into Partial Fractions where P/Q is a Proper Fraction

If the given rational function is improper it should first be expressed as the sum of a polynomial and a proper fraction. We now have the following theorem.

THEOREM IX.—(i) To the non-repeated factor $x - a$ of Q there corresponds a fraction of the form $A/(x - a)$.

(ii) To the repeated factor $(x - a)^r$ corresponds a group of terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \ldots + \frac{A_r}{(x-a)^r}.$$

(iii) To the non-repeated factor $x^2 + px + q$ corresponds a fraction of the form $(Bx + C)/(x^2 + px + q)$.

(iv) To the **repeated factor** $(x^2 + px + q)^r$ **corresponds a group of terms**

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \ldots + \frac{B_rx + C_r}{(x^2 + px + q)^r}.$$

The A's, B's and C's are constants, *i.e.* are independent of x.

(i) Let $Q = (x - a) R$, where R is a polynomial which has not $(x - a)$ as a factor. Then since R is prime to $x - a$,

$$\frac{P}{Q} = \frac{X}{x - a} + \frac{Y}{R}$$

where $X/(x - a)$, Y/R are proper fractions. (Theorem VI.) Since $X/(x - a)$ is a proper fraction it follows that X must be a constant.

(ii) Let $Q = (x - a)^rR$, where $r > 1$ and R is prime to $(x - a)^r$, i.e. $x - a$ is not a factor of R. Then as before

$$\frac{P}{Q} = \frac{X}{(x - a)^r} + \frac{Y}{R}.$$

where $X/(x - a)^r$, Y/R are proper fractions. Thus X is of degree $(r - 1)$ at most and may be expressed in the form

$$A_1 (x - a)^{r-1} + A_2 (x - a)^{r-2} + \ldots + A_r,$$

where some of the coefficients $A_1, A_2, \ldots A_r$ may be zero. (§ 9·14.)

$$\text{Hence } \frac{X}{(x - a)^r} = \frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_r}{(x - a)^r}.$$

(iii) Let $Q = (x^2 + px + q) R$, where R is prime to $x^2 + px + q$.

$$\text{Then } \frac{P}{Q} = \frac{X}{x^2 + px + q} + \frac{Y}{R},$$

where each $X/(x^2 + px + q)$ and Y/R are proper fractions. Since $x^2 + px + q$ is of the second degree the former fraction implies that X is of the first degree, *i.e.* of the form $Bx + C$.

(iv) This case is similar to (ii) except that the numerators involved will be of the first degree.

9·20. Methods for Determining Coefficients

There are various methods for determining equations for the purpose of calculating the coefficients in the identity which expresses

the rational function as the sum of partial fractions. In any particular case it may be convenient to use more than one method.

(i) The substitution of particular real values of the variable. In many cases the choice, as determined by the denominators of the partial fractions, can provide equations which contain only one coefficient.

(ii) The substitution of a complex number to provide an equation in which only two coefficients remain. These, being real, can then be found by equating real and imaginary parts.

(iii) The equating of coefficients of corresponding powers of the variable.

(iv) The use of the formula $f(a)/\phi'(a)$ for the coefficient of $1/(x-a)$ where $f(x)/\phi(x)$ is a proper fraction and $(x-a)$ is a non-repeated factor of $\phi(x)$.

To prove (iv) we may proceed as follows.

Write $\phi(x) = (x-a)\,\lambda(x)$ where $\lambda(a) \neq 0$.

Then $\phi'(x) = (x-a)\,\lambda'(x) + \lambda(x)$ giving $\phi'(a) = \lambda(a)$.

Write $\dfrac{f(x)}{\phi(x)} = \dfrac{A}{x-a} + \dfrac{\psi(x)}{\lambda(x)}$, i.e. $f(x) = A\lambda(x) + (x-a)\,\psi(x)$,

where A denotes a constant which is different from zero.

Then $f(a) = A\lambda(a) = A\phi'(a) \neq 0$. Hence $A = f(a)/\phi'(a)$.

Examples.—(1) *Express*

$$\{(x-a)(x-b)(x-c)(x-d)\}/\{(x+a)(x+b)(x+c)(x+d)\}$$

as the sum of a polynominal and partial fractions, (i) *when a, b, c, d are all unequal,* (ii) *when they are all equal.* [*Camb. Sch.*]

(i) Clearly

$$\frac{(x-a)(x-b)(x-c)(x-d)}{(x+a)(x+b)(x+c)(x+d)} \equiv 1 + \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c} + \frac{D}{x+d},$$

i.e. $(x-a)(x-b)(x-c)(x-d)$

$$\equiv (x+a)(x+b)(x+c)(x+d) + A(x+b)(x+c)(x+d)$$
$$+ B(x+a)(x+c)(x+d) + C(x+a)(x+b)(x+d)$$
$$+ D(x+a)(x+b)(x+c).$$

Put $x = -a$ in this identity.

$(-2a)(-a-b)(-a-c)(-a-d)$

$$= 0 + A(-a+b)(-a+c)(-a+d) + 0.B + 0.C + 0.D,$$

i.e $A = 2a(b+a)(c+a)(d+a)/(b-a)(c-a)(d-a)$.

Similarly $B = 2b(a+b)(c+b)(d+b)/(a-b)(c-b)(d-b)$,

$C = 2c(a+c)(b+c)(d+c)/(a-c)(b-c)(d-c)$.

$D = 2d(a+d)(b+d)(c+d)/(a-d)(b-d)(c-d)$.

(ii) $a = b = c = d$. The expression becomes $(x - a)^4/(x + a)^4$.

$$(x - a)^4 = (x + a - 2a)^4$$
$$= (x + a)^4 - 4(x + a)^3 . 2a + 6(x + a)^2 (2a)^2 - 4(x + a)(2a)^3 + (2a)^4.$$

Hence $\dfrac{(x - a)^4}{(x + a)^4} = 1 - \dfrac{8a}{x + a} + \dfrac{24a^2}{(x + a)^2} - \dfrac{32a^3}{(x + a)^3} + \dfrac{16a^4}{(x + a)^4}$.

(2) *Express* $\dfrac{x^2}{(x + 1)^2 (x^2 + 1)}$ *as the sum of three partial fractions.*

[*Camb. Sch.*]

$$\frac{x^2}{(x + 1)^2 (x^2 + 1)} = \frac{a}{x + 1} + \frac{b}{(x + 1)^2} + \frac{cx + d}{x^2 + 1},$$

i.e. $x^2 = a(x + 1)(x^2 + 1) + b(x^2 + 1) + (cx + d)(x + 1)^2$.

To determine b, write $x = -1$, giving $b = \frac{1}{2}$.

As c and d are real numbers we can determine them by writing $x = i$, observing that $x^2 + 1 = (x - i)(x + i)$. Thus

$i^2 = (ci + d)(1 + i)^2$ or $-1 = -2c + 2id$, giving $c = \frac{1}{2}$, $d = 0$.

We can find a by equating coefficients of x^3. Thus $0 = a + c$ giving $a = -\frac{1}{2}$.

Hence $$\frac{x^2}{(x + 1)^2 (x^2 + 1)} = \frac{-\frac{1}{2}}{x + 1} + \frac{\frac{1}{2}}{(x + 1)^2} + \frac{\frac{1}{2}x}{x^2 + 1}.$$

(3) *Express in partial fractions* $2x^5/(x^2 - 1)(x^2 - 4)$.

We first observe that the numerator is of higher degree than the denominator, so that it is first necessary to reduce it to lower degree. By division it is easily seen that

$$\frac{2x^5}{(x^2 - 1)(x^2 - 4)} = 2x + \frac{10x^3 - 8x}{(x^2 - 1)(x^2 - 4)}.$$

Write $$\frac{10x^3 - 8x}{(x^2 - 1)(x^2 - 4)} = \frac{a}{x - 1} + \frac{b}{x + 1} + \frac{c}{x - 2} + \frac{d}{x + 2},$$

i.e. $10x^3 - 8x$
$$= a(x + 1)(x^2 - 4) + b(x - 1)(x^2 - 4)$$
$$+ c(x + 2)(x^2 - 1) + d(x - 2)(x^2 - 1).$$

In this identity write $x = 1, -1, 2, -2$ in succession. We obtain $a = -\frac{1}{3}$, $b = -\frac{1}{3}$, $c = \frac{16}{3}$, $d = \frac{16}{3}$. Thus

$$\frac{2x^5}{(x^2 - 1)(x^2 - 4)} = 2x - \frac{1}{3(x - 1)} - \frac{1}{3(x + 1)} + \frac{16}{3(x - 2)} + \frac{16}{3(x + 2)}.$$

(4) *Prove that if* $(1 + x)^n = c_0 + c_1 x + \ldots + c_n x^n$, *then*

$$\frac{c_0}{y} - \frac{c_1}{y + 1} + \frac{c_2}{y + 2} - \ldots \pm \frac{c_n}{y + n} = \frac{n!}{y(y + 1)(y + 2)\ldots(y + n)},$$

where y is not zero or a negative integer. [*Camb. Sch.*]

Write $$F(y) = \frac{n!}{y(y + 1)(y + 2)\ldots(y + n)} = \sum_{r=0}^{n} \frac{n!}{\phi'(-r)} . \frac{1}{y + r},$$

where $\phi(y) = \prod_{r=0}^{n} (y + r)$ [method (iv)].

Differentiating logarithmically,

$$\frac{\phi'(y)}{\phi(y)} = \sum_{r=0}^{n} \frac{1}{y + r}.$$

Write $\Pi'(y+r) = y(y+1)\ldots(y+r-1)(y+r+1)\ldots(y+n)$.

Then $\displaystyle \phi'(y) = \prod_{r=1}^{n}(y+r) + \sum_{r=1}^{n-1}\{\Pi'(y+r)\} + \prod_{r=0}^{n-1}(y+r)\}$.

Hence $\phi'(0) = n!$; $\phi'(-n) = (-n)(-n+1)\ldots(-1) = (-1)^n n!$;

$$\begin{aligned}\phi'(-r) &= (-r)(-r+1)\ldots(-1)(1)\ldots(-r+n)\\ &= (-1)^r r!\,(n-r)!,\ 1 < r \leqslant n-1.\end{aligned}$$

$$\begin{aligned}F(y) &= \frac{n!}{n!}\cdot\frac{1}{y} + \sum_{r=1}^{n-1}(-1)^r\frac{n!}{r!\,(n-r)!}\cdot\frac{1}{y+r} + (-1)^n\frac{n!}{n!}\cdot\frac{1}{y+n}\\ &= c_0 + \sum_{r=1}^{n-1}\frac{c_r}{y+r} + (-1)^n\frac{c_n}{y+n}.\end{aligned}$$

(5) *Express $x/(x^5-1)$ in partial fractions.*

If a is a root of $x^5 = 1$ then using method (iv) the coefficient of $1/(x-a)$ in the series of partial fractions is $1/5a^3 = a^2/5$. The roots are 1, $\cos\dfrac{2r\pi}{5} \pm i\sin\dfrac{2r\pi}{5}$, $r = 1, 2$.. The complex roots may be grouped in pairs in the form a, a^{-1} where $a = \cos\dfrac{2r\pi}{5} + i\sin\dfrac{2r\pi}{5}$. Taking such a pair the corresponding partial fractions are

$$\frac{1}{5}\left\{\frac{a^2}{x-a} + \frac{a^{-2}}{x-a^{-1}}\right\} = \frac{1}{5}\frac{x(a^2+a^{-2}) - (a+a^{-1})}{x^2 - x(a+a^{-1}) + 1} = \frac{2}{5}\frac{x\cos\frac{4r\pi}{5} - \cos\frac{2r\pi}{5}}{x^2 - 2x\cos\frac{2r\pi}{5} + 1}.$$

Hence $$\frac{x}{x^5-1} = \frac{1}{5}\frac{1}{x-1} + \frac{2}{5}\frac{x\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5}}{x^2 - 2x\cos\frac{2\pi}{5} + 1} + \frac{2}{5}\frac{x\cos\frac{8\pi}{5} - \cos\frac{4\pi}{5}}{x^2 - 2x\cos\frac{4\pi}{5} + 1}.$$

NOTE.—The method of this example extends directly to functions of the form $x^m/(x^n-1)$, where m and n are positive integers and $m < n$.

9·21. Calculus Method of Determining the Coefficients

Consider the proper fraction P/Q and write $P = f(x)$, $Q = F(x)$. Suppose that $F(x) = (x-a)^n F_1(x)$ where $F_1(x)$ is prime to $(x-a)$. Then

$$\frac{f(x)}{F(x)} = \frac{a_1}{x-a} + \frac{a_2}{(x-a)^2} + \ldots + \frac{a_{n-1}}{(x-a)^{n-1}} + \frac{a_n}{(x-a)^n} + \frac{f_1(x)}{F_1(x)},$$

where $f_1(x)/F_1(x)$ is a proper fraction.

Multiplying both sides of the equation by $F(x)$,

$$f(x) - F_1(x)\{a_n + a_{n-1}(x-a) + a_{n-2}(x-a)^2 + \ldots + a_1(x-a)^{n-1}\} = (x-a)^n f_1(x).$$

In this equation write $x = a$. Then

$$f(a) - a_n F_1(a) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Now differentiate both sides of the equation $(n - 1)$ times and in each of the equations obtained write $x = a$. Since $(x - a)^n$ is a factor of the right-hand side of the equation, each of the $n - 1$ differential coefficients will contain $x - a$ as a factor and hence the expression will vanish for $x = a$. Thus it is only necessary to consider the contribution of the terms on the left-hand side.

Denoting this side by $\phi(x)$, and the polynomial

$$a_n + a_{n-1}(x - a) + a_{n-2}(x - a)^2 + \ldots + a_1(x - a)^{n-1}$$

by $\psi(x)$ we require the differential coefficients of

$$\phi(x) = f(x) - F_1(x)\,\psi(x).$$

Applying Leibnitz's theorem on the differentiation of a product, and denoting by $D^r\phi$ the rth differential coefficient of $\phi(x)$,

$$D^r\phi = D^rf - F_1 . D^r\psi - \binom{r}{1} DF_1 . D^{r-1}\psi - \binom{r}{2} D^2F_1 . D^{r-2}\psi$$
$$- \ldots - \binom{r}{r-1} D^{r-1}F_1 . D\psi - \binom{r}{r} D^rF_1 . \psi,$$

the coefficients being the coefficients of the binomial expansion. Now writing $x = a$ and observing that $D^r\psi = r!\,a_{n-r}$ this equation gives $0 = D^rf(a) - r!\,a_{n-r}F_1(a) - r.(r - 1)!a_{n-r+1}DF_1(a)$

$$- \frac{r(r-1)}{2!}.(r - 2)!a_{n-r+2}D^2F_1(a)$$
$$- \ldots - ra_{n-1}D^{r-1}F_1(a) - a_nD^rF_1(a)$$

$$= D^rf(a) - r! \sum_{p=0}^{r} \frac{1}{p!} a_{n-r+p}D^pF_1(a).$$

Thus writing $r = 1, 2, 3, \ldots$ in succession,

$$0 = Df(a) - a_{n-1}F_1(a) - a_nDF_1(a) \quad \ldots\ldots \text{(ii)}$$
$$0 = D^2f(a) - 2a_{n-2}F_1(a) - 2a_{n-1}DF_1(a) - a_nD^2F_1(a) \quad \ldots \text{(iii)}$$
$$0 = D^3f(a) - 6a_{n-3}F_1(a) - 6a_{n-2}DF_1(a)$$
$$- 3a_{n-1}D^2F_1(a) - a_nD^3F_1(a) \ldots \text{(iv)}$$

. . . .

Thus the n equations (i), (ii), (iii), (iv) .... provide a series of recurrence relations from which $a_n, a_{n-1}; a_{n-2}, \ldots$ can be calculated in turn.

Example.—*Find the partial fractions of* $\dfrac{x}{(x-1)(2x-1)(x-2)^n}$ *where n is a positive integer.* [*M.T.*]

In the given expression write $y = x - 2$. Then

$$\frac{x}{(x-1)(2x-1)(x-2)^n} = \frac{2+y}{(1+y)(3+2y)y^n}$$
$$= \frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \ldots + \frac{a_{n-1}}{y} + \frac{b}{1+y} + \frac{c}{3+2y}$$
$$= \sum_{r=0}^{n-1} \frac{a_r}{y^{n-r}} + \frac{b}{1+y} + \frac{c}{3+2y}.$$

Hence

$$2 + y = (1+y)(3+2y)y^n \sum_{r=0}^{n-1} \frac{a_r}{y^{n-r}} + by^n(3+2y) + cy^n(1+y).$$

Putting $y = 0, -1, -3/2$ in succession we have

$$a_0 = \tfrac{2}{3}, \quad b = (-1)^n, \quad c = (-1)^{n+1}(\tfrac{2}{3})^n.$$

Next consider the form

$$\frac{2+y}{(1+y)(3+2y)} = \frac{1}{1+y} - \frac{1}{3+2y} = y^n \sum_{r=0}^{n-1} \frac{a_r}{y^{n-r}} + \frac{by^n}{1+y} + \frac{cy^n}{3+2y}.$$

To find a_r, $1 \leqslant r < n-1$, differentiate both sides of this identity r times and put $y = 0$ in the result. From Leibnitz's theorem it is clear that there will be no contribution from $by^n(1+y)^{-1}$ or $cy^n(3+2y)^{-1}$ since the rth differential coefficient for $1 < r < n-1$ will contain y as a factor. The only contribution from $y^n \sum_{r=0}^{n-1} \frac{a_r}{y^{n-r}} = \sum_{r=0}^{n-1} a_r y^r$ is $a_r r!$.

Also $D^r(1+y)^{-1} = (-1)^r r!(1+y)^{-r-1} = (-1)^r r!$ when $y = 0$.
$D(3+2y)^{-1} = (-1)^r r! 2^r (3+2y)^{-r-1} = (-1)^r r! 2^r 3^{-r-1}$ when $y = 0$.

Hence $\qquad r! a_r = (-1)^r r! - (-1)^r r! \, 2^r 3^{-r-1}$ giving
$$a_r = (-1)^r(1 - 2^r 3^{-r-1}), \quad 1 \leqslant r < n-1.$$

Thus
$$\frac{2+y}{(1+y)(3+2y)y^n}$$
$$= \frac{\frac{2}{3}}{y^n} + \sum_{r=1}^{n-1} \frac{(-1)^r(1-2^r 3^{-r-1})}{y^{n-r}} + \frac{(-1)^n}{1+y} + \frac{(-1)^{n+1}2^n 3^{-n}}{3+2y},$$

or
$$\frac{x}{(x-1)(2x-1)(x-2)^n}$$
$$= \frac{\frac{2}{3}}{(x-2)^n} + \sum_{r=1}^{n-1} (-1)^r \frac{1 - 2^r 3^{-r-1}}{(x-2)^{n-r}} + \frac{(-1)^n}{x-1} + \frac{(-1)^{n+1}2^n 3^{-n}}{(2x-1)}.$$

Alternative method.

Writing $y = x - 2$ as before, we have

$$\frac{2+y}{(1+y)(3+2y)y^n} = \frac{1}{y^n}\left\{\frac{1}{1+y} - \frac{1}{3+2y}\right\}.$$

Now $\quad (1+y)^{-1} = 1 - y + y^2 - \ldots + \dfrac{(-1)^n y^n}{1+y}$
$$= \sum_{r=0}^{n-1} (-1)^r y^r + \frac{(-1)^n y^n}{1+y}.$$

Similarly, $\quad (3+2y)^{-1} = \frac{1}{3} \displaystyle\sum_{r=0}^{n-1} (-1)^r (\tfrac{2}{3})^r y^r + \frac{(-1)^n (\frac{2}{3})^n y^n}{3+2y}.$

Hence $$\frac{2 + y}{(1 + y)(3 + 2y)\,y^n}$$
$$= \frac{1}{y^n}\left\{\sum_{r=0}^{n-1}(-1)^r\left[1 - \tfrac{1}{3}\left(\tfrac{2}{3}\right)^r\right]y^r\right\} + \frac{(-1)^n}{1 + y} - \frac{(-1)^n\left(\frac{2}{3}\right)^n}{3 + 2y}$$
$$= \frac{\frac{2}{3}}{y^n} + \sum_{r=1}^{n-1}(-1)^r\frac{1 - \frac{1}{3}\left(\frac{2}{3}\right)^r}{y^{n-r}} + \frac{(-1)^n}{1 + y} + \frac{(-1)^{n+1}\left(\frac{2}{3}\right)^n}{3 + 2y},$$
as before.

9·3. Expansion of a Rational Fraction

Any proper fraction P/Q may be written in the form
$$\frac{P}{Q} = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_mx^m}{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n} \quad \text{where } m < n.$$
Further, if $b_0 \neq 0$, the fraction may be expressed in the form
$$\frac{a_0 + a_1x + a_2x^2 + \ldots + a_mx^m}{1 + c_1x + c_2x^2 + \ldots + c_nx^n}.$$

Suppose that it is possible to expand the rational function in powers of x by more than one method. To make the argument explicit suppose that by two different methods the series $\Sigma\lambda_nx^n$, $\Sigma\mu_nx^n$ are obtained. Then *provided the interval of convergence of the power series is positive, the two expansions must be identical.* For since the expansions represent the same function for the same range of x, say, $|x| < \rho$, where $\rho > 0$, it follows that
$$\Sigma\lambda_nx^n = \Sigma\mu_nx^n, \quad |x| < \rho.$$
From the theorem on identical equality of power series it follows that $\lambda_n = \mu_n$ for all values of n.

9·31. Method of Expansion

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the n roots of
$$1 + c_1x + c_2x^2 + \ldots + c_nx^n = 0.$$
These roots will not all be distinct if the equation has repeated roots, and in general some will be real and some complex.

Let λ be the least value of $|\lambda_r|$, $r = 1, 2, \ldots, n$. *Then the rational function*
$$(a_0 + a_1x + a_2x^2 + \ldots + a_mx^m)/(1 + c_1x + c_2x^2 + \ldots + c_nx^n)$$
can be expanded in a series of ascending powers of x if and only if $|x| < \lambda$.
$$\frac{a_0 + a_1x + a_2x + \ldots + a_mx^n}{1 + c_1x + c_2x^2 + \ldots + c_nx^n} = \Sigma\frac{k}{(x - \lambda_r)^p},$$
where p is a positive integer $\geqslant 1$, k is a constant depending on λ_r.

The typical term written down may be expanded by the binomial theorem if, and only if, $|x| < |\lambda_r|$, i.e. when we consider all the terms, if, and only if, $|x| < \lambda$.

Expanding each term we obtain a *finite* number of *absolutely* convergent series. Thus in particular it is legitimate to rearrange the terms of the series and group them in powers of x. Thus we obtain the required expansion in powers of x.

Examples.—(1) *Express in partial fractions* $\dfrac{4x^3 - 12x^2 + 14x - 7}{(x^2 - 3x + 2)^3}$ *and show that the coefficient of* x^n *in the expansion of this expression in ascending powers of* x *is*

$$-\frac{n(n+1)}{2} - \frac{(n+1)(5n+14)}{2^{n+4}}.$$

State the range of values of x *for which this expansion is valid.* [*M.T.*]

Now $x^2 - 3x + 2 = (x-1)(x-2)$.

Write $\dfrac{4x^3 - 12x^2 + 14x - 7}{(x^2 - 3x + 2)^3}$

$$\equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{F}{(x-2)^3}$$

i.e. $4x^3 - 12x^2 + 14x - 7 \equiv A(x-1)^2(x-2)^3 + B(x-1)(x-2)^3$
$+ C(x-2)^3 + D(x-2)^2(x-1)^3 + E(x-2)(x-1)^3$
$+ F(x-1)^3.$

In this identity write $x = 1, 2$ in succession. Then

$$4 - 12 + 14 - 7 = (-1)^3 C, \text{ i.e. } C = 1.$$

$$\text{Also } 32 - 48 + 28 - 7 = 1^3 . F, \text{ i.e. } F = 5.$$

Equating coefficients of x^5 and the constant terms on both sides of the identity,

$$0 = A + D,$$

$$-7 = -8A + 8B - 8C - 4D + 2E - F$$

$$\text{i.e.} \quad 6 = -8A + 8B - 4D + 2E.$$

To obtain two other equations write $x = -2, -1$ in both sides of the identity. These substitutions give

$$-115 = -756A + 192B - 64 - 432D + 108E - 135,$$

$$-37 = -108A + 54B - 27 - 72D + 24E - 40.$$

Solving these equations we obtain $A = -D = 0$, $B = 1$, $E - 1$. Thus

$$\frac{4x^3 - 12x^2 + 14x - 7}{(x^2 - 3x + 2)^3} = \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} - \frac{1}{(x-2)^2} + \frac{5}{(x-2)^3}.$$

$$\text{Now } \frac{1}{(x-1)^2} = (1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n, \ |x| < 1.$$

$$\frac{1}{(x-1)^3} = -(1-x)^{-3} = -\sum_{n=0}^{\infty} \tfrac{1}{2}(n+1)(n+2)x^n, \ |x| < 1.$$

$$\frac{1}{(x-2)^2} = \tfrac{1}{4}(1-\tfrac{1}{2}x)^{-2} = \tfrac{1}{4}\sum_{n=0}^{\infty} (n+1)(\tfrac{1}{2}x)^n, \ |\tfrac{1}{2}x| < 1.$$

$$\frac{1}{(x-2)^3} = -\tfrac{1}{8}(1-\tfrac{1}{2}x)^{-3} = -\tfrac{1}{8}\sum_{n=0}^{\infty} \tfrac{1}{2}(n+1)(n+2)(\tfrac{1}{2}x)^n, \ |\tfrac{1}{2}x| < 1.$$

Hence, provided $|x| < 1$,

$$\frac{4x^3 - 12x^2 + 14x - 7}{(x^2 - 3x + 2)^3} = \sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=0}^{\infty} \tfrac{1}{2}(n+1)(n+2)x^n$$

$$- \tfrac{1}{4}\sum_{n=0}^{\infty} (n+1)(\tfrac{1}{2}x)^n - \tfrac{5}{8}\sum_{n=0}^{\infty} \tfrac{1}{2}(n+1)(n+2)(\tfrac{1}{2}x)^n.$$

Thus the coefficient of x^n in the expansion is

$$(n+1) - \tfrac{1}{2}(n+1)(n+2) - (n+1)2^{-n-2} - 5(n+1)(n+2)2^{-n-4}$$

$$= -\frac{n(n+1)}{2} - \frac{(n+1)(5n+14)}{2^{n+4}}.$$

(2) *Resolve* $1/(1-ax)(1-bx)$ *into partial fractions. If this function is expanded as a series of ascending powers of* x, (i) *by utilising the partial fractions already found;* (ii) *by multiplying the expansions of* $(1-ax)^{-1}$ and $(1-bx)^{-1}$, *show that the coefficients of* x^n *obtained by the two methods are identical.*

[*Lond., B.A.*]

$$\text{Write } \frac{1}{(1-ax)(1-bx)} \equiv \frac{A}{1-ax} + \frac{B}{1-bx},$$

$$\text{i.e. } 1 \equiv A(1-bx) + B(1-ax).$$

Putting $x = 1/a$ and $1/b$ in succession we find that

$$A = a/(a-b), \ B = -b/(a-b).$$

$$\text{Now } (1-ax)^{-1} = \sum_{n=0}^{\infty} a^n x^n, \ |ax| < 1,$$

$$(1-bx)^{-1} = \sum_{n=0}^{\infty} b^n x^n, \ |bx| < 1.$$

Using the partial fractions the coefficient of x^n is

$$\frac{a^{n+1}}{a-b} - \frac{b^{n+1}}{a-b}.$$

$$\text{Again, } (1-ax)^{-1}(1-bx)^{-1} = \left(1 + \sum_{n=1}^{\infty} a^n x^n\right)\left(1 + \sum_{n=1}^{\infty} b^n x^n\right).$$

The coefficient of x^n in the product is

$$a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + ab^{n-1} + b^n.$$

This is a geometrical progression of $n+1$ terms whose first term is a^n and whose common ratio is b/a. The sum of the terms is

$$a^n\left\{1-\left(\frac{b}{a}\right)^{n+1}\right\}\Big/\left(1-\frac{b}{a}\right),\ \text{i.e. } (a^{n+1}-b^{n+1})/(a-b).$$

(3) *Prove that if a, b, c are unequal, it is impossible for the coefficients of two consecutive terms in the expansion of* $(1-ax)/(1-bx)(1-cx)$ *to be both zero.* [*Madras, B.A.*]

$$\frac{1-ax}{(1-bx)(1-cx)}=\frac{1}{b-c}\left\{\frac{b-a}{1-bx}+\frac{a-c}{1-cx}\right\}$$

Now provided $|bx|<1$, $|cx|<1$.

$$(1-bx)^{-1}=\sum_{r=0}^{\infty}b^rx^r,\ (1-cx)^{-1}=\sum_{r=0}^{\infty}c^rx^r.$$

Hence $\dfrac{1-ax}{(1-bx)(1-cx)}=\dfrac{1}{b-c}\displaystyle\sum_{r=0}^{\infty}[(b-a)b^r+(a-c)c^r]x^r.$

Suppose that the coefficients of x^r and x^{r+1} are both zero. Then

$$(b-a)b^r+(a-c)c^r=0,$$
$$(b-a)b^{r+1}+(a-c)c^{r+1}=0.$$

Hence $\left(\dfrac{b}{c}\right)^r=\left(\dfrac{b}{c}\right)^{r+1}$ since $b-a\neq 0$, $a-c\neq 0$. This requires $b=c$, which is not true.

(4) *Resolve* $\dfrac{2x^2+3x+4}{(x-1)(x^2+2)}$ *into partial fractions, and hence find the value of the function, correct to 6 places of decimals, when* $x=-0{\cdot}02$. [*Camb. Sch.*]

$$\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2},$$

i.e. $2x^2+3x+4=A(x^2+2)+(Bx+C)(x-1).$

Put $x=1$; then $2+3+4=3A$, $A=3$. Putting $x=0,\ -1$ we obtain $4=2A-C$, $2-3+4=3A+(-B+C)(-2)$,

i.e. $3=3A+2B-2C.$

Solving these equations we obtain $C=2$, $B=-1$. Thus

$$\begin{aligned}\frac{2x^2+3x+4}{(x-1)(x^2+2)}&=\frac{3}{x-1}+\frac{-x+2}{x^2+2}\\&=-3(1-x)^{-1}-\tfrac{1}{2}(x-2)(1+\tfrac{1}{2}x^2)^{-1}\\&=-3(1+x+x^2+x^3+x^4+\dots)\\&\quad-\tfrac{1}{2}(x-2)(1-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-\dots),\ |x|<1\\&=-2-\tfrac{7}{2}x-\tfrac{7}{2}x^2-\tfrac{11}{4}x^3-\tfrac{11}{4}x^4\dots\end{aligned}$$

Now when

$$x=-0{\cdot}02,\ x^2=0{\cdot}0004,\ x^3=-0{\cdot}000008,\ x^4=0{\cdot}00000016,\dots$$

It is clear that higher powers of x will not affect the result correct to 6 decimal places.

The value is

$$-2+{\cdot}07-{\cdot}0014+{\cdot}000022-{\cdot}00000044\dots=-1{\cdot}931378$$

correct to 6 decimal places.

(5) *Prove that*

$$\frac{(x-1)(x-2)\dots(x-n)}{x(x+1)(x+2)\dots(x+n)} = \sum_{r=0}^{n} (-1)^{n-r} \frac{(n+r)!}{r!\,r!\,(n-r)!\,(x+r)};$$

and deduce that

$$-\frac{(n+1)!}{(n-1)!} + \frac{(n+2)!}{1!\,2!\,(n-2)!} - \frac{(n+3)!}{2!\,3!\,(n-3)!} + \frac{(n-4)!}{3!\,4!\,(n-4)!} - \dots$$
$$+ (-1)^n \frac{(2n)!}{(n-1)!\,n!} = (-1)^n n(n+1). \quad \text{[Camb. Sch.]}$$

Write $\dfrac{(x-1)(x-2)\dots(x-n)}{x(x+1)(x+2)\dots(x+n)} = \displaystyle\sum_{r=0}^{n} \frac{\lambda_r}{x+r}$. Then

$$(x-1)(x-2)\dots(x-n)$$
$$= \sum_{r=0}^{n} \lambda_r x(x+1)\dots(x+r-1)(x+r+1)\dots(x+n).$$

In this identity write $x = -r$. Then

$$(-r-1)\dots(-r-n) = \lambda_r(-r)(-r+1)\dots(-1)(1)\dots(-r+n),$$

$$\text{i.e.}\quad (-1)^n \frac{(r+n)!}{r!} = \lambda_r(-1)^r r!\,(n-r)!,$$

$$\text{i.e.}\quad \lambda_r = (-1)^{n-r} \frac{(r+n)!}{r!\,r!\,(n-r)!}.$$

If $x > n$, then $\dfrac{\lambda_r}{x+r} = \dfrac{\lambda_r}{x}\left(1+\dfrac{r}{x}\right)^{-1} = \dfrac{\lambda_r}{x}\left(1-\dfrac{r}{x}+\dots\right), r = 1, 2, 3, \dots, n.$

Hence the coefficient of $1/x^2$ in $\displaystyle\sum_{r=0}^{n} \frac{\lambda_r}{(x+r)}$ is $-\displaystyle\sum_{r=0}^{n} r\lambda_r$.

Now consider the coefficient of $1/x^2$ in $\dfrac{(x-1)\dots(x-n)}{x(x+1)\dots(x+n)}$,

$$\text{i.e. the coefficient of } 1/x \text{ in } \frac{\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right)\dots\left(1-\frac{n}{x}\right)}{\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)\left(1+\frac{3}{x}\right)\dots\left(1+\frac{n}{x}\right)},$$

$$\text{i.e. in}\left\{1 - \tfrac{1}{2}n(n+1).\frac{1}{x} + \dots\right\}\left\{1 + \tfrac{1}{2}n(n+1)\frac{1}{x}\dots\right\}^{-1}.$$

This coefficient is $-n(n+1)$.

$$\text{Hence } -n(n+1) = -\sum_{r=0}^{n} r\lambda_r,$$

$$\text{i.e. } n(n+1) = (-1)^{n-1}\frac{(n+1)!}{1!\,(n-1)!} + (-1)^{n-2}\frac{(n+2)!}{1!\,2!\,(n-2)!} + \dots$$
$$+ \frac{(2n)!}{(n-1)!\,n!}.$$

Multiplying throughout by $(-1)^n$ we obtain the required result.

(6) *Express* $\{x(x+1)\ldots(x+n)\}^{-1}$ *in partial fractions. Hence, show that*

$$\sum_{r=0}^{n} \frac{a^r}{x(x+1)\ldots(x+r)} = \sum_{s=0}^{n} \frac{(-1)^s A_s}{s!\,(n-s)!\,(x+s)},$$

where the A_s *are polynomials of degree* n *in* a *with integral coefficients. Find* A_s *and show that*

$$\lim_{n\to\infty} \frac{A_s}{(n-s)!} = a^s e^a. \qquad \text{[Lond. B.Sc.]}$$

Proceeding as in § 9·20, Ex. 4, we have

$$\frac{a^r}{x(x+1)\ldots(x+r)} = \frac{1}{r!}\frac{a^r}{x} - \frac{1}{1!\,(r-1)!}\frac{a^r}{x+1}$$
$$+ \frac{1}{2!\,(r-2)!}\frac{a^r}{x+2} - \ldots + \frac{(-1)^r}{r!}\frac{a^r}{x+r}.$$

Hence $\displaystyle\sum_{r=0}^{n} \frac{a^r}{x(x+1)\ldots(x+r)}$

$$= \frac{1}{x}\left\{1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots + \frac{a^n}{n!}\right\}$$
$$- \frac{1}{x+1}\left\{a + \frac{a^2}{1!} + \frac{a^3}{2!} + \ldots + \frac{a^n}{(n-1)!}\right\}$$
$$+ \frac{1}{x+2}\left\{\frac{a^2}{2!} + \frac{a^3}{2!\,1!} + \frac{a^4}{2!\,2!} + \ldots + \frac{a^n}{2!\,(n-2)!}\right\}$$
$$- \frac{1}{x+3}\left\{\frac{a^3}{3!} + \frac{a^4}{3!\,1!} + \frac{a^5}{3!\,2!} + \ldots + \frac{a^n}{3!\,(n-3)!}\right\}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{(-1)^s}{x+s}\left\{\frac{a^s}{s!} + \frac{a^{s+1}}{s!\,1!} + \frac{a^{s+2}}{s!\,2!} + \ldots + \frac{a^n}{s!\,(n-s)!}\right\}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$= \sum_{s=0}^{n} \frac{(-1)^s A_s}{s!\,(n-s)!}\cdot\frac{1}{x+s}$$

where $A_s = (n-s)!\,a^s\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^{n-s}}{(n-s)!}\right)$. Clearly A_s is a polynomial of degree n in a and all the coefficients are integers.

Again, $\displaystyle\lim_{n\to\infty} \frac{A_s}{(n-s)!} = a^s \lim_{n\to\infty}\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots \frac{a^{n-s}}{(n-s)!}\right)$

$$= a^s\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots \frac{a^n}{n!} + \ldots \text{to infinity}\right)$$

$$\therefore \lim_{n\to\infty} \frac{A_s}{(n-s)!} = a^s e^a.$$

EXERCISES IX

1. Express in partial fractions $\dfrac{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}{(x-a)(x-b)(x-c)(x-d)}$. Hence, or otherwise show that

$$\frac{(a-\alpha)(a-\beta)(a-\gamma)(a-\delta)}{(a-b)(a-c)(a-d)}+\frac{(b-\alpha)(b-\beta)(b-\gamma)(b-\delta)}{(b-a)(b-c)(b-d)}$$
$$+\text{ two similar terms } = a+b+c+d-\alpha-\beta-\gamma-\delta.$$

[*Camb. Sch.*]

2. Find constants a, b, c such that

$$\frac{x^2-5x+1}{(x+1)(x+2)(x+3)} \equiv \frac{a}{x+1}+\frac{b}{(x+1)(x+2)}+\frac{c}{(x+1)(x+2)(x+3)}.$$

[*Madras, B.A.*]

3. Express the following functions as partial fractions:

$$\frac{x^2-1}{(x-2)(x-3)},\ \frac{1}{x^4+1},\ \frac{1}{x(x-1)^2}.$$

[*Camb. Sch.*]

4. Prove the identity:

$$16(x+2)(x-4)(x-6)$$
$$=(x-1)(x-3)(x-5)(x-7)\left[\frac{-15}{x-1}+\frac{15}{x-3}+\frac{7}{x-5}+\frac{9}{x-7}\right].$$

[*Camb. Sch.*]

5. Split into partial fractions:

(i) $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$, (ii) $\dfrac{x+1}{(x-1)^2(x-2)}$.

[*Madras, B.A.*]

6. Split into partial fractions:

$$\frac{1}{x^2(x+2)},\ \frac{x^2}{(1+2x)(1+3x)},\ \frac{x^2-x-1}{x^3-8}.$$

[*Madras, B.A.*]

7. Resolve the expansion $\dfrac{9}{(x-1)(x+2)^2}$ into partial fractions and hence prove that the coefficient of x^n in the expansion of this expression in ascending powers of x is

$$-1-(-\tfrac{1}{2})^{n+2}(3n+5).$$

[*Sc.T. Prelim.*]

8. If $\phi(x)$ is a polynomial of degree not greater than that of a polynomial $f(x)$ show that

$$\frac{\phi(x)}{(x-a)f(x)} \equiv \frac{\phi(a)}{(x-a)f(a)}+\frac{\text{a polynomial in } x}{f(x)},$$

provided $f(a) \neq 0$. Discuss the case of $f(a)=0$.

Expand $(2x^6-5x^4+2x^3+6x^2-2)/\{(x+2)(x^2-1)^2\}$ in a series of ascending powers of x, stating carefully the general term. [*Camb. Sch.*]

9. Prove that the coefficient of x^{4n} in the expansion of

$$1/\{(1-x)(1-x^2)(1-x^4)\}$$

in ascending powers of x is $(n+1)^2$. [*Camb. Sch.*]

10. Resolve the expression $\dfrac{25 + 8x}{(1 + 2x)^2 (3 - x)}$ into partial fractions. Find the first three terms in the expansion of the expression in ascending powers of x.

11. Expand $\dfrac{x^2}{(1 + x)^2 (1 - 2x)}$ in a series of ascending powers of x, giving an expression for the coefficient of the general term. Show that the sum of all the coefficients up to that of x^n inclusive is $\frac{1}{18}(2^{n+2} - 3n - 5)$ or $\frac{1}{18}(2^{n+2} + 3n - 4)$ according as n is odd or even.

12. Resolve $\dfrac{4x^2 - 3x + 5}{(2 - x)(1 + x^2)}$ into partial fractions and hence find the value of the function correct to 6 decimal places, when x $0{\cdot}05$. [*Camb. Sch.*]

13. Express $\dfrac{57x^3 - 25x^2 + 9x - 1}{(x - 1)^2 (2x - 1)(5x - 1)}$ as a sum of partial fractions; and expand in ascending powers of x as far as the term in x^4. [*Camb. Sch.*]

14. If $\dfrac{1}{(x - 1)^2 (x - 2)} = a_0 + a_1 x + a_2 x^2 + \ldots$ to ∞, where $|x| < 1$, find a_n. Show that a_n is the sum of the first $(n + 1)$ coefficients in the expansion of $1/(1 - x)(x - 2)$ in ascending powers of x, where $|x| < 1$. [*Madras, B.Sc.*]

15. Express $\dfrac{x + 4}{(x^2 - 4)(x + 1)}$ in partial fractions, and find the coefficient of x^{2n-1} in its expansion in ascending powers of x. State the conditions under which the expansion is valid.

16. Observing that

$$\frac{1}{(1 - ax)(1 - x/a)} = \frac{a^2}{a^2 - 1}\left(\frac{1}{1 - ax} - \frac{1}{a^2}\frac{1}{1 - x/a}\right),$$

show that $a^n + a^{n-2} + a^{n-4} + \ldots + a^{-n} = (a^{2n+2} - 1)/a^n (a^2 - 1)$ by obtaining the coefficient of x^n on both sides of the identity.

17. Express $\dfrac{x}{(1 + 2x)^2 (1 - 3x)}$ in partial fractions and hence expand it in ascending powers of x. [*Madras, B.Sc.*]

18. Find the coefficient of x^n in the expansion of $\dfrac{x + 3}{(x + 2)(x - 1)^2}$. [*Madras, B.A.*]

19. Express $(3 - 7x^2)/(1 - 3x)(1 + 2x)(1 + x)$ in partial fractions, and hence expand the function as a series of ascending powers of x. State for what range of values of x the expansion is valid. [*Lond. B.Sc.*]

20. The expression $a + \dfrac{b}{1 + 2x} + \dfrac{c}{1 - 3x^2}$ can be expanded in the form $1 + x + 2x^2 + px^3 + \ldots$ where x is small. Find a, b and c. [*Madras, B.Sc.*]

21. Express in partial fractions $1/(x-a)^2(x-b)^2$. Hence, or otherwise, prove that

$$(r+1)b^r + 2r\,ab^{r-1} + 3(r-1)a^2b^{r-2} + \ldots + (r+1)a^r$$
$$= \{(r+1)(a^{r+3}-b^{r+3}) - (r+3)ab(a^{r+1}-b^{r+1})\}/(a-b)^3,$$

where r is any positive integer. [*Lond. B.A.*]

22. If x is so small that its fourth and higher powers may be neglected, express $(1-x)/(1+x)^4(1-x+x^2)$ in ascending powers of x. [*Madras, B.Sc.*]

23. Express $10(1-3x)/(2x+1)(x-2)^2$ in the form of three partial fractions. Expand the latter by the binomial theorem in ascending powers of x as far as the term in x^3.

24. Express $x/(1+x^2)(3-2x)$ in partial fractions. Prove that, if this expression is expanded in ascending powers of x, in the form $a_0 + a_1x + \ldots$, the sum of the coefficients of the first $(4n+1)$ terms is $\frac{4}{13}\{1-(\frac{2}{3})^{4n}\}$. [*Lond., B.A.*]

25. Resolve into partial fractions $\dfrac{1+2x+3x^2+4x^3}{(1-x)(1+x)(1+x^2)}$ and then transform into a series in ascending powers of x as far as the term involving x^4. [*Sc.T.*]

26. Express $\{(x-1)(x+1)^2\}^{-1}$ in partial fractions. Hence, find the coefficient of x^{-n} in the expansion of this expression in negative powers of x, stating the values of x for which the expansion is valid. [*Lond. B.A.*]

9·4. Recurring Series

Suppose that the rational proper fraction

$$\frac{P}{Q} = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_mx^m}{1 + c_1x + c_2x^2 + \ldots + c_nx^n}$$

has been expanded in an absolutely convergent power series

$$u_0 + u_1x + u_2x^2 + \ldots$$

Then $a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$

$$= (1 + c_1x + c_2x^2 + \ldots + c_nx^n)(u_0 + u_1x + u_2x^2 + \ldots)$$

Since the series is absolutely convergent the product on the right can be rearranged in ascending powers of x. Further, from the theorem on identical equality of power series it follows that the coefficients of the corresponding powers of x on both sides of the equation are equal. Thus

$$u_0 = a_0,\ u_1 + c_1u_0 = a_1,\ \ldots,$$
$$u_r + c_1u_{r-1} + c_2u_{r-2} + \ldots + c_ru_0 = a_r,\ r < n,$$
$$u_r + c_1u_{r-1} + c_2u_{r-2} + \ldots + c_nu_{r-n} = 0,\ r \geqslant n.$$

A series $\Sigma u_r x^r$ possessing the property that beyond a certain point the coefficients u_r are connected by a linear relation of the form

$$u_r + c_1 u_{r-1} + c_2 u_{r-2} + \ldots + c_n u_{r-n} = 0,$$

where $c_1, c_2, \ldots c_n$ are constants, and n is a fixed integer, is called a **Recurring Series.**

9·41. Scale of Relation

It is convenient to adopt here a slight change of notation. Let m be a fixed positive integer, $p_1, p_2, p_3, \ldots p_m$ be m constants and $u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n + \ldots$ a series such that any $m + 1$ successive coefficients are related by an equation of the form

$$u_n + p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_m u_{n-m} = 0 \quad \ldots\ldots \text{(i)}$$

Such a series is called a **Recurring Series of the *m*th Order.**

If we know the first m coefficients in the series, i.e. $u_0, u_1, u_2, \ldots u_{m-1}$, the successive coefficients may be determined by (i). Since this relation itself contains m constants, it follows that a recurring series of the mth order depends on $2m$ constants.

Thus, in particular, if we are given the first $2m$ terms of a series, then the series may be continued as a recurring series of the *mth* order in one and only one way.

It is clear from § 9·4 that the relation (i) is derived from a rational fraction whose denominator is the polynomial

$$1 + p_1 x + p_2 x^2 + \ldots + p_m x^m \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

The relation (i) between the coefficients, and the polynomial (ii) have both been called the **Scale of Relation** by different writers. In what follows we shall call *either* of the equations the scale of relation.

9·51. Convergence of Recurring Series

If x be sufficiently small, any recurring series $\Sigma u_n x^n$ is absolutely convergent.

We use the scale of relation

$$u_n + p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_m u_{n-m} = 0$$

and denote by ρ the greater of the numbers $|p_1| + |p_2| + \ldots + |p_m|$ and 1, and by k the upper bound of $|u_r|$, $r = 0, 1, 2, \ldots m$. Then

$$|u_n| \leqslant |p_1| \cdot |u_{n-1}| + |p_2| \cdot |u_{n-2}| + \ldots + |p_m| \cdot |u_{n-m}|$$
$$\leqslant \rho\, |u_{n_1}|,$$

where $n - m \leqslant n_1 \leqslant n - 1$, and $|u_{n_1}|$ is the greatest value of $|u_r|$, $n - m \leqslant r \leqslant n - 1$.

Again

$$|u_{n_1}| \leqslant |p_1|.|u_{n_1-1}| + |p_2|.|u_{n_1-2}| + \ldots + |p_m|.|u_{n_1-m}|$$
$$\leqslant \rho\,|u_{n_2}|,$$

where $|u_{n_2}|$ is the greatest value of $|u_r|$, $n_1 - m \leqslant r \leqslant n_1 - 1$.

Continuing this process we obtain a sequence of values u_{n_1}, u_{n_2}, $u_{n_3}, \ldots u_{n_p}$, such that

$$|u_n| \leqslant \rho\,|u_{n_1}|, \quad |u_{n_1}| \leqslant \rho\,|u_{n_2}|,$$
$$|u_{n_2}| \leqslant \rho\,|u_{n_3}|, \ldots, \quad |u_{n_{p-1}}| \leqslant \rho\,|u_{n_p}|.$$

It should be observed that there can only be a finite number of members of the sequence and that if $|u_{n_p}|$ be the last member then $|u_{n_p}|$ is one of the values of

$$|u_0|, \ |u_1|, \ |u_2|, \ \ldots, \ |u_m|.$$

Thus $|u_{n_p}|$ is a constant which is *independent of* n.

Multiplying the inequalities together $|u_n| \leqslant k\rho^p$, where k is independent of n.

Clearly the number p must lie between n/m and n, for the above process can be repeated at least $[n/m]$ times where $[n/m]$ denotes the integral part of n/m.

From the definition of ρ it follows that

$$|u_n| < K\rho^n \text{ where } K \text{ is independent of } n.$$

$$\text{Also } \lim_{n \to \infty} |u_n|^{\frac{1}{n}} < \lim_{n \to \infty} (K^{\frac{1}{n}}\rho) = \rho.$$

Hence the series certainly converges if $|x| < 1/\rho$.

9·52. The Generating Function of a Recurring Series

If $\Sigma u_n x^n$ is a convergent recurring series then its sum is a rational proper fraction. The sum of the series is called the *Generating Function.*

Let s be the sum of the recurring series whose scale of relation is

$$u_n + p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_m u_{n-m} = 0.$$
$$s = u_0 + u_1 x + u_2 x^2 + \ldots$$

Multiply s by $p_1 x$, $p_2 x^2$, $\ldots$ $p_m x^m$ and add. Then

$$s(1 + p_1 x + p_2 x^2 + \ldots + p_m x^m) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-1} x^{m-1},$$

where $a_0 = u_0$, $a_1 = u_1 + p_1 u_0$, $\ldots$, and in general

$$a_r = u_r + p_1 u_{r-1} + \ldots + p_r u_0,$$

$r = 0, 1, 2, \ldots m - 1$.

The coefficients of the higher powers of x vanish because of the scale of relation. Thus the generating function is the proper fraction

$$\frac{a_0 + a_1x + a_2x^2 + \ldots + a_{m-1}x^{m-1}}{1 + p_1x + p_2x^2 + \ldots + p_mx^m}.$$

Thus we have shown that *the necessary and sufficient condition that a power series in x be a recurring series is that it be an expansion of a proper rational function of x.*

The method used in this section is that for summing to infinity any recurring series whose scale of relation is given. In any particular series we can write down immediately the denominator of the generating function and then determine the numerator by equating coefficients. If, however, we are only given the series it is first necessary to determine the scale of relation. The method may be seen by a study of the following question.

Example.—*Show that the series whose nth term is $2^nn^2x^n$ is a recurring series and find its generating function. For what range of values of x does the latter represent the sum to infinity of the series.* [*Camb. Sch.*]

Consider the series

$$u_0 + u_1x + u_2x^2 + \ldots + u_nx^n + \ldots$$

where $u_0 = 0$, $u_n = 2^nn^2$, $n > 1$. The given series is $\sum_{n=0}^{\infty} u_nx^n$.

Suppose that the coefficients u_n satisfy a relation of the form

$$u_n + p_1u_{n-1} + p_2u_{n-2} + p_3u_{n-3} = 0, \quad n > 3.$$

We now show that it is possible to find values of p_1, p_2 and p_3 satisfying the equation. Substituting for u_n, u_{n-1}, u_{n-2}, u_{n-3} this relation becomes

$$2^nn^2 + p_12^{n-1}(n-1)^2 + p_22^{n-2}(n-2)^2 + p_32^{n-3}(n-3)^2 = 0.$$

This equation gives on simplification:

$$n^2(8 + 4p_1 + 2p_2 + p_3) - 2n(4p_1 + 4p_2 + 3p_3) + 4p_1 + 8p_2 + 9p_3 = 0.$$

If this condition is satisfied for all $n > 3$ it follows that

$$8 + 4p_1 + 2p_2 + p_3 = 0,$$
$$4p_1 + 4p_2 + 3p_3 = 0,$$
$$4p_1 + 8p_2 + 9p_3 = 0.$$

Thus there are three equations to determine the three unknowns p_1, p_2, p_3. Solving them we find

$$p_1 = -6, \quad p_2 = 12, \quad p_3 = -8.$$

It follows that the given series is a recurring series whose scale of relation is

$$1 - 6x + 12x^2 - 8x^3 = (1 - 2x)^3.$$

The generating function has the form

$$\frac{a_0 + a_1x + a_2x^2}{(1 - 2x)^3} = 2x + 16x^2 + 72x^3 + \ldots,$$

since $u_0 = 0$, $u_1 = 2$, $u_2 = 16$, $u^3 = 72$, ... Multiplying both sides by $(1 - 2x)^3$ it is easily seen that

$$a_0 + a_1x + a_2x^2 = 2x + 4x^2,$$
i.e. $a_0 = 0$, $a_1 = 2$, $a_2 = 4$.

Hence the sum of the series, *i.e.* the generating function is

$$(2x + 4x^2)/(1 - 2x)^3 = 2x(1 + 2x)/(1 - 2x)^3.$$

The factor $(1 - 2x)^3$ in the denominator of the rational fraction shows that the expansion is valid for $|x| < \frac{1}{2}$,

$$\text{i.e. } 2x(1 + 2x)/(1 - 2x)^3 = \sum_{n=0}^{\infty} 2^n n^2 x^n, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

9·6. Determination of the General Term of a Recurring Series

If the generating function be expressed as the sum of partial fractions of the form

$$\lambda/(1 + \mu x)^r$$

where λ, μ are constants and r is a positive integer, then we can determine u_n by expanding $\lambda/(1 + \mu x)^r$ in ascending powers of x and collecting coefficients of x^n. The expansion will only be valid provided $|\mu x| < 1$, i.e. $|x| < 1/|\mu|$. The method is readily seen from a study of the following examples.

Examples.—(1) *The sequence* $u_0, u_1, u_2, \ldots$ *is defined by* $u_0 = 0$, $u_1 = 1$, $u_n = u_{n-1} + u_{n-2}$, $(n = 2, 3, \ldots)$. *Obtain a general formula for* u_n *and show that* u_n *is the integer nearest to* $\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n$. [*Camb. Sch.*]

The scale of relation is $1 - x - x^2$ and the generating function has the form

$$\frac{a_0 + a_1 x}{1 - x - x^2} = x + u_2 x^2 + \ldots$$

$$\text{i.e. } a_0 + a_1 x = (x + u_2 x^2 + \ldots)(1 - x - x^2).$$

Equating coefficients it follows that $a_0 = 0$, $a_1 = 1$.

$$\text{Again } \frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha x)(1 - \beta x)},$$

where $\alpha = -\frac{\sqrt{5}-1}{2}$, $\beta = \frac{\sqrt{5}+1}{2}$.

Expressing as partial fractions the generating function

$$\frac{x}{1 - x - x^2} = \frac{-1/\sqrt{5}}{1 - \alpha x} + \frac{1/\sqrt{5}}{1 - \beta x} = \frac{1}{\sqrt{5}}\Sigma(\beta^n - \alpha^n)x^n, \quad |x| < 2/(\sqrt{5} + 1).$$

$$\text{Hence } u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n + (-1)^{n+1}\left(\frac{\sqrt{5}-1}{2}\right)^n\right\}.$$

From the definition of u_n it is clear that u_n is an integer.

$$\text{Now } \frac{\sqrt{5}-1}{2} < \frac{2{\cdot}24 - 1}{2} = 0{\cdot}62,$$

$$\text{and } \left(\frac{\sqrt{5}-1}{2}\right)^2 < ({\cdot}62)^2 < 0{\cdot}5.$$

$$\text{Also for } n > 2, \left(\frac{\sqrt{5}-1}{2}\right)^n < \left(\frac{\sqrt{5}-1}{2}\right)^2 < 0{\cdot}5.$$

It follows that u_n must be the integer nearest to $\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n$.

(2) *A sequence of terms* $u_0, u_1, u_2, \ldots u_n, \ldots$ *is such that any three consecutive terms are connected by the relation*

$$6u_{n+1} - 5u_n + u_{n-1} = 0.$$

If $u_0 = 1$, $u_1 = \frac{1}{6}$, *find an expression for* u_n *and show that the infinite series*

$$u_0 + u_1 + u_2 + \ldots$$

converges to the sum unity. [*Camb. Sch.*]

Proceeding as in Ex. 1 it is easily seen that the generating function is

$$\frac{1 - \frac{2}{3}x}{1 - \frac{5}{6}x + \frac{1}{6}x^2} = \frac{1 - \frac{2}{3}x}{(1 - \frac{1}{2}x)(1 - \frac{1}{3}x)} = \frac{-1}{1 - \frac{1}{2}x} + \frac{2}{1 - \frac{1}{3}x}$$

$$= -(1 - \tfrac{1}{2}x)^{-1} + 2(1 - \tfrac{1}{3}x)^{-1}$$

$$= -\sum^{\infty} (\tfrac{1}{2}x)^n + 2\sum_{n=0}^{\infty} (\tfrac{1}{3}x)^n, \quad |x| < 2$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{3^n} - \frac{1}{2^n}\right) x^n.$$

The infinite series $u_0 + u_1 + u_2 + \ldots$ is the expansion when $x = 1$.

Since the power series $\Sigma u_n x^n$ converges uniformly for $x = 1$, the sum of the series Σu_n is obtained by substituting $x = 1$ in the sum function of $\Sigma u_n x^n$. Also when $x = 1$,

$$\frac{1 - \frac{2}{3}x}{1 - \frac{5}{6}x + \frac{1}{6}x^2} = 1.$$

(3) *Find the nth term of the recurring series whose scale of relation is*

$$u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 0.$$

and whose first three terms are $1, 0, -5$. [*M.T.*]

The generating function is

$$\frac{a_0 + a_1x + a_2x^2}{1 - 4x + 5x^2 - 2x^3} = 1 - 5x^2 + u_3x^3 + \ldots$$

Multiplying both sides of the equation by $1 - 4x + 5x^2 - 2x^3$ and equating corresponding coefficients we find that $a_0 = 1$, $a_1 = -4$, $a_2 = 0$.

The generating function is

$$\frac{1 - 4x}{1 - 4x + 5x^2 - 2x^3} = \frac{1 - 4x}{(1 - x)^2(1 - 2x)} = \frac{A}{1 - x} + \frac{B}{(1 - x)^2} + \frac{C}{1 - 2x}.$$

Evaluating A, B, C in the usual way, we find $A = 2$, $B = 3$, $C = -4$. Hence

$$\frac{1 - 4x}{(1 - x)^2(1 - 2x)}$$

$$= 2(1 - x)^{-1} + 3(1 - x)^{-2} - 4(1 - 2x)^{-1}$$

$$= 2\sum_{n=0}^{\infty} x^n + 3\sum_{n=0}^{\infty} (n + 1)x^n - 4\sum_{n=0}^{\infty} 2^n x^n, \qquad |x| < \tfrac{1}{2}.$$

$$= \sum_{n=0}^{\infty} \{2 + 3(n + 1) - 2^{n+2}\} x^n.$$

Thus $u_n = 3n + 5 - 2^{n+2}$.

Since the first term of the series is u_0 it follows that the nth term is $u_{n-1}x^{n-1}$ where $u_{n-1} = 3n + 2 - 2^{n+1}$.

(4) *A sequence of numbers* $b_0, b_1, b_2, \ldots$ *is defined by* $b_0 = \frac{2}{3}$, $b_1 = -\frac{7}{9}$, $3b_r + 5b_{r-1} + 2b_{r-2} = 0$, $r > 2$. *Find the value of* b_r.

Proceeding as in the previous examples it will be found that the generating function is

$$\frac{2+x}{3+5x+2x^2} = \frac{-1}{3+2x} + \frac{1}{1+x} = -\tfrac{1}{3}(1+\tfrac{2}{3}x)^{-1} + (1+x)^{-1}$$

$$= -\tfrac{1}{3}\sum_{r=0}^{\infty}(-1)^r(\tfrac{2}{3})^r x^r + \sum_{r=0}^{\infty}(-1)^r x^r, \quad |x| < 1,$$

$$= \sum_{r=0}^{\infty}(-1)^r\{-\tfrac{1}{3}(\tfrac{2}{3})^r + 1\}x^r.$$

Hence $b_r = (-1)^r\{1 - \frac{1}{3}(\frac{2}{3})^r\}$.

This result may be used to calculate the coefficients a_{n-r} in § 9·21 Ex. Change the notation of the latter example by writing $b_r = a_{n-r}$, so that $b_0 = a_n$, $b_1 = a_{n-1}, \ldots$ Then $b_0 = \frac{2}{3}$, $b_1 = -\frac{7}{9}$ and so on.

9·7. The Sum to a Finite Number of Terms

If the general term of the series is known, then the sum to a finite number of terms may be obtained by using the method of § 9·52. On the other hand, if u_n is not given and it is necessary to calculate it by the method of § 9·6, it may be more convenient to sum the series directly by considering the terms which make up u_n. The methods are illustrated by the two following examples.

Examples.—(1) *Find the sum to n terms of the series whose nth term is* $2^n n^2 x^n$.

In § 9·52 Ex. it is shown that the scale of relation is

$$(1-2x)^3 = 1 - 6x + 12x^2 - 8x^3. \quad \text{Now write}$$

$$S = 2x + 16x^2 + 72x^3 + 256x^4 + \ldots + 2^n n^2 x^n.$$

$$-6x\,S = \quad -12x^2 - 96x^3 - 432x^4 - \ldots - 6.2^{n-1}(n-1)^2x^n - 6.2^n n^2 x^{n+1}$$

$$+12x^2\,S = \quad +24x^3 + 192x^4 + \ldots + 12.2^{n-2}(n-2)^2x^n + 12.2^{n-1}(n-1)^2x^{n+1} + 12.2^n n^2 x^{n+2}$$

$$-8x^3\,S = \quad -16x^4 - \ldots - 8.2^{n-3}(n-3)^2x^n - 8.2^{n-2}(n-2)^2x^{n+1} - 8.2^{n-1}(n-1)^2x^{n+2} - 8.2^n n^2 x^{n+3}.$$

Adding, $S(1 - 6x + 12x^2 - 8x^3)$

$$= 2x + 4x^2 + x^{n+1}\{-6.2^n n^2 + 12.2^{n-1}(n-1)^2 - 8.2^{n-2}(n-2)^2\}$$
$$+ x^{n+2}\{12.2^n n^2 - 8.2^{n-1}(n-1)^2\} - 8.2^n n^2 x^{n+3}$$
$$= 2x + 4x^2 - 2^{n+1}(n+1)^2x^{n+1} + 2^{n+2}(2n^2+2n-1)x^{n+2} - 2^{n+3}n^2x^{n+3}.$$

This equation determines S.

(2) *Find the sum to n terms of the recurring series*

$$1 + 2x + 3x^2 + 9x^3 + \ldots$$

for which the scale of relation is

$$u_n = -u_{n-1} + 10u_{n-2} - 8u_{n-3}. \qquad \text{[Camb. Sch.]}$$

The generating function has the form

$$(a_0 + a_1x + a_2x^2)/(1 + x - 10x^2 + 8x^3)$$

Proceeding as in § 9·6 we find, $a_0 = 1$, $a_1 = 3$, $a_2 = -5$.

Again expressing the generating function as partial fractions

$$\frac{1 + 3x - 5x^2}{1 + x - 10x^2 + 8x^3} = \frac{A}{1 - x} + \frac{B}{1 - 2x} + \frac{C}{1 + 4x},$$

where $A = \frac{1}{5}$, $B = \frac{5}{6}$, $C = -\frac{1}{30}$. Expanding each of the partial fractions separately and collecting coefficients of like powers of x,

$$\frac{1 + 3x - 5x^2}{1 + x - 10x^2 + 8x^3} = \sum_{r=0} \{A + B.2^r + (-1)^r C.4^r\}\, x^r, \quad |x| < \tfrac{1}{4}.$$

Hence the $(r + 1)$th term of the given series is

$$\{A + B.2^r + (-1)^r C.4^r\}\, x^r.$$

The sum to n terms is

$$A \sum_{r=0}^{n-1} x^r + B \sum_{r=0}^{n-1} (2x)^r + C \sum_{r=0}^{n-1} (-4x)^r$$

$$= A\,\frac{1 - x^n}{1 - x} + B\,\frac{1 - 2^n x^n}{1 - 2x} + C\,\frac{1 - (-4x)^n}{1 + 4x}$$

$$= \frac{1}{5}\,\frac{1 - x^n}{1 - x} + \frac{5}{6}\,\frac{1 - 2^n x^n}{1 - 2x} - \frac{1}{30}\,\frac{1 - (-1)^n 4^n x^n}{1 + 4x}.$$

9·8. Coefficients which are Polynomials in n

If u_n is a polynomial of degree p in n then series $\Sigma u_n x^n$ is a recurring series whose scale of relation is $(1 - x)^{p+1}$. The assumption is that

$$u_n = \lambda_0 + \lambda_1 n + \lambda_2 n^2 + \ldots + \lambda_p n^p,$$

where $\lambda_0, \lambda_1, \ldots, \lambda_p$ are constants independent of n.

Write $S_0 = u_0 + u_1x + u_2x^2 + u_3x^3 + \ldots$ Then

$$(1 - x)\, S_0 = u_0 + (u_1 - u_0)\, x + (u_2 - u_1)\, x^2 + (u_3 - u_2)\, x^3 + \ldots$$
$$= u_0 + v_0x + v_1x^2 + v_2x^3 + \ldots$$

where $v_n = u_{n+1} - u_n$. Now if $n \geqslant 0$,

$$v_n = \{\lambda_0 + \lambda_1(n + 1) + \lambda_2(n + 1)^2 + \ldots + \lambda_p(n + 1)^p\}$$
$$- \{\lambda_0 + \lambda_1 n + \lambda_2 n^2 + \ldots + \lambda_p n^p\}$$
$$= \mu_0 + \mu_1 n + \mu_2 n^2 + \ldots + \mu_{p-1} n^{p-1},$$

where $\mu_0, \mu_1, \mu_2, \ldots \mu_{p-1}$ are constants independent of n. Thus v_n is a polynomial in n of degree $p - 1$.

Write $xS_1 = (1 - x)\, S_0 - u_0$. Then $S_1 = v_0 + v_1x + v_2x^2 + \ldots$

Similarly by considering $(1 - x)\, S_1$ and writing

$$x\, S_2 = (1 - x)\, S_1 - v_0$$

it follows that $S_2 = w_0 + w_1x + w_2x^2 + \ldots$ where w_n is polynomial in n of degree $p - 2$.

Proceeding in this way it follows that

$$S_{p-1} = \rho_0 + \rho_1 x + \rho_2 x^2 + \ldots + \rho_n x^n + \ldots$$

where ρ_n is of degree unity in n, i.e.

$$\rho_n = \nu_0 + \nu_1 n,$$

where ν_0 and ν_1 are constants independent of n.

$$\begin{aligned}(1 - x) S_{p-1} &= \rho_0 + (\rho_1 - \rho_0) x + (\rho_2 - \rho_1) x^2 + \\ &\qquad + (\rho_n - \rho_{n-1}) x^n + \ldots \\ &= \rho_0 + \nu_1 (x + x^2 + \ldots + x^n + \ldots) \\ &= \nu_0 + \nu_1 x/(1 - x).\end{aligned}$$

Hence $S_{p-1} = \{v_0 + (v_1 - v_0) x\}/(1 - x)^2$. This is a proper rational fraction whose denominator is $(1 - x)^2$.

From the definition of S_{p-1}, i.e. $xS_{p-1} = (1 - x) S_{p-2} - k$, where k is constant it follows that S_{p-2} is a proper rational fraction whose denominator is $(1 - x)^3$.

Retracing the steps by which the sums S were defined it follows that S_0 is a proper rational fraction whose denominator is $(1 - x)^{p+1}$.

It follows that the given series $\Sigma u_n x^n$ is a recurring series whose scale of relation is $(1 - x)^{p+1}$.

It should be observed that this result provides us with a direct method of summing a certain kind of power series.

Examples.—(1) *Prove that the sum of the infinite series whose nth term is* $(-1)^n n^2 x^n$ *is* $-\dfrac{x(1 - x)}{(1 + x)^3}$ *where x is positive and less than* 1.

[*Lond. B.Sc.*]

The series may be written in the form $\Sigma u_n(-x)^n$ where $u_n = n^2$. It follows that the series is a recurring one whose generating function is

$$\{1 - (-x)\}^3, \text{ i.e. } (1 + x)^3.$$

Let S denote the sum of the series. Then

$$\begin{aligned}S &= -1^2x + 2^2x^2 - 3^2x^3 + 4^2x^4 - \ldots + (-1)^n n^2 x^n + \ldots \\ (1 + x) S &= -x + 3x^2 - 5x^3 + 7x^4 - \ldots + (-1)^n(2n - 1) x^n + \ldots \\ (1 + x)^2 S &= -x + 2x^2 - 2x^3 + 2x^4 - \ldots + (-1)^n 2x^n + \ldots \\ &= -x + 2x^2(1 - x + x^2 - x^3 + \ldots) \\ &= -x + 2x^2/(1 + x).\end{aligned}$$

Hence $S = \{-x(1 + x) + 2x^2\}/(1 + x)^3 = -x(1 - x)/(1 + x)^3$.

(2) *Sum to n terms the series* $1^3 + 2^3.x + 3^3.x^2 + \ldots$ [*Camb. Sch.*]

The question requires the sum to n terms of the recurring series $\Sigma u_r x^r$ where $u_r = (r + 1)^3$. The generating function is $(1 - x)^4$.

Write $S = 1^3 + 2^3x + 3^3x^2 + 4^3x^3 + 5^3x^4 + \ldots + n^3x^{n-1}$

$$(1 - x)\,S = 1 + 7x + 19x^2 + 37x^3 + 61x^4 + \ldots + (3n^2 - 3n + 1)\,x^{n-1} - n^3x^n$$

$$(1 - x)^2\,S = 1 + 6x + 12x^2 + 18x^3 + 24x^4 + \ldots + (6n - 6)\,x^{n-1} - (n^3 + 3n^2 - 3n + 1)\,x^n + n^3x^{n+1}$$

$$\begin{aligned}(1 - x)^3\,S &= 1 + 5x + 6x^2 + 6x^3 + 6x^4 + \ldots + 6x^{n-1} - (n^3 + 3n^2 + 3n - 5)\,x^n \\ &\quad + (2n^3 + 3n^2 - 3n + 1)\,x^{n+1} - n^3x^{n+2} \\ &= 1 + 5x + \frac{6x^2\,(1 - x^{n-2})}{1 - x} - (n^3 + 3n^2 + 3n - 5)\,x^n \\ &\quad + (2n^3 + 3n^2 - 3n + 1)\,x^{n+1} - n^3x^{+2}.\end{aligned}$$

$$\therefore\; S = \{1 + 4x + x^2 - (n + 1)^3x^n + (3n^3 + 6n^2 - 4)\,x^{n+1} - (3n^3 + 3n^2 - 3n + 1)\,x^{n+2} + n^3x^{n+3}\}/(1 + x)^4.$$

9·9. Linear Finite Difference Equations

The scale of relation

$$u_n + p_1u_{n-1} + p_2u_{n-2} + \ldots + p_{m+1}u_{n-m+1} + p_mu_{n-m} = 0$$

may be written in the form

$$u_{n+m} + p_1u_{n+m-1} + p_2u_{n+m-2} + \ldots + p_{m-1}u_{n+1} + p_mu_n = 0.$$

In either form $p_1, p_2, \ldots, p_{m-1}, p_m$ represent m constants.

We may regard this as an equation determining u_n and speak about u_n as the solution of the equation which may be referred to as a **finite difference equation.** Further, since the coefficients $p_1, p_2, \ldots, p_m$ are constants and $u_n, u_{n+1}, \ldots$ only occur *linearly* it may be called more precisely, a *linear finite difference equation with constant coefficients.*

The *general* solution u_n will involve m arbitrary constants, for clearly u_0, u_1, u_{m-1} are arbitrary. A method of finding u_n by applying the method of partial fractions to the generating function has already been considered in § 9·6.

It is convenient at this stage to introduce the characteristic equation

$$\rho^m + p_1\rho^n \quad p_2\rho^{m2} + \ldots + p_{m-1}\rho + p_m = 0.$$

The relation between the characteristic equation and the second form of the scale of relation is easily seen. With the help of the characteristic equation we can write down the general solution of the given finite difference equation. Methods of solving such equations will be considered in Chapter X.

We give here some results on the form of u_n.

CASE I. ($m = 2$).—*Let α and β be the roots of the characteristic equation*

$$\rho^2 + p_1\rho + p_2 = 0.$$

Then the general solution of the difference equation

$$u_{n+2} + p_1 u_{n+1} + u_n = 0 \textit{ is}$$

$$u_n = c_1\alpha^n + c_2\beta^n, \quad \alpha \neq \beta;$$

$$u_n = (c_1 + nc_2)\,\alpha^n, \quad \alpha = \beta,$$

where c_1 and c_2 are arbitrary constants.

If the roots are complex the solution can be written in the form

$$u_n = c_1 r^n \cos n\phi + c_2 r^n \sin n\phi$$

where $r = |\alpha|$, $\phi = amp.\ \alpha$.

Let u_0, u_1 be arbitrary. Then

$$u_0 + u_1x + u_2x^2 + \ldots + u_nx^n + \ldots = \frac{a_0 + a_1x}{1 + p_1x + p_2x^2}.$$

In this equation a_0, a_1 are functions of u_0, u_1 and are completely determined when u_0 and u_1 are given. Since α and β are the roots of $\rho^2 + p_1\rho + p_2 = 0$, the equation whose roots are $1/\alpha$, $1/\beta$ is $1 + p_1x + p_2x^2 = 0$. Hence

$$1 + p_1x + p_2x^2 = (1 - \alpha x)(1 - \beta x).$$

Then if $\alpha \neq \beta$, $$\frac{a_0 + a_1x}{1 + p_1x + p_2x^2} = \frac{c_1}{1 - \alpha x} + \frac{c_2}{1 - \beta x},$$

where c_1, c_2 can be regarded as arbitrary constants, since a_0, a_1 are arbitrary. Thus

$$\Sigma u_nx^n = \frac{c_1}{1 - \alpha x} + \frac{c_2}{1 - \beta x}$$

$$= c_1 \sum_{n=0}^{\infty} \alpha^nx^n + c_2 \sum_{n=0}^{\infty} \beta^nx^n, \quad |\alpha x| < 1, \ |\beta x| < 1,$$

$$= \sum_{n=0}^{\infty} (c_1\alpha^n + c_2\beta^n)\,x^n.$$

Equating corresponding coefficients,

$$u_n = c_1\alpha^n + c_2\beta^n.$$

If α and β are complex, α and β will be conjugate numbers and we can write $\alpha = re^{\iota\phi}$, $\beta = re^{-\iota\phi}$, where $r = |\alpha|$, $\phi = \text{amp. } \alpha$.

In practice it is convenient to take the principal value of ϕ. Then

$$\begin{aligned} u_n &= c_1 r^n e^{n\iota\phi} + c_2 r^n e^{-n\iota\phi} \\ &= (c_1 + c_2)\, r^n \cos n\phi + i\,(c_1 - c_2)\, r^n \sin n\phi \\ &= C_1 r^n \cos n\phi + C_2 r^n \sin n\phi \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Next suppose $\alpha = \beta$. Then

$$\begin{aligned} \Sigma u_n x^n &= \frac{C_1}{1 - ax} + \frac{C_2}{(1 - ax)^2} \\ &= C_1 \sum_{n=0}^{\infty} a^n x^n + C_2 \sum_{n=0}^{\infty} (n \quad 1)\, a^n x^n, \quad |ax| < 1, \\ &= \sum_{n=0}^{\infty} \{C_1 + (n + 1)\, C_2\}\, a^n x^n. \end{aligned}$$

Hence, $u_n = (C_1 + C_2 + nC_2)\, \alpha^n$ and writing $C_1 + C_2 = c_1$, $C_2 = c_2$, we have the required form.

CASE II. $(m > 2)$. The argument used in Case I can be generalised so as to apply to cases in which $m > 2$. We state results for $m = 3$.

Let α, β, γ be the three roots of the characteristic equation $\rho^3 + p_1\rho^2 + p_2\rho + p_3 = 0$. Then the general solution of the equation

$$u_{n+3} + p_1 u_{n+2} + p_2 u_{n+1} + p_3 u_n = 0$$

takes one if the forms

$u_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$, $\alpha \neq \beta \neq \gamma$, *all roots real*;

$u_n = c_1 r^n \cos n\phi + c_2 r^n \sin n\phi + c_3\gamma^n$, γ real, α and β *complex*, $r = |\alpha|$, $\phi =$ *amp.* α;

$u_n = c_1\alpha^n + (c_2 + nc_3)\,\beta^n$, $\alpha \neq \beta$, $\beta = \gamma$;

$u_n = (c_1 + nc_2 + n^2c_3)\,\alpha^n$, $\alpha = \beta = \gamma$.

Examples.—(1) *Find the nth term of the recurring series whose scale of relation is $u_{n+2} - u_{n+1} - u_n = 0$, where $u_0 = 0$, $u_1 = 1$.* [See § 9.6, Ex. 1.]

The roots of the characteristic equation $\rho^2 - \rho - 1 = 0$ are $\frac{1}{2}(1 \pm \sqrt{5})$. Hence the general solution of the difference equation is

$$u_n = c_1 \{\tfrac{1}{2}(1 + \sqrt{5})\}^n + c_2 \{\tfrac{1}{2}(1 - \sqrt{5})\}^n.$$

Substituting for $n = 0$, 1 we find $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$. Hence

$$u_n = \{\tfrac{1}{2}(1 + \sqrt{5})\}^n/\sqrt{5} - \{\tfrac{1}{2}(1 - \sqrt{5})\}^n/\sqrt{5}.$$

(2) *Find the value of u_n which satisfies the relation*

$$u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 0,$$

where $u_0 = 1,\ u_1 = 0,\ u_2 = -5$. [See § 9·6, Ex. 3.]

The characteristic equation is

$$\rho^3 - 4\rho^2 + 5\rho - 2 = 0 = (\rho - 1)^2 (\rho - 2)$$

The general solution is

$$u_n = (c_1 + nc_2)\, 1^n + c_3 2^n = c_1 + nc_2 + c_3 2^n$$

Substituting for $n = 0, 1, 2$, and solving for c_1, c_2, c_3 we find

$$c_1 = 5,\ c_2 = 3,\ c_3 = -4.$$

Hence $$u_n = 5 + 3n - 2^{n+2}.$$

(3) *A sequence of numbers u_n satisfies the relation $u_n = u_{n-1} - u_{n-2}$. If $u_0 = 1,\ u_1 = (1 + \sqrt{3})/2$, find the value of u_n.*

The roots of the characteristic equation $\rho^2 - \rho + 1 = 0$ are

$$\tfrac{1}{2}(1 \pm \sqrt{-3}) = \tfrac{1}{2}(1 \pm i\sqrt{3}) = \cos \tfrac{1}{3}\pi \pm i \sin \tfrac{1}{3}\pi$$

$$u_n = c_1 (\cos \tfrac{1}{3}\pi + i \sin \tfrac{1}{3}\pi)^n + c_2 (\cos \tfrac{1}{3}\pi - i \sin \tfrac{1}{3}\pi)^n$$

$$= C_1 \cos \frac{n\pi}{3} + C_2 \sin \frac{n\pi}{3},$$

where $C_1 = c_1 + c_2,\ C_2 = i(c_1 - c_2)$ are arbitrary constants.

The result could have been written down from the general formula quoted above.

Substitution for $n = 0, 1$ gives $C_1 = C_2 = 1$.

Hence $u_n = \cos \dfrac{n\pi}{3} + \sin \dfrac{n\pi}{3}$.

EXERCISES IX

27. The series $1 + 3x + 7x^2 + \ldots + p_n x^n + \ldots$ is such that

$$p_{n+1} = 3p_n - 2p_{n-1};$$

find the value of p_n. [*Camb. Sch.*]

28. Find the generating function and the general term of the series

$$2 + 3x + 5x^2 + 9x^3 + \ldots$$

29. If u_n satisfies the difference equation

$$u_{n+3} - 2u_{n+2} + 4u_{n+1} - 3u_n = 0,$$

and $u_0 = 1,\ u_1 = 2,\ u_2 = 3$, prove that the sum of the series $\Sigma u_n x^n$ is

$$(1 + 3x^2)/(1 - 2x + 4x^2 - 3x^3).$$

30. Find the sum to n terms of the series

$$2 + 7x + 25x^2 + 91x^3 + \ldots,$$

on the assumption that the series is a recurring one.

31. If $u_0, u_1, u_2, \ldots$ are numbers connected by the recurrence formula

$$u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 0,$$

find an expression for u_n, given $u_0 = 0,\ u_1 = 2,\ u_2 = 5$. [*Camb. Sch.*]

32. Prove that $1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \ldots$ is a recurring series whose scale of relation is $1 - 2x\cos\theta + x^2$. Find the sum to n terms and to infinity.

33. Find the generating function and the sum to n terms of the recurring series $\sum_{n=1}^{\infty} x^n \sin n\theta$.

34. In the series $u_0 + u_1x + u_2x^2 + \ldots$ any three successive coefficients are connected by the relation

$$u_{r+1} + pu_r + qu_{r-1} = 0;$$

show how to find the sum to n terms. Assuming that the series

$$2 + \tfrac{7}{5} + 1 + \tfrac{91}{125} + \ldots$$

is of this type, find the nth term and the sum to infinity. [*Camb. Sch.*]

35. Find the sum of the series $a_0 + a_1x + a_2x^1 + \ldots$ whose coefficients satisfy the relation

$$3a_n - 7a_{n-1} + 5a_{n-2} - a_{n-3} = 0 \text{ and } a_0 = 1,\ a_1 = 8,\ a_2 = 17,$$

proving that $2a_n = 20n - 7 + 3^{2-n}$. [*Camb. Sch.*]

36. Given that $\Sigma u_n x^n$, $\Sigma v_n x^n$ are recurring series whose scales of relation are $1 + px + qx^2 = 0$, $1 + rx + sx^2 = 0$ respectively, prove that $\Sigma(u_n + v_n)x^n$ is also a recurring series and find its scale of relation.

37. If $\Sigma u_n x^n$, $\Sigma v_n x^n$ are recurring series whose scales of relation are $1 + px + qx^2 = 0$ and $1 + rx + sx^2 = 0$, respectively and $p^2 \neq 4q$, $r^2 \neq 4s$, prove that $\Sigma u_n v_n x^n$ is a recurring series whose scale of relation is

$$1 - prx + (p^2s + rq^2 - 2qs)x^2 - pqrsx^3 + q^2s^2x^4 = 0.$$

38. If $\sum_{n=0}^{\infty} u_n x^n$ is a recurring series of the mth order, and $v_n = \sum_{r=0}^{n} u_r$, prove that $\sum_{n=0}^{\infty} v_n x^n$ is a recurring series of order $m + 1$.

39. Find the general solution of the difference equation

$$u_{n+4} - u_{n+3} - 8u_{n+1} - 8u_n = 0.$$

Hence, find u_n for all positive integral values of n, if $u_0 = 2$, $u_1 = 9$, $u_2 = 5$, $u_3 = 9$.

CHAPTER X

FINITE DIFFERENCES

IN previous chapters we have been concerned mainly with functions of a *continuous variable,* i.e. the variable could take any value inside a given interval or domain. Thus given any sub-interval of the given interval there exists values of the variable inside the sub-interval, however small it may be. In this chapter we consider functions of a *discontinuous* variable, *i.e.* the function is given for a certain specified set of values in the interval. Some of the elementary theorems of finite differences will be developed and certain operators used in the calculus of finite differences will be introduced.

10·1. Divided Differences

Let $y = f(x)$ be a function of x defined for $x = x_0, x_1, x_2, \ldots, x_n$.

Define $[x_0 x_1]$ by the equation

$$[x_0 x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

Then $[x_0 x_1]$ is called the divided difference of $f(x)$ for the values x_0, x_1. Clearly the order of x_0, x_1 is immaterial since

$$[x_0 x_1] = \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_1 - y_0}{x_1 - x_0} = [x_1 x_0].$$

If x_0, x_1, x_2 denote three values of the argument then define the divided difference of $f(x)$ with respect to x_0, x_1, x_2 by the equation.

$$[x_0 x_1 x_2] = \frac{[x_0 x_1] - [x_1 x_2]}{- x_0 - x_2}.$$

Substitution for $[x_0 x_1]$, $[x_1 x_2]$ gives

$$[x_0 x_1 x_2] = \frac{x_0 y_1 - x_1 y_0 + x_1 y_2 - x_2 y_1 + x_2 y_0 - x_0 y_2}{(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)}$$

$$= \frac{y_0 (x_2 - x_1) + y_1 (x_0 - x_2) + y_2 (x_1 - x_0)}{(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)}$$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}.$$

This divided difference is thus a symmetric function of x_0, x_1, x_2 and hence we can interchange any two values of the argument without affecting the value of the divided difference, in other words $[x_0x_1x_2]$ is independent of the order of x_0, x_1, x_2.

In general, the divided difference of $n + 1$ arguments is defined by the equation

$$[x_0x_1x_2\ldots x_n] = \frac{[x_0x_1x_2\ldots x_{n-1}] - [x_1x_2x_3\ldots x_n]}{x_0 - x_n}.$$

Expressed as a symmetric function of x_0, x_1, x_2, $\ldots$, x_n this can be written as

$$\frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots + \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

The cases $n = 1$, $n = 2$ have already been proved above, and the general case is easily proved by induction. Hence the divided difference for each value of n is such that any two values x_r, x_s can be interchanged without altering the value of the divided difference. The results may be expressed in the form of a difference table as follows:—

$$\begin{array}{cccccc}
x_0 & y_0 & & & & \\
 & & [x_0x_1] & & & \\
x_1 & y_1 & & [x_0x_1x_2] & & \\
 & & [x_1x_2] & & [x_0x_1x_2x_3] & \\
x_2 & y_2 & & [x_1x_2x_3] & & \cdot \\
 & & [x_2x_3] & & [x_1x_2x_3x_4] & \cdot \\
x_3 & y_3 & & [x_2x_3x_4] & & \cdot \\
 & & [x_3x_4] & & \cdot & \cdot \\
x_4 & y_4 & & \cdot & & \cdot \\
 & & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
 & & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
 & & \cdot & & [x_{n-3}x_{n-2}x_{n-1}x_n] & \\
\cdot & \cdot & & [x_{n-2}x_{n-1}x_n] & & \\
 & & [x_{n-1}x_n] & & & \\
x_n & y_n & & & &
\end{array}$$

10·11. Interpolation Formulae in Terms of Divided Differences

I. We can express $f(x_0)$ in terms of $f(x_1)$ and divided differences in the following way. By definition

$$[x_0x_1x_2\ldots x_n] = -\frac{[x_1x_2\ldots x_n]}{x_0 - x_n} + \frac{[x_0x_1\ldots x_{n-1}]}{x_0 - x_n}$$

$$[x_0x_1\ldots x_{n-1}] = -\frac{[x_1x_2\ldots x_{n-1}]}{x_0 - x_{n-1}} + \frac{[x_0x_1\ldots x_{n-2}]}{x_0 - x_{n-1}}$$

$$[x_0x_1\ldots x_{n-2}] = -\frac{[x_1x_2\ldots x_{n-2}]}{x_0 - x_{n-2}} + \frac{[x_0x_1\ldots x_{n-3}]}{x_0 - x_{n-2}}$$

$$\begin{matrix} \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \end{matrix}$$

$$[x_0x_1x_2x_3] = -\frac{[x_1x_2x_3]}{x_0 - x_3} + \frac{[x_0x_1x_2]}{x_0 - x_3}$$

$$[x_0x_1x_2] = -\frac{[x_1x_2]}{x_0 - x_2} + \frac{[x_0x_1]}{x_0 - x_2}$$

$$[x_0x_1] = -\frac{f(x_1)}{x_0 - x_1} + \frac{f(x_0)}{x_0 - x_1}.$$

Repeated substitution for the right-hand member gives

$$[x_0x_1x_2] = -\frac{[x_1x_2]}{x_0 - x_2} - \frac{f(x_1)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$[x_0x_1x_2x_3] = -\frac{[x_1x_2x_3]}{x_0 - x_3} - \frac{[x_1x_2]}{(x_0 - x_2)(x_0 - x_3)} - \frac{f(x_1)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$\begin{matrix} \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \end{matrix}$$

$$[x_0x_1x_2\ldots x_n] = -\frac{[x_1x_2\ldots x_n]}{x_0 - x_n} - \frac{[x_1x_2\ldots x_{n-1}]}{(x_0 - x_{n-1})(x_0 - x_n)} - \ldots$$

$$- \frac{[x_1x_2]}{(x_0 - x_2)\ldots(x_0 - x_n)} - \frac{f(x_1)}{(x_0 - x_1)\ldots(x_0 - x_n)}$$

$$+ \frac{f(x_0)}{(x_0 - x_1)\ldots(x_0 - x_n)}.$$

Hence $f(x_0) = f(x_1)$

$$+ \sum_{r=1}^{n-1} (x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_r)[x_1x_2\ldots x_{r+1}]$$

$$+ (x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)[x_0x_1\ldots x_n].$$

Changing the notation by writing x instead of x_0, we can say that if x is any given value of the argument

$$f(x) = f(x_1) + \sum_{r=1}^{n-1} (x - x_1)(x - x_2)\ldots(x - x_r)[x_1 x_2 \ldots x_{r+1}] + R_n(x)$$

where $\quad R_n(x) = (x - x_1)(x - x_2)\ldots(x - x_n)[x x_1 \ldots x_n].$

Thus $f(x)$ has been expressed as a polynomial in x with a remainder term $R_n(x)$. The polynomial depends on the values of the function at a finite number of points only. The result which is an identity, is known as *Newton's general interpolation formula.* In its present form it depends on a series of divided differences with reference to arbitrary arguments of the variable x. No restriction has been placed on these values, except that they must lie in the range of values for which $f(x)$ is defined. If remainder term $R_n(x)$ is so small that it can be neglected, then $f(x)$ can be calculated from Newton's formula.

II. Using the symmetrical formula for $[x_0 x_1 x_2 \ldots x_n]$ and replacing x_0 by x we have

$$[x x_1 x_2 \ldots x_n] = \frac{f(x)}{(x - x_1)(x - x_2)\ldots(x - x_n)} +$$

$$\frac{f(x_1)}{(x_1 - x)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots + \frac{f(x_n)}{(x_n - x)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

Hence

$$f(x) = f(x_1)\frac{(x - x_2)(x - x_3)\ldots(x - x_n)}{(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)} +$$

$$f(x_2)\frac{(x - x_1)(x - x_3)\ldots(x - x_n)}{(x_2 - x_1)(x_2 - x_3)\ldots(x_2 - x_n)} + \ldots +$$

$$f(x_n)\frac{(x - x_1)(x - x_2)\ldots(x - x_{n-1})}{(x_n - x_1)(x_n - x_2)\ldots(x_n - x_{n-1})} +$$

$$(x - x_1)(x - x_2)\ldots(x - x_n)[x x_1 x_2 \ldots x_n]$$

$$= \sum_{r=1} f(x_r)\frac{(x - x_1)(x - x_2)\ldots(x - x_{r-1})(x - x_{r+1})\ldots(x - x_n)}{(x_r - x_1)(x_r - x_2)\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots(x_r - x_n)} + R_n(x)$$

where $\quad R_n(x) = (x - x_1)(x - x_2)\ldots(x - x_n)[x x_1 x_2 \ldots x_n].$

The result may be expressed more succinctly as follows.

Write $\psi(x) = \prod_{r=1}^{n} (x - r_r)$. Then by logarithmic differentiation

$$\psi'(x) = \psi(x) \sum_{r=1}^{n} \frac{1}{x - x_r},$$

$$\psi'(x_r) = (x_r - x_1)(x - x_2)\ldots(x_r - x_{r-1})(x_r + x_{r+1})\ldots (x_r - x_n).$$

The formula then becomes

$$f(x) = \sum_{=1} \frac{f(x_r)\,\psi(x)}{(x - x_r)\,\psi'(x_r)} + \psi(x)\,[xx_1x_2\ldots x_n].$$

This is known as *Lagrange's interpolation formula*. The remainder term $R_n(x)$ is the same as that which occurs in Newton's formula.

One difference between the two interpolation formulae, which is important from the point of view of applications, should be noted. In the Lagrange form the terms of the approximating function do not depend on divided differences, whereas in Newton's formula each term depends explicitly on a divided difference.

If the points $x, x_1, x_2, \ldots, x_n$ can be arranged in the form of an arithmetical progression with constant difference ω, i.e. $x_r = x + r\omega$, $r = 1, 2, \ldots, n$ then $[xx_1x_2\ldots x_n]$ can be expressed in a simple form in terms of the values of the function at the given points. For substitution in the formula gives

$$[xx_1x_2\ldots x_n] = \frac{f(x_n)}{(x_n - x)(x_n - x_1)\ldots(x_n - x_{n-1})}$$

$$+ \frac{f(x_{n-1})}{(x_{n-1} - x)(x_{n-1} - x_1)\ldots(x_{n-1} - x_n)}$$

$$+ \frac{f(x_{n-2})}{(x_{n-2} - x)(x_{n-2} - x_1)\ldots(x_{n-2} - x_n)}$$

$$\ldots + \frac{f(x)}{(x - x_1)(x - x_2)\ldots(x - x_n)}$$

$$= \frac{f(x + n\omega)}{n\omega.(n-1)\,\omega\ldots\omega} + \frac{f(x + \overline{n - 1}\,\omega)}{(n-1)\,\omega.(n-2)\,\omega\ldots\omega\,(-\omega)}$$

$$+ \frac{f(x + \overline{n - 2}\,\omega)}{(n-2)\,\omega.(n-3)\,\omega\ldots(-\omega)(-2\omega)}$$

$$+ \dots + \frac{f(x)}{(-\omega)(-2\omega)\dots(-n\omega)}$$

$$= \frac{f(x+n\omega)}{\omega^n n!} - \frac{f(x+\overline{n-1}\omega)}{\omega^n(n-1)!1!} + \frac{f(x+\overline{n-2}\omega)}{\omega^n(n-2)!2!} - \dots$$

$$+ (-1)^n \frac{f(x)}{\omega^n n!}.$$

Hence $n!\ \omega^n [xx_1x_2\dots x_n]$

$$= f(x+n\omega) - \binom{n}{1} f(x+\overline{n-1}\omega) + \binom{n}{2} f(x+\overline{n-2}\omega) - \dots$$

$$+ (-1)^n f(x).$$

10·2. The Operator Δ

Let $y = f(x)$ be a function of x defined for a set of values in the interval (a, b) and let x, $x + h$ be two such values of the argument. Then the operator Δ as applied to $f(x)$ is defined by the equation

$$\Delta f(x) = f(x+h) - f(x).$$

$\Delta f(x)$ is called the first difference of $f(x)$ with respect to h. The symbol Δ is then an operator which has a meaning only when applied to a function of a given independent variable with reference to a particular increment h. Thus the symbol Δ is not complete and it would be more precise to represent it as

$$\underset{x,\,h}{\Delta}$$

In general, however, there is no ambiguity in the use of Δ since x and h are known and understood in particular cases. Where there is any doubt they must be stated.

Now let x_0, x_1, x_2, $\dots$, x_n be a monotonic increasing sequence of numbers such that $a \leqslant x_0 < x_n \leqslant b$, $h_r = x_{r+1} - x_r$ and $y_r = f(x_r)$ for $r = 0, 1, 2, \dots, n$. Then $\Delta f(x_r) = f(x_{r+1}) - f(x_r)$ $= y_{r+1} - y_r$.

The most important cases occur when h_r is constant, when we can write $x_r = x_0 + r\omega$, $r = 1, 2, \dots, n$, ω denoting the constant increment.

It follows from definition that if $z = \phi(x)$ is another function defined in (a, b) then

$$\Delta(y+z) = \Delta y + \Delta z.$$

In particular, if we take the set of values defined above,

$$\Delta(y_r + z_r) = \Delta y_r + \Delta z_r,\ r = 0, 1, 2, \dots, n.$$

Clearly the result extends directly to any finite number of different functions, in other words, the operator Δ obeys *the distributive law for addition.*

Observe in this connection that if s is a positive integer, $\sum\limits_{r=1}^{s} (\Delta y_r)$ is *not, in general, equal* to $\Delta\left(\sum\limits_{r=1}^{s} y_r\right)$. Here we are dealing with operations on the same function. Thus

$$\sum_{r=1}^{s} \Delta y_r = \sum_{r=1}^{s} (y_{r+1} - y_r) = y_{s+1} - y_1$$

$$\Delta\left(\sum_{r=1}^{s} y_r\right) = \sum_{r=1}^{s+1} y_r - \sum_{r=1}^{s} y_r = y_{s+1}.$$

The two expressions will be equal, if and only if, $y_1 = 0$.

Next consider how the index laws apply to Δ. Define $\Delta^m y$, where m is a positive integer, by the equation

$$\Delta^m y = \Delta\,(\Delta^{m-1} y).$$

Thus, *e.g.* using the increments defined above,

$$\Delta^2 y_r = \Delta\,(\Delta y_r) = \Delta\,(y_{r+1} - y_r) = y_{r+2} - 2y_{r+1} + y_r.$$

From the definition it follows that Δ obeys *the index law for multiplication*, i.e.

$$\Delta^m(\Delta^p y) = \Delta^{m+p} y = \Delta^p(\Delta^m y).$$

Further, Δ obeys *the commutative law with regard to constants.* For, if c denotes a constant,

$$\Delta\, cy = c\,\Delta\, y.$$

10·21. Table of Differences

Differences may be expressed in the form of a table as follows:—

Argument	*Function*	Δ	Δ^2	Δ^3	..
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		..
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		..
		Δy_3		$\Delta^3 y_2$	
x_4	y_4		$\Delta^2 y_3$		
		Δy_4			
x_5	y_5				

Each entry in a vertical column of differences is obtained from the two nearest entries in the adjacent left vertical column by subtracting the entry above from the one below.

As a numerical example take $f(x) = x^4$, $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$, $x_5 = 3$, $x_6 = 4$.

Argument	*Function*	Δ	Δ^2	Δ^3	Δ^4	Δ^5
-2	16					
		-15				
-1	1		14			
		-1		-12		
0	0		2		24	
		1		12		0
1	1		14		24	
		15		36		0
2	16		50		24	
		65		60		
3	81		110			
		175				
4	256					

Observe that for the function x^4, the fourth differences are constant.

10·22. Properties of the Operator Δ

In this section we assume that r denotes a positive integer or zero and that ω is a constant, different from zero.

I. It has already been shown that Δ obeys the following laws:—

(i) *the distributive law for addition,*

(ii) *the index law for positive integers,*

(iii) *the commutative law with regard to constants,*

i.e. $$\Delta\, cf(x) = c\,\Delta f(x).$$

The result (iii) is also true if c is replaced by a function $g(x)$ which is periodic and of period ω. Thus

$$\begin{aligned}\Delta\, g(x) f(x) &= g(x+\omega) f(x+\omega) - g(x) f(x)\\ &= g(x)\{f(x+\omega) - f(x)\}, \text{ since } g(x+\omega) = g(x),\\ &= g(x)\,\Delta f(x).\end{aligned}$$

II. *If $u(x)$ and $v(x)$ are functions of x defined for $x = x_r = x + r\omega$ and $v(x) = \Delta u(x)$ then*

$$v(x) + v(x+\omega) + \ldots + v(x + \overline{n-1}\,\omega) = u(x+n\omega) - u(x).$$

We have

$$v(x) + v(x+\omega) + \ldots + v(x + \overline{n-1}\,\omega)$$
$$= \Delta u(x) + \Delta u(x+\omega) + \ldots + \Delta u(x + \overline{n-1}\,\omega)$$
$$= \{u(x+\omega) - u(x)\} + \{u(x+2\omega) - u(x+\omega)\} + \ldots + \{u(x+n\omega) - u(x + \overline{n-1}\,\omega)\}$$
$$= u(x+n\omega) - u(x).$$

A particular case of this formula has been used in the summation of series. Thus if we write $x = 1$, $\omega = 1$, $v(r) = v_r$; $u(r) = u_r$ the result takes the form $\sum_{r=1}^{n} v_r = u_{n+1} - u_1$, where $v_r = u_{r+1} - u_r$.

III. *If m is a positive integer, then*

$$\Delta^m u(x) = u(x+m\omega) - \binom{m}{1} u(x + \overline{m-1}\,\omega) + \binom{m}{2} u(x + \overline{m-2}\,\omega) - \ldots + (-1)^m u(x).$$

The result may be proved by induction.

$$\Delta^2 u(x) = \Delta\{\Delta u(x)\} = u(x+2\omega) - 2u(x+\omega) + u(x)$$
$$= u(x+2\omega) - \binom{2}{1} u(x+\omega) + (-1)^2 u(x).$$

Hence the result is true for $m = 2$.

Assume it holds for $m = p$ where p is a positive integer, $p \geqslant 2$.

Then

$$\Delta^{p+1} u(x) = \Delta\{\Delta^p u(x)\}$$
$$= \Delta\left\{u(x+p\omega) - \binom{p}{1} u(x + \overline{p-1}\,\omega) + \binom{p}{2} u(x + \overline{p-2}\,\omega) - \ldots + (-1)^p u(x)\right\}$$
$$= \{u(x + \overline{p+1}\,\omega) - u(x+p\omega)\} - \binom{p}{1}\{u(x+p\omega) - u(x + \overline{p-1}\,\omega)\} + \binom{p}{2}\{u(x + \overline{p-1}\,\omega) - u(x + \overline{p-2}\,\omega)\} - \ldots + (-1)^p\{u(x+\omega) - u(x)\}.$$

Since $\binom{p}{r-1} + \binom{p}{r} = \binom{p+1}{r+1}$, r a positive integer, it follows that

$$\Delta^{p+1} u(x) = u(x + \overline{p+1}\omega) - \binom{p+1}{1} u(x + p\omega)$$
$$+ \binom{p+1}{2} u(x + \overline{p-1}\omega) - \ldots + (-1)^{p+1} u(x).$$

Thus if the result is true for $m = p$, it holds also for $m = p + 1$. The formula has been proved for $m = 2$ so that it is true in general.

Observe that the coefficients in the expansion of $\Delta^m u(x)$ are the same as those in the binomial expansion of $(1 - x)^m$.

If in the general result we write $x = n$, $u(n) = u_n$, $\omega = 1$, we have

$$\Delta^m u_n = u_{m+n} - \binom{m}{1} u_{m+n-1} + \binom{m}{2} u_{m+n-2} - \ldots + (-1)^m u_n$$

which is in a form directly applicable to series.

The general result can be used to express differences in terms of divided differences. Thus

$$\Delta^n f(x) = f(x + n\omega) - \binom{n}{1} f(x + \overline{n-1}\omega) +$$
$$\binom{n}{2} f(x + \overline{n-2}\omega) - \ldots + (-1)^n f(x)$$
$$= \omega^n n!\,[x x_1 x_2 \ldots x_n] \text{ from § 10·11.}$$

IV. *If $f(x)$ is a polynomial of degree p then $\Delta^p f(x)$ is constant and $\Delta^{p+1} f(x) = 0$.*

Write $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_p x^p$, where $a_p \neq 0$. Then

$$\Delta f(x) = \{a_0 + a_1(x + \omega) + a_2(x + \omega)^2 + \ldots + a_p(x + \omega)^p\}$$
$$- \{a_0 + a_1 x + a_2 x^2 + \ldots + a_p x^p\}$$
$$= b_0 + b_1 x + b_2 x^2 + \ldots + b_{p-1} x^{p-1},$$

where $b_0, b_1, b_2, \ldots, b_{p-1}$ are constants depending only on $a_0, a_1, a_2, \ldots, a_p$, and $b_{p-1} = \omega p a_p$. Since $\omega \neq 0$, $a_p \neq 0$, it follows that $b_{p-1} \neq 0$.

Thus $\Delta f(x)$ is a polynomial of degree $p - 1$.

Similarly, $\Delta^2 f(x)$ is a polynomial of degree $p - 2$. Proceeding in this way it follows that $\Delta^p f(x)$ is a constant and hence $\Delta^{p+1} f(x) = 0$.

10·23. Errors in Table of Differences

Let $y_0, y_1, y_2, \ldots, y_r, \ldots, y_n$ be the values of the function corresponding to the values $x_0, x_1, x_2, \ldots, x_r, \ldots, x_n$ of the argument x. Suppose that an entry, say y_r, is incorrectly inserted in the table as $y_r + \delta$ and consider the effect on successive columns of differences. Table A shows part of a table of differences and the terms between the diagonal dotted lines show those which will be affected by the introduction of δ. It will be seen that in general the number of terms affected in the column for $\Delta^s y$ will be $s + 1$ in number, these being $\Delta^s y_{r-s}, \Delta^s y_{r-s+1}, \ldots, \Delta^s y_{r-1}, \Delta^s y_r$, provided $s \leqslant r$. If $s > r$ the number of terms affected will be $r + 1$ and these will be $\Delta^s y_0, \Delta^s y_1, \ldots, \Delta^s y_r$.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
·		·		·	
	·		·		·
y_{r-5}		·		·	
	Δy_{r-5}		·		·
y_{r-4}		$\Delta^2 y_{r-5}$		·	
	Δy_{r-4}		$\Delta^3 y_{r-5}$		·
y_{r-3}		$\Delta^2 y_{r-4}$		$\Delta^4 y_{r-5}$	
	Δy_{r-3}		$\Delta^3 y_{r-4}$		$\Delta^5 y_{r-5}$
y_{r-2}		$\Delta^2 y_{r-3}$		$\Delta^4 y_{r-4}$	
	Δy_{r-2}		$\Delta^3 y_{r-3}$		$\Delta^5 y_{r-4}$
y_{r-1}		$\Delta^2 y_{r-2}$		$\Delta^4 y_{r-3}$	
	Δy_{r-1}		$\Delta^3 y_{r-2}$		$\Delta^5 y_{r-3}$
y_r		$\Delta^2 y_{r-1}$		$\Delta^4 y_{r-2}$	
	Δy_r		$\Delta^3 y_{r-1}$		$\Delta^5 y_{r-2}$
y_{r+1}		$\Delta^2 y_r$		$\Delta^4 y_{r-1}$	
	Δy_{r+1}		$\Delta^3 y_r$		$\Delta^5 y_{r-1}$
y_{r+2}		$\Delta^2 y_{r+1}$		$\Delta^4 y_r$	
	Δy_{r+2}		$\Delta^3 y_{r+1}$		$\Delta^5 y_r$
y_{r+3}		$\Delta^2 y_{r+2}$		$\Delta^4 y_{r+1}$	
	Δy_{r+3}		$\Delta^3 y_{r+2}$		·
y_{r+4}		$\Delta^2 y_{r+3}$		·	
	Δy_{r+4}		·		
y_{r+5}		·			
	·				
·					

TABLE A

The numerical effect of the introduction of δ on each of the terms is illustrated in Table B which records only the errors involved. Thus, *e.g.* Δy_{r-1} will exceed its true value by δ, whereas Δy_r will be less than its true value by δ.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
·		·		·	
	·		·		·
·		·		·	
	·		·		·
·		·		·	
	·		·		·
·		·		·	
	·		·		δ
·		·		δ	
	·		δ		-5δ
·		δ		-4δ	
	δ		-3δ		10δ
$(y_r)\ \delta$		-2δ		6δ	
	$-\delta$		3δ		-10δ
·		δ		-4δ	
	·		$-\delta$		5δ
·		·		δ	
	·		·		$-\delta$
·		·		·	
	·		·		·
·		·		·	
	·		·		
·		·			
	·				
·					

TABLE B

Assuming $s < r$ we can write down the magnitudes of the errors in the $s + 1$ terms of $\Delta^s y$. Corresponding to $\Delta^s y_{s-r}$, $\Delta^s y_{s-r+1}$, $\Delta^s y_{s-r+2}$, $\ldots$, $\Delta^s y_r$ the respective errors will be δ, $-\binom{s}{1}\delta$, $\binom{s}{2}\delta$, $\ldots$, $(-1)^s\delta$. In these terms the coefficients of δ are the binomial coefficients in the expansion of $(1-\delta)^s$. Obvious modifications will be necessary if $s > r$. In this case the coefficients involved will be those of the last $r + 1$ terms of the binomial expansion.

Example.—*Show that if an entry in the following table is corrected the function tabulated is of the third degree in x.*

x	$f(x)$	x	$f(x)$	x	$f(x)$
1·4	5·704	1·8	10·072	2·2	16·488
1·5	6·625	1·9	11·469	2·3	18·457
1·6	7·656	2·0	13·031	2·4	20·584
1·7	8·803	2·1	14·671	2·5	22·875

First construct a table of differences.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1·4	5·704			
		0·921		
1·5	6·625		0·110	
		1·031		0·006
1·6	7·656		0·116	
		1·147		0·006
1·7	8·803		0·122	
		1·269		0·006
1·8	10·072		0·128	
		1·397		0·037
1·9	11·469		0·165	
		1·562		− 0·087
2·0	13·031		0·078	
		1·640		0·099
2·1	14·671		0·177	
		1·817		− 0·025
2·2	16·488		0·152	
		1·969		0·006
2·3	18·457		0·158	
		2·127		0·006
2·4	20·584		0·164	
		2·291		
2·5	22·875			

The function tabulated will be of the third degree in x provided $\Delta^3 f(x)$ is constant. Five of the values are equal to 0·006 and the remaining four are different. An incorrect entry of one value in the column for $f(x)$ will produce four incorrect values in $\Delta^3 f(x)$. From the position of the four values it follows that the incorrect entry is 13·031, corresponding to $x = 2·0$. Since $0·037 - 0·006 = 0·031$, this suggests that the error δ is 0·031 and that the correct value corresponding to $x = 2·0$ is 13·0. Since

$$-0·087 = 0·006 - 3 \times 0·031, \quad 0·099 = 0·006 + 3 \times 0·031,$$
$$-0·025 = 0·006 - 0·031$$

the result verifies.

Thus changing 13·031 into 13·0 has the effect of making $\Delta^3 f(x)$ constant and equal to 0·006. Hence $f(x)$ is of the third degree in x.

10·24. The Norlünd Operator $\underset{\omega}{\Delta}$

The operator $\underset{\omega}{\Delta} f(x)$ is defined by the equation

$$\underset{\omega}{\Delta} f(x) = \frac{f(x+\omega) - f(x)}{\omega}$$

and $\underset{\omega}{\Delta} f(x)$ is called the *first difference quotient* of $f(x)$.

Similarly the second difference quotient is defined by

$$\underset{\omega}{\overset{2}{\Delta}} f(x) = \underset{\omega}{\Delta}\{\underset{\omega}{\Delta} f(x)\} = \underset{\omega}{\Delta}\left\{\frac{f(x+\omega)-f(x)}{\omega}\right\}$$

$$= \frac{f(x+2\omega) - 2f(x+\omega) + f(x)}{\omega^2}.$$

In general $\underset{\omega}{\overset{m}{\Delta}} f(x) = \underset{\omega}{\Delta}\left\{\underset{\omega}{\overset{m-1}{\Delta}} f(x)\right\} = \dfrac{\underset{\omega}{\overset{m-1}{\Delta}} f(x+\omega) - \underset{\omega}{\overset{m-1}{\Delta}} f(x)}{\omega}.$

Changing m into $m + 1$, this equation can be written as

$$\underset{\omega}{\overset{m}{\Delta}} f(x+\omega) = \omega \underset{\omega}{\overset{m-1}{\Delta}} f(x) + \underset{\omega}{\overset{m}{\Delta}} f(x).$$

From the definition it follows that the operators Δ and $\underset{\omega}{\Delta}$ are related by the equation

$$\Delta^m f(x) = \omega^m \underset{\omega}{\overset{m}{\Delta}} f(x).$$

Using the symbol $\doteqdot$ to denote symbolical equivalence,

$$\Delta^m \doteqdot \omega^m \underset{\omega}{\overset{m}{\Delta}}.$$

If $\omega = 1$, the two operators become the same.

From definition it follows that $\underset{\omega}{\Delta}$ obeys the distributive law for addition, the index laws for positive integers, and the commutative law for constants and periodic functions of period ω as in the case of the operator Δ.

10·3. The Operator E

This operator is defined by the equations

$$Ef(x) = f(x+\omega),\quad E^m f(x) = E\{E^{m-1}f(x)\} = f(x + m\,\omega).$$

As in the case of the operator Δ, E depends on x and ω, and in applying it we must assume that these are known. Hence the symbol E should be regarded as an abbreviation of $\underset{x,\,\omega}{E}$.

From definition it follows that E obeys the same laws of combination as Δ and $\underset{\omega}{\Delta}$.

Thus if k is a constant, then

$$(E + k) f(x) = Ef(x) + kf(x) = f(x + \omega) + kf(x)$$
$$= kf(x) + f(x + \omega) = (k + E) f(x).$$
$$(kE) f(x) = E\{kf(x)\} = kf(x + \omega) = (Ek) f(x).$$

It is easy to extend the operation E for negative indices.

Thus $$\frac{1}{E} f(x) = E^{-1} f(x) = f(x - \omega)$$

$$\frac{1}{E^m} f(x) = E^{-m} f(x) = f(x - m\omega).$$

The operator E^{-m} behaves exactly like E^m. Thus

$$E^{-m} E^n f(x) = E^{-m}\{E^n f(x)\} = E^{-m} f(x + n\omega)$$
$$= f(x + n\omega - m\omega) = E^n E^{-m} f(x).$$

Thus in the case of E division or multiplication by negative power of E is permissible.

Further, E obeys the following law for multiplication, a property which is not shared by Δ. Thus

$$E\{f(x)\,\phi(x)\,\psi(x)\} = f(x + \omega)\,\phi(x + \omega)\,\psi(x + \omega)$$
$$= Ef(x) \,.\, E\phi(x) \,.\, E\psi(x).$$

10·31. Symbolic Methods

Since $$Ef(x) = f(x + \omega) = \{f(x + \omega) - f(x)\} + f(x)$$
$$= \Delta f(x) + f(x) = (\Delta + 1) f(x),$$

we can write $E \doteqdot \Delta + 1$ or $\Delta \doteqdot E - 1$.

If we *assume* that $f(x + \omega)$ can be expanded by Taylor's theorem, then

$$Ef(x) = f(x + \omega) = f(x) + \omega Df(x) + \frac{\omega^2}{2!} D^2 f(x) + \frac{\omega^3}{3!} D^3 f(x) + \dots$$

where $$D = \frac{d}{dx}.$$

The Taylor expansion may be written symbolically as

$$\left\{1 + \omega D + \frac{\omega^2}{2!} D^2 + \frac{\omega^3}{3!} D^3 + \dots\right\} f(x) = e^{\omega D} f(x).$$

Thus $$Ef(x) = e^{\omega D} f(x), \text{ or } E \doteqdot e^{\omega D}$$

Summarising the formulae for symbolic equivalence we have

$$E \doteqdot 1 + \Delta \doteqdot 1 + \omega \mathbin{\Delta\!\!\!\!\;\cdot} = e^{\omega D}$$

or

$$\Delta \doteqdot \omega \mathbin{\Delta} \doteqdot E - 1 \doteqdot e^{\omega D} - 1.$$

Symbolic calculus provides a powerful method of obtaining important formulae. As far as addition, subtraction, and multiplication are concerned the symbols E, Δ, $\underset{\omega}{\Delta}$, $e^{\omega D}$ behave as if they were algebraic quantities. A polynomial whose terms consist of these symbols, the coefficients in the polynomial being constants, represents an operation. If there is more than one polynomial they may be combined by multiplication, addition, and subtraction to form a single polynomial. The proof of the two following theorems are examples of symbolic calculation.

A. $\Delta^m f(x) = (E - 1)^m f(x)$

$$= \left\{E^m - \binom{m}{1} E^{m-1} + \binom{m}{2} E^{m-2} - \ldots + (-1)^m\right\} f(x)$$

$$= E^m f(x) - \binom{m}{1} E^{m-1} f(x) + \binom{m}{2} E^{m-2} f(x) - \ldots + (-1)^m f(x),$$

$$= f(x + m\omega) - \binom{m}{1} f(x + \overline{m-1}\,\omega) + \binom{m}{2} f(x + \overline{m-2}\,\omega) - \ldots + (-1)^m f(x).$$

B. $f(x + m) = E^m f(x) = (1 + \Delta)^m f(x)$

$$= \left\{1 + \binom{m}{1} \Delta + \binom{m}{2} \Delta^2 + \ldots + \binom{m}{m} \Delta^m\right\} f(x)$$

$$= 1 + \binom{m}{1} \Delta f(x) + \binom{m}{2} \Delta^2 f(x) + \ldots + \binom{m}{m} \Delta^m f(x).$$

We now consider how the symbolic methods may be used to obtain the formal expansion of a function as a series. We have shown in B that if $\omega = 1$ and x is *a positive integer*,

$$E^x \doteqdot \sum_{r=0}^{x+1} \binom{x}{r} \Delta^r.$$

But if x is not a positive integer we can still attach a meaning to E^x when applied to any function. Thus, e.g. $E^x f(0) = f(x)$.

Now the series $\Sigma \binom{x}{r} \Delta^r$ applied to any function $f(x)$ has an infinite number of terms if x is not a positive integer, unless $\Delta^r f(x) = 0$ for $r > n$, where n is a fixed positive integer. In the case of every polynomial there exists an integer n with this property and the operator E^x can be extended so as to apply to a polynomial, where now x is no longer restricted to a positive integer.

If $f(x)$ is not a polynomial then E^x applied to $f(0)$ gives the infinite series

$$E^x f(0) = (1 + \Delta)^x f(0),$$

i.e. $$f(x) = f(0) + \binom{x}{1} \Delta f(0) + \binom{x}{2} \Delta^2 f(0) + \ldots.$$

Clearly the expansion cannot be valid if the series is divergent; even if the series is convergent we cannot say without further consideration that the expansion is valid. The discussion of the conditions under which the expansion is true, is beyond the scope of the present book.

We can obtain *Newton's backward formula* by symbolic reasoning as follows:

Since $E - \Delta \doteqdot 1$,

$$E^x \doteqdot \left(\frac{E}{E - \Delta}\right)^x \doteqdot \left(1 - \frac{\Delta}{E}\right)^{-x}.$$

Expanding as an infinite series

$$E^x \doteqdot 1 + \binom{x}{1} \frac{\Delta}{E} + \binom{x+1}{2} \cdot \frac{\Delta^2}{E^2} + \ldots$$

$$\doteqdot \sum_{r=0}^{\infty} \binom{x+r-1}{r} \frac{\Delta^r}{E^r}.$$

Applying the operator to $f(0)$ we have

$$E^x f(0) = f(x) = f(0) + \binom{x}{1} \Delta f(-1) + \binom{x+1}{2} \Delta^2 f(-2) +$$

$$\binom{x+2}{3} \Delta^2 f(-3) + \ldots$$

If x is a positive integer or if $f(x)$ is a polynomial the formula will apply. It will only be valid in other cases if the series is convergent and $f(x)$ satisfies certain conditions.

If in the symbolic formula $E^x \doteq \sum_{r=0}^{\infty} \binom{x+r-1}{r} \frac{\Delta^r}{E^r}$ we write $x = 1$, we obtain

$$E \doteq \sum_{r=0}^{\infty} \frac{\Delta^r}{E^r}.$$

Applying the formula to $f(x)$ and taking the increment to be ω,

$$f(x+\omega) = f(x) + \Delta f(x-\omega) + \Delta^2 f(x-2\omega) + \Delta^3 f(x-3\omega) + \ldots$$

10·4. Factorial Expressions

A *descending factorial* expression of degree m is a product of m factors of the form

$$f(x)\, f(x-\omega)\, f(x-2\omega) \ldots f(x-\overline{m-1}\,\omega)$$

where m is a positive integer. Examples of such expressions are $x(x-1)(x-2)(x-3)$, $x^2(x-2)^2(x-4)^2$,

$(3x+1)(3x-8)(3x-17)$, $1/(2x+3)(2x+1)(2x-1)$.

Similarly, an *ascending factorial* expression is a product of the form $\quad f(x+\omega)\, f(x+2\omega)\, f(x+3\omega) \ldots f(x+m\omega)$.

Examples are

$$(x+2)(x+3)(x+4),\quad 1/(2x+5)(2x+9)(2x+13).$$

Clearly by reversing the order of the factors a descending factorial can be arranged as an ascending factorial and vice versa.

Two important factorial expressions are

$$x(x-\omega)(x-2\omega)\ldots(x-\overline{m-1}\,\omega)$$

$$(x+\omega)^{-1}(x+2\omega)^{-1}\ldots(x+m\omega)^{-1}$$

and these may be denoted by $(x)_{m,\,\omega}$ and $(x)_{-m,\,\omega}$ respectively.* They are obtained by putting $f(x) = x$ and $f(x) = 1/x$ in the general factorial expression.

If $\omega = 1$ we write

$$(x)_{m,1} = (x)_m = x(x-1)(x-2)\ldots(x-m+1)$$

$$(x)_{-m,1} = (x)_{-m} = 1/\{(x+1)(x+2)\ldots(x+m)\}.$$

* Other notations are in use for these factorials, and a summary of these is given in Jordan: *Calculus of Finite Differences*, Chap. II.

If m and n are positive integers it follows from the definition of $(x)_{m,\omega}$ that if $m > n$,

$$(x)_{m,\omega} = x(x-\omega)\ldots(x-\overline{n-1}\omega)(x-n\omega)\ldots(x-\overline{m-1}\omega)$$
$$= (x)_{n,\omega}(x-n\omega)_{m-n,\omega}.$$

This formula will be *assumed* valid for any value of m and n.

Then if we write $n = 0$,

$$(x)_{m,\omega} = (x)_{0,\omega}(x)_{m,\omega}$$

giving $$(x)_{0,\omega} = 1.$$

Next, if we put $m = 0$,

$$1 = (x)_{n,\omega}(x-n\omega)_{-n,\omega} = (x+n\omega)_{n,\omega}(x)_{-n,\omega}$$

or $$(x)_{-n,\omega} = 1/(x+n\omega)_{n,\omega}$$
$$= 1/\{(x+n\omega)(x+\overline{n-1}\omega)\ldots(x+\omega)\}$$
$$= 1/\{(x+\omega)(x+2\omega)\ldots(x+n\omega)\}.$$

Hence the assumption of the validity of

$$(x)_{m,\omega} = (x)_{n,\omega}(x-n\omega)_{m-n,\omega}$$

is consistent with the notation adopted initially for $(x)_{m,\omega}$ and $(x)_{-m,\omega}$.

We can obtain a more general form for $(x)_{m,\omega}$ by replacing x by $ax+b$. Thus

$$(ax+b)_{m,\omega} = (ax+b)(ax+b-a\omega)(ax+b-2a\omega)\ldots(ax+b-\overline{m-1}\omega)$$

$$(ax+b)_{-m,\omega} = (ax+b+a\omega)^{-1}(ax+b+2a\omega)^{-1}(ax+b+3a\omega)^{-1}\ldots(ax+b+ma\omega)^{-1}.$$

10·41. Difference Quotients for Factorial Expressions

(i) We have $\omega\,\Delta\,\{f(x)f(x-\omega)f(x-2\omega)\ldots f(x-\overline{m-1}\omega)$

$$= f(x+\omega)f(x)f(x-\omega)\ldots f(x+m-2\omega)$$
$$- f(x)f(x-\omega)\ldots f(x-\overline{m}-2\omega)f(x-m-1\omega)$$
$$= f(x)f(x-\omega)\ldots f(x-\overline{m-2}\omega)\{f(x+\omega)-f(x-\overline{m-1}\omega)\}.$$

We can consider in particular the forms corresponding to $f(x) = ax+b$.

The factorial expression becomes

$$(ax + b)\,(ax + b - a\omega)\,(ax + b - 2a\omega)\ldots(ax + b - \overline{am - 1}\,\omega) = (ax + b)_{m,\,\omega}.$$

Hence $\omega \underset{\omega}{\Delta}(ax + b)_{m,\,\omega} = (ax + b)_{m-1,\,\omega}\{ax + b + a\omega - (ax + b - am\omega + a\omega)\}$

$$\underset{\omega}{\Delta}(ax + b)_{m,\,\omega} = am\,(ax + b)_{m-1,\,\omega}$$

Proceeding in this way we find

$$\underset{\omega}{\overset{2}{\Delta}}(ax + b)_{m,\,\omega} = a^2m\,(m - 1)\,(ax + b)_{m-2,\,\omega},$$

$$\underset{\omega}{\overset{3}{\Delta}}(ax + b)_{m,\,\omega} = a^3m\,(m - 1)\,(m - 2)\,(ax + b)_{m-3,\,\omega},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\underset{\omega}{\overset{m-1}{\Delta}}(ax + b)_{m,\,\omega} = a^{m-1}m\,(m - 1)\ldots2\,(ax + b)_{1,\,\omega},$$

$$= a^{m-1}m!\,(ax + b),$$

$$\underset{\omega}{\overset{m}{\Delta}}(ax + b)_{m,\,\omega} = a^m m!,$$

$$\underset{\omega}{\overset{n}{\Delta}}(ax + b)_{m,\,\omega} = 0,\quad n > m.$$

In particular, if $a = 1$, $b = 0$, we have

$$\underset{\omega}{\overset{n}{\Delta}}\frac{(x)_{m,\,\omega}}{m!} = \frac{(x)_{m-n,\,\omega}}{(m - n)!},\quad n \leqslant m,$$

$$= 0,\quad n > m.$$

If $\omega = 1$, $\dfrac{(x)_m}{m!} = \dfrac{x\,(x - 1)\,(x - 2)\ldots(x - m + 1)}{m!} = \dbinom{x}{m}.$

Hence
$$\underset{\omega}{\overset{m}{\Delta}}\binom{x}{m} = \binom{x}{m - n}.$$

Example.—*Show that if $r > 1$ is a positive integer, then*

$$\sum_{x=1}^{n}(x)_r = \frac{(n + 1)_{r+1}}{r + 1}.$$

Since $\Delta\,(x)_{r+1} = (r + 1)\,(x)_r$, we can write

$$(r + 1)\,(x)_r = (x + 1)_{r+1} - (x)_{r+1}.$$

Writing down the corresponding expressions for $x = 1, 2, \ldots, n$, and adding,

$$(r+1)\sum_{x=1}^{n}(x)_r$$
$$= \{(2)_{r+1} - (1)_{r+1}\} + \{(3)_{r+1} - (2)_{r+1}\} + \ldots + \{(n+1)_{r+1} - (n)_{r+1}\}$$
$$= (n+1)_{r+1}, \text{ since } (1)_{r+1} = 0,\ r > 1,$$

which is the required result.

(ii) Next consider the factorial

$$1/\{f(x+\omega)\,f(x+2\omega)\ldots f(x+m\omega)\}.$$

Proceeding as before

$$\omega \mathop{\Delta}_{\omega} \frac{1}{f(x+\omega)\,f(x+2\omega)\ldots f(x+m\omega)}$$

$$= \frac{f(x+\omega) - f(x+\overline{m+1}\,\omega)}{f(x+\omega)\,f(x+2\omega)\ldots f(x+m\omega)\,f(x+\overline{m+1}\,\omega)}.$$

If $f(x) = (ax+b)$ the factorial becomes $(ax+b)_{-m,\,\omega}$ and the result gives

$$\Delta\,(ax+b)_{-m,\,\omega} = -\,ma\,(ax+b)_{-(m+1),\,\omega}$$

10·42. Expansion of a Polynomial in Factorials

Let $\phi(x)$ be a polynomial of degree m in x. Then we can represent $\phi(x)$ in the form

$$\phi(x) = a_0 + a_1\frac{(x)_{1,\,\omega}}{1!} + a_2\frac{(x)_{2,\,\omega}}{2!} + \ldots + a_m\frac{(x)_{m,\,\omega}}{m!}$$

for the right-hand side is a polynomial of degree m with $m+1$ arbitrary constants. Substituting $x = 0$ we see that $a_0 = \phi(0)$. Next, applying the operator $\mathop{\Delta}_{\omega}$ in succession we have

$$\mathop{\Delta}_{\omega}\phi(x) = a_1 + \frac{a_2}{1!}(x)_{1,\,\omega} + \frac{a_3}{2!}(x)_{2,\,\omega} + \ldots + \frac{a_m}{(m-1)!}(x)_{m-1,\,\omega}$$

$$\mathop{\Delta}_{\omega}^{2}\phi(x) = a_2 + \frac{a_3}{1!}(x)_{1,\,\omega} + \ldots + \frac{a_m}{(m-2)!}(x)_{m-2,\,\omega}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathop{\Delta}_{\omega}^{m}\phi(x) = a_m.$$

Writing $x = 0$ in each equation, we have $a_s = \overset{s}{\underset{\omega}{\Delta}} \phi(0)$, $s = 1, 2, \ldots, m$, and

$$\phi(x) = \phi(0) + \frac{(x)_{1,\omega}}{1!} \underset{\omega}{\Delta} \phi(0) + \frac{(x)_{2,\omega}}{2!} \overset{2}{\underset{\omega}{\Delta}} \phi(0) + \ldots$$
$$+ \frac{(x)_{m,\omega}}{m!} \overset{m}{\underset{\omega}{\Delta}} \phi(0).$$

The coefficients $\overset{s}{\underset{\omega}{\Delta}} \phi(0)$ can be obtained by writing down the values of $\phi(x)$ for $x = 0, \omega, 2\omega, \ldots$, and then calculating the difference quotients in succession. The expansion can also be written in the form

$$\phi(x) = \phi(0) + \frac{(x)_{1,\omega}}{1!\,\omega} \Delta \phi(0) + \frac{(x)_{2,\omega}}{2!\,\omega^2} \Delta^2 \phi(0) + \ldots$$
$$+ \frac{(x)_{m,\omega}}{m!\,\omega^m} \Delta^m \phi(0).$$

The function $\phi(x)$ can be expanded in terms of a more general factorial as follows. Write

$$\phi(x) = b_0 + b_1 \frac{(x-a)_{1,\omega}}{1!} + b_2 \frac{(x-a)_{2,\omega}}{2!} + \ldots + b_m \frac{(x-a)_{m,\omega}}{m!}.$$

Proceeding as before, but writing $x = a$ instead of $x = 0$, and observing that $\underset{\omega}{\Delta} (x-a)_{r,\omega} = r\,(x-a)_{r-1,\omega}$ we see that

$$b_r = \overset{r}{\underset{\omega}{\Delta}} \phi(a) = \frac{1}{\omega^r} \Delta^r \phi(a).$$

Hence

$$\phi(x) = \phi(a) + \frac{(x-a)_{1,\omega}}{1!} \underset{\omega}{\Delta} \phi(a) + \frac{(x-a)_{2,\omega}}{2!} \overset{2}{\underset{\omega}{\Delta}} \phi(a) + \ldots$$
$$+ \frac{(x-a)_{m,\omega}}{m!} \overset{m}{\underset{\omega}{\Delta}} \phi(a)$$
$$= \phi(a) + \frac{(x-a)_{1,\omega}}{1!\,\omega} \Delta \phi(a) + \frac{(x-a)_{2,\omega}}{2!\,\omega^2} \Delta^2 \phi(a) + \ldots$$
$$+ \frac{(x-a)_{m,\omega}}{m!\,\omega^m} \Delta^m \phi(a).$$

This is the general form of Newton's formula for a polynomial. It can be used to find $\phi(x)$ when $\phi(a)$ and the differences for $\phi(a)$ are known.

Examples.—(1) *Express* $2x^4 - 11x^3 + 22x^2 - 12x + 2$ *in the form* $a_0 + a_1x + a_2x(x-1) + a_3x(x-1)(x-2) + a_4x(x-1)(x-2)(x-3)$.

The form required is

$$\phi(x) = \phi(0) + \frac{(x)_{1,\,\omega}}{1!}\underset{\omega}{\Delta}\phi(0) + \frac{(x)_{2,\,\omega}}{2!}\overset{2}{\underset{\omega}{\Delta}}\phi(0) + \frac{(x)_{3,\,\omega}}{3!}\overset{3}{\underset{\omega}{\Delta}}\phi(0) + \frac{(x)_{4,\,\omega}}{4!}\overset{4}{\underset{\omega}{\Delta}}\phi(0),$$

where $\omega = 1$.

x	$\phi(x)$	$\underset{\omega}{\Delta}\phi(x)$	$\overset{2}{\underset{\omega}{\Delta}}\phi(x)$	$\overset{3}{\underset{\omega}{\Delta}}\phi(x)$	$\overset{4}{\underset{\omega}{\Delta}}\phi(x)$
0	2				
		1			
1	3		6		
		7		6	
2	10		12		48
		19		54	
3	29		66		
		85			
4	114				

$$\phi(x) = 2 + \frac{(x)_{1,\,\omega}}{1!}.1 + \frac{(x)_{2,\,\omega}}{2!}.6 + \frac{(x)_{3,\,\omega}}{3!}.6 + \frac{(x)_{4,\,\omega}}{4!}.48.$$

Hence $\phi(x) = 2 + x + 3x(x-1) + x(x-1)(x-2)$
$+ 2x(x-1)(x-2)(x-3)$.

(2) *Express* $1 - 3x^2 - x^3$ *in the form*

$$a_0 + a_1x + a_2x(x+2) + a_3x(x+2)(x+4).$$

In the usual notation

$$x = (x)_{1,\,\omega},\quad x(x+2) = (x)_{2,\,\omega},\quad x(x+2)(x+4) = (x)_{3,\,\omega},$$

where $\omega = -2$.

x	$\phi(x)$	$\underset{-2}{\Delta}$	$\overset{2}{\underset{-2}{\Delta}}$	$\overset{3}{\underset{-2}{\Delta}}$
0	1			
		2		
-2	-3		6	
		-10		-6
-4	17		18	
		-46		
-6	109			

Hence $1 - 3x^2 - x^3$

$$= 1 + \frac{(x)_{1,\,\omega}}{1!}.2 + \frac{(x)_{2,\,\omega}}{2!}.6 + \frac{(x)_3}{3!}(-6)$$

$$= 1 + 2x + 3x(x+2) - x(x+2)(x+4).$$

(3) *Express* $8x^3 + 28x^2 + 16x - 8$ *in the form*
$a_0 + a_1(2x+5) + a_2(2x+5)(2x+3) + a_3(2x+5)(2x+3)(2x+1)$.

The required form is

$$\phi(x) = a_0 + a_1 (2x+5)_1 + a_2 (2x+5)_2 + a_3 (2x+5)_3.$$

This can be reduced to the standard form by the substitution $t = 2x + 5$ or $2x = t - 5$. The given polynomial then becomes

$$f(t) = t^3 - 8t^2 + 13t + 2$$

and the form required is

$$a_0 + a_1 t + a_2 t(t-2) + a_3 t(t-2)(t-4)$$
$$= f(0) + \frac{(t)_{1,2}}{1!} \underset{2}{\Delta} f(0) + \frac{(t)_{2,2}}{2!} \underset{2}{\overset{2}{\Delta}} f(0) + \frac{(t)_{3,2}}{3!} \underset{2}{\overset{3}{\Delta}} f(0).$$

t	$f(t)$	$\underset{2}{\Delta} f(x)$	$\underset{2}{\overset{2}{\Delta}} f(x)$	$\underset{2}{\overset{3}{\Delta}} f(x)$
0	2			
		1		
2	4		-4	
		-7		6
4	-10		8	
		9		
6	8			

Hence $f(t)$

$$= 2 + \frac{t \cdot 1}{2!} + \frac{t(t-2)}{2!}(-4) + \frac{t(t-2)(t-4)}{3!} . 6$$
$$= 2 + (2x+5) - 2(2x+5)(2x+3) + (2x+5)(2x+3)(2x+1).$$

(4) *Find the polynomial of the third degree such that*

$$\phi(0) = 1,\ \phi(1) = 4,\ \phi(2) = 15,\ \phi(3) = 40.$$

Use the formula

$$\phi(x) = \phi(0) + \frac{(x)_{1,1}}{1!} \Delta \phi(0) + \frac{(x)_{2,1}}{2!} \Delta^2 \phi(0) + \frac{(x)_{3,1}}{3!} \Delta^3 \phi(0).$$

x	$\phi(x)$	Δ	Δ^2	Δ^3
0	1			
		3		
1	4		8	
		11		6
2	15		14	
		25		
3	40			

$$\phi(x) = 1 + \frac{x}{1!} . 3 + \frac{x(x-1)}{2!} . 8 + \frac{x(x-1)(x-2)}{3} . 6$$
$$= 1 + x + x^2 + x^3.$$

(5) *Find the polynomial of the third degree in x which takes the values* 8, 1, 0, -19 *when $x = -1, 0, 1, 2$ respectively.*

Use the formula

$$\phi(x) = \phi(-1) + \frac{(x+1)_{1,1}}{1!} \Delta \phi(-1) + \frac{(x+1)_{2,1}}{2!} \Delta^2 \phi(-1)$$
$$+ \frac{(x+1)_{3,1}}{3!} \Delta^3 \phi(-1).$$

x	$\phi(x)$	Δ	Δ^2	Δ^3
-1	8			
		-7		
0	1		6	
		-1		-24
1	0		-18	
		-19		
2	-19			

$$\phi(x) = 8 + (x+1)(-7) + \frac{(x+1)x}{2!}(6) + \frac{(x+1)x(x-1)}{3!}(-24)$$
$$= 1 + 3x^2 - 4x^3.$$

(6) *Find $\phi(x)$ where $\phi(x)$ is a polynomial of the third degree such that*
$\phi(-2) = 85,\ \phi(-1\cdot5) = 40,\ \phi(-1) = 15,\ \phi(-0\cdot5) = 4.$

Use the formula

$$\phi(x) = \phi(-2) + \frac{(x+2)_{1,\,\omega}}{1!\,\omega}\Delta\phi(-2) + \frac{(x+2)_{2,\,\omega}}{2!\,\omega^2}\Delta^2\phi(-2) + \frac{(x+2)_{3,\,\omega}}{3!\,\omega^3}\Delta^3\phi(-2),$$

where $\omega = 0\cdot5$.

x	$\phi(x)$	Δ	Δ^2	Δ^3
$-2\cdot0$	85			
		-45		
$-1\cdot5$	40		20	
		-25		-6
$-1\cdot0$	15		14	
		-11		
$-0\cdot5$	4			

$$\phi(x) = 85 + \frac{(x+2)}{1!\,(0\cdot5)}(-45) + \frac{(x+2)(x+1\cdot5)}{2!\,(0\cdot5)^2}(20) + \frac{(x+2)(x+1\cdot5)(x+1)}{3!\,(0\cdot5)^3}(-6)$$
$$= 1 - 2x + 4x^2 - 8x^3.$$

(7) *Find the polynomial of degree four such that $f(-2) = 72$, $f(-1) = 44$, $f(0) = 40$, $f(1) = 96$, $f(2) = 272$. Write down the most general form of the polynomial $\phi(x)$ of degree six which has the same values as $f(x)$ when $x = 0$, ± 1, ± 2, and determine $\phi(x)$ when $\phi'(1) = 0$ and $\phi'(-1) = 0$ also.*

[*Lond. B.Sc.*]

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
-2	72				
		-28			
-1	44		24		
		-4		36	
0	40		60		24
		56		60	
1	96		120		
		176			
2	272				

$$f(x) = f(-2) + \frac{(x+2)_{1,1}}{1!}\Delta f(-2) + \frac{(x+2)_{2,1}}{2!}\Delta^2 f(-2)$$
$$+ \frac{(x+2)_{3,1}}{3!}\Delta^3 f(-2) + \frac{(x+2)_{4,1}}{4!}\Delta^4 f(-2)$$

$$= 72 + (x+2)(-28) + \frac{(x+2)(x+1)}{2!}(24) + \frac{(x+2)(x+1)x}{3!}(36)$$
$$+ \frac{(x+2)(x+1)x(x-1)}{4!}(24)$$

$$= 40 + 18x + 29x^2 + 8x^3 + x^4.$$

Since $\phi(x)$ is a polynomial of degree 6, it may be written in the form

$$\phi(x) = a_0 + a_1(x+2)_{1,1} + a_2(x+2)_{2,1} + a_3(x+2)_{3,1} + a_4(x+2)_{4,1}$$
$$+ a_5(x+2)_{5,1} + a_6(x+2)_{6,1}.$$

Also since $\phi(x) = f(x)$ for $x = 0, \pm 1, \pm 2$,

$$a_0 = f(-2),\quad a_1 = \Delta f(-2),\quad a_2 = \Delta^2 f(-2)/2!,\quad a_3 \quad \Delta^3 f(-2)/3!,$$
$$a_4 = \Delta^4 f(-2)/4!$$

$$\phi(x) = f(x) + a_5(x+2)_{5,1} + a_6(x+2)_{6,1}.$$

$$\phi'(x) = f'(x) + a_5 D(x+2)_{5,1} + a_6 D(x+2)_{6,1}, \text{ where } D = d/dx.$$

Using logarithmic differentiation,

$$D(x+2)_{5,1} = D\{(x+2)(x+1)x(x-1)(x-2)\}$$
$$= (x+1)x(x-1)(x-2) + (x+2)x(x-1)(x-2)$$
$$+ (x+2)(x+1)(x-1)(x-2) + (x+2)(x+1)x(x-2)$$
$$+ (x+2)(x+1)x(x-1).$$

$$D(x+2)_{6,1} = D\{(x+2)(x+1)x(x-1)(x-2)(x-3)\}$$
$$= (x+1)x(x-1)(x-2)(x-3)$$
$$+ (x+2)x(x-1)(x-2)(x-3)$$
$$+ (x+2)(x+1)(x-1)(x-2)(x-3)$$
$$+ (x+2)(x+1)x(x-2)(x-3)$$
$$+ (x+2)(x+1)x(x-1)(x-3)$$
$$+ (x+2)(x+1)x(x-1)(x-2).$$

When $x = 1$, $D(x+2)_{5,1} = 3.2.1.-1. = -6$,

$$D(x+2)_{6,1} = 3.2.1.-1.-2 = 12.$$

When $x = -1$, $D(x+2)_{5,1} = 1.-1.-2.-3 = -6$,

$$D(x+2)_{6,1} = 1.-1.-2.-3.-4 = 24.$$

Also, since $f'(x) = 18 + 58x + 24x^2 + 4x^3$, $f'(1) = 104$, $f'(-1) = -20$.

Since $\phi'(1) = \phi'(-1) = 0$,

$$0 = 104 - 6a_5 + 12a_6.$$
$$0 = -20 - 6a_5 + 24a_6.$$

Solving these equations, $a_5 = 38$, $a_6 = 31/3$. Hence

$$\phi(x) = 40 + 18x + 29x^2 + 8x^3 + x^4 + 38(x+2)(x+1)x(x-1)(x-2)$$
$$+ \tfrac{31}{3}(x+2)(x+1)x(x-1)(x-2)(x-3)$$

$$= 40 + 46x + \tfrac{211}{3}x^2 - 27x^3 - \tfrac{152}{3}x^4 + 7x^5 + \tfrac{31}{3}x^6.$$

10·43. Successive Differences of a Polynomial

The application of the operator $\underset{\omega}{\Delta}$ to a polynomial lowers its degree by one and hence *if the degree of the polynomial is m, the mth difference quotient and the mth difference will both be constant and differences of higher order will be zero.*

To obtain the differences in a particular case we express the polynomial in factorials and then apply the results

$$\underset{\omega}{\overset{n}{\Delta}} \frac{(x)_{m,\,\omega}}{m!} = \frac{(x)_{m-n,\,\omega}}{(m-n)!}, \; n < m; \quad \underset{\omega}{\overset{m}{\Delta}} \frac{(x)_{m,\,\omega}}{m!} = 1, \; m = n;$$

$$\underset{\omega}{\overset{n}{\Delta}} \frac{(x)_{m,\,\omega}}{m!} = 0, \; n > m.$$

Example.—*Find successive differences for* $\phi(x) = 1 - 3x^2 - x^3$, *for* $\omega = -2$.

We have seen in § 10·42 that

$$\phi(x) = 1 + 2\,(x)_{1,\,\omega} + 3\,(x)_{2,\,\omega} - (x)_{3,\,\omega}, \text{ where } \omega = -2.$$

$$\underset{\omega}{\Delta}\,\phi(x) = 2 + 6\,(x)_{1,\,\omega} - 3(x)_{2,\,\omega} = 2 - 3x^2$$

$$\underset{\omega}{\overset{2}{\Delta}}\,\phi(x) = 6 - 6\,(x)_{1,\,\omega} = 6 - 6x$$

$$\underset{\omega}{\overset{3}{\Delta}}\,\phi(x) = -6, \quad \underset{\omega}{\overset{4}{\Delta}}\,\phi(x) = 0.$$

Since $\omega^n \underset{\omega}{\overset{n}{\Delta}} \doteqdot \Delta^n$ it follows that

$$\Delta\,\phi(x) = -4 + 6x^2, \; \Delta^2\,\phi(x) = 24 - 24x, \; \Delta^3\,\phi(x) = 48.$$

10·5. Generating Functions

Let $f(t)$ be a function of t which can be expanded in a power series whose general term is $u_x t^x$. Then

$$f(t) = \Sigma\, u_x t^x.$$

If the series contains an infinite number of terms it must be assumed to be convergent.

Then $f(t)$ is called the generating function of u_x and is written as

$$f(t) = G u_x.$$

The term generating function as applied to a recurring series has already been considered in § 9·52.

Examples of generating functions are provided by well known expansions.

(i) $(1-t)^{-1} = 1 + t + t^2 + \ldots + t^x + \ldots$; $(1-t)^{-1} = G1$.

(ii) $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^x}{x!} + \ldots$; $e^t = G\frac{1}{x!}$.

(iii) $\frac{e^t}{t} = \frac{1}{t} + \frac{1}{1!} + \frac{t}{2!} + \ldots \frac{t^x}{(x+1)!} + \ldots$; $\frac{e^t}{t} = G\frac{1}{(x+1)!}$

(iv) $-\log(1-t) = t + \frac{t}{2} + \frac{t}{3} + \ldots + \frac{t^x}{x} + \ldots$;

$$-\log(1-t) = G\frac{1}{x}.$$

(v) $(1+t)^m = 1 + \binom{m}{1}t + \binom{m}{2}t^2 + \ldots + \binom{m}{x}t^x + \ldots$;

$$(1+t)^m = G\binom{m}{x}.$$

New generating functions may be obtained from known expansions by various operations such as differentiation and integration term by term or they may be obtained by summation of a given series. If an operation is used it will be necessary to justify what is done. In particular, if differentiation term by term is used questions of uniform convergence will arise.

Thus, for example, the series $\sum_{x=0}^{\infty} t^x$ converges uniformly to $1/(1-t)$ for $|t| \leqslant a < 1$. Hence the series obtained by differentiating $\sum_{x=0}^{\infty} t^x$ with respect to t will converge to the differential coefficient of $1/(1-t)$ and moreover the convergence is uniform for the same range of values of t. This gives

$$1 + 2t + \ldots + xt^{x-1} + \ldots = \frac{1}{(1-t)^2}.$$

Multiplying throughout by t,

$$\sum_{x=0}^{\infty} xt^x = \frac{t}{(1-t)^2}, \text{ and } Gx = \frac{t}{(1-t)^2}.$$

If we differentiate r times and multiply throughout by $t^r/r!$ we obtain

$$\sum_{x=0}^{\infty}\binom{x}{r}t^x = \frac{t^r}{(1-t)^{r+1}}, \text{ and } G\binom{x}{r} = \frac{t^r}{(1-t)^{r+1}}.$$

We can regard G as an operator which obeys certain laws as in the case of operators already considered. We have the following properties.

(i) $G\{f(x) + \phi(x)\} = Gf(x) + G\phi(x)$.

This following immediately from definition and the result clearly extends to the sum of a finite number of functions.

(ii) $G\{cf(x)\} = cGf(x)$ where c is a constant

This follows from definition. As an example, consider a polynomial of degree m. Then in accordance with the Newton formula

$$f(x) = f(0) + \binom{x}{1}\Delta f(0) + \binom{x}{2}\Delta^2 f(0) + \dots$$
$$+ \binom{x}{r}\Delta^r f(0) + \dots + \binom{x}{m}\Delta^m f(0).$$

We have just seen that

$$G\binom{x}{r} = \frac{t^r}{(1-t)^{r+1}}.$$

Hence $$Gf(x) = \sum_{r=0}^{m}\Delta^r f(0).\frac{t^r}{(1-t)^{r+1}}.$$

(iii) If $Gf(x)$ is given then $Gf(x+1)$ may be found as follows:—

Write $Gf(x) = \phi(t)$ so that $\phi(t) = \sum_{x=0}^{\infty} f(x)\,t^x$, or

$$\phi(t) = f(0) + f(1)\,t + f(2)\,t^2 + \dots + f(x+1)\,t^{x+1} + \dots$$

$$\frac{\phi(t) - f(0)}{t} = f(1) + f(2)\,t + \dots + f(x+1)\,t^x + \dots$$
$$= \sum_{x=0}^{\infty} f(x+1)\,t^x.$$

Hence $$Gf(x+1) = \frac{\phi(t) - f(0)}{t}.$$

From this result it follows that

$$G\Delta f(x) = G\{f(x+1) - f(x)\} = Gf(x+1) - Gf(x)$$
$$= \{(1-t)\,\phi(t) - f(0)\}/t.$$

Proceeding as above we find

$$Gf(x+2) = \frac{\phi(t) - f(0) - tf(1)}{t^2}$$

$$Gf(x+m) = \frac{\phi(t) - f(0) - tf(1) - \ldots - t^{m-1}f(m-1)}{t^m}.$$

We can now find the generating function of

$$\Delta^m f(x) = f(x+m) - \binom{m}{1} f(x+m-1)$$
$$+ \binom{m}{2} f(x+m-2) - \ldots + (-1)^m f(x)$$
$$= \sum_{r=0}^{m} (-1)^r \binom{m}{r} f(x+m-r).$$

Hence $G\,\Delta^m f(x) =$

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \left\{ \frac{\phi(t) - f(0) - tf(1) - \ldots - t^{m-r-1} f(m-r-1)}{t^{m-r}} \right\}.$$

10·6. Interpolation Using Differences

If we are given the values of a function $f(x)$ corresponding to a finite set of values of the argument x then the problem of interpolation is concerned with the determination of the value of the function for some other value of the argument. The problem is, in general, indeterminate for the value of the function can be assigned arbitrarily. To obtain a useful solution some restriction on the form of the function is necessary. In particular, if the function can be represented by a polynomial of degree n, then the problem has a unique solution provided, $n+1$ values of the function are known. If we approximate to the function by a polynomial then we can obtain an approximate solution, but it will be essential to consider the error term involved in the approximation.

Simple interpolation formulae are based on Newton's formula proved in § 10·11. It has been shown that

$$n!\,\omega^n [xx_1x_2\ldots x_n] = f(x+n\omega) - \binom{n}{1} f(x+\overline{n-1}\,\omega) + \ldots$$
$$+ (-1)^n f(x)$$
$$= \Delta^n f(x),$$

and also

$$f(x) = f(x_1) + \sum_{r=1}^{n-1} \{(x - x_1)(x - x_2)\ldots(x - x_r)[x_1 x_2 \ldots x_{r+1}]\} + R_n(x)$$

where $R_n(x) = (x - x_1)(x - x_2)\ldots(x - x_n)[x x_1 x_2 \ldots x_n]$.

In the last equation write

$$x_1 = a,\ x_r = a + (r - 1)\omega,\ r = 1, 2, 3, \ldots.$$

Then

$$f(x) = f(a) + \sum_{r=1}^{n-1}\left\{\frac{(x-a)(x-a-\omega)\ldots(x-a-\overline{r-1}\,\omega)}{r!\,\omega^r}\,\Delta^r f(a)\right\} + R_n(x)$$

since $r!\omega^r [x_1 x_2 \ldots x_{r+1}] = \Delta^r f(x_1) = \Delta^r f(a)$,

$$= f(a) + \frac{(x-a)}{1!\,\omega}\Delta f(a) + \frac{(x-a)(x-a-\omega)}{2!\,\omega^2}\Delta^2 f(a) + \ldots$$

$$+ \frac{(x-a)(x-a-\omega)\ldots(x-a-\overline{n-2}\,\omega)}{(n-1)!\,\omega^{n-1}}\Delta^{n-1} f(a) + R_n(x).$$

If we omit the remainder term we obtain Newton's interpolation polynomial for forward differences.

$$f(x) = f(a) + \frac{(x-a)}{1!\,\omega}\Delta f(a) + \frac{(x-a)(x-a-\omega)}{2!\,\omega^2}\Delta^2 f(a) + \ldots$$

The degree of approximation obtained by the use of this formula will depend on the magnitude of the remainder term which has been neglected. This formula for a polynomial has already been proved in § 10·42.

Example.—*Given* $\log_e 2{\cdot}0 = 0{\cdot}6931$, $\log_e 2{\cdot}1 = 0{\cdot}7419$, $\log_e 2{\cdot}2 = 0{\cdot}7885$, $\log_e 2{\cdot}3 = 0{\cdot}8329$, *find an approximate value of* $\log_e 2{\cdot}175$.

In Newton's formula take $x = 2{\cdot}175$, $a = 2$, $\omega = 0{\cdot}1$. Then

$$\log 2{\cdot}175 = \log 2 + \frac{0{\cdot}175}{1!\,(0{\cdot}1)}\Delta \log 2 + \frac{(0{\cdot}175)(0{\cdot}175 - 0{\cdot}1)}{2!\,(0{\cdot}1)^1}\Delta^2 \log 2 + \ldots$$

Argument	*Function*	Δ	Δ^2	Δ^3
2·0	0·6931			
		0·0488		
2·1	0·7419		− 0·0022	
		0·0466		0
2·2	0·7885		− 0·0022	
		0·0444		
2·3	0·8329			

$$\therefore \log 2{\cdot}175 = 0{\cdot}6931 + 1{\cdot}75 \times 0{\cdot}0488 + \frac{1{\cdot}75 \times 0{\cdot}75 \times (-0{\cdot}0022)}{2!} = 0{\cdot}7771.$$

10·61. Newton's Formula for Backward Differences

If in the formula for divided differences

$$[x_1x_2\ldots x_{r+1}] = \frac{f(x_1)}{(x_1-x_2)\ldots(x_1-x_{r+1})} + \frac{f(x_2)}{(x_2-x_1)\ldots(x_2-x_{r+1})}$$

$$+ \frac{f(x_3)}{(x_3-x_1)\ldots(x_3-x_{r+1})} + \ldots + \frac{f(x_{r+1})}{(x_{r+1}-x_1)\ldots(x_{r+1}-x_r)}$$

we write $x_r = a - (r-1)\,\omega$, then

$$[x_1x_2\ldots x_{r+1}] = \frac{f(a)}{r!\,\omega^r} - \frac{f(a-\omega)}{1!\,(r-1)!\,\omega^r} + \frac{f(a-2\omega)}{2!\,(r-2)!\,\omega^r} - \ldots + (-1)^r\frac{f(a-r\omega)}{r!\,\omega^r},$$

or $r!\,\omega^r\,[x_1x_2\ldots x_{r+1}] = f(a) - \binom{r}{1} f(a-\omega) + \binom{r}{2} f(a-2\omega) - \ldots + (-1)^r \binom{r}{r} f(a-r\omega)$

$$= f(a) - \binom{r}{1} E^{-1} f(a) + \binom{r}{2} E^{-2} f(a) - \ldots + (-1)^r \binom{r}{r} E^{-r} f(a)$$

$$= \left\{1 - \binom{r}{1} E^{-1} + \binom{r}{2} E^{-2} - \ldots + (-1)^r \binom{r}{r} E^{-r}\right\} f(a)$$

$$= \{1 - E^{-1}\}^r f(a) = (E-1)^r E^{-r} f(a)$$

$$= \Delta^r f(a - r\omega).$$

Now make the substitution $x_r = a - (r-1)\,\omega$, $r = 1, 2, \ldots,$ in the formula

$$f(x) = f(x_1) + \sum_{r=1}^{n-1} \{(x-x_1)(x-x_2)\ldots(x-x_r)\,[x_1x_2\ldots x_{r+1}]\} + R_n(x).$$

Then $f(x) = f(a) +$

$$\sum_{r=1}^{n-1} \left\{\frac{(x-a)(x-a+\omega)\ldots(x-a+\overline{r-1}\,\omega)}{r!\,\omega^r}\,\Delta^r f(a-r\omega)\right\} + R_n(x).$$

If we neglect the remainder term we obtain the polynomial

$$f(x) = f(a) + \frac{(x-a)}{1!\,\omega} \Delta f(a-\omega)$$
$$+ \frac{(x-a)(x-a+\omega)}{2!\,\omega^2} \Delta^2 f(a-2\omega)$$
$$+ \frac{(x-a)(x-a+\omega)(x-a+2\omega)}{3!\,\omega^3} \Delta^3 f(a-3\omega) + \ldots$$

In particular, if $a = 0$, $\omega = 1$,

$$f(x) = f(0) + x\,\Delta f(-1) + \frac{x(x+1)}{2!} \Delta^2 f(-2)$$
$$+ \frac{x(x+1)(x+2)}{3!} \Delta^3 f(-3) + \ldots$$

a formula which has already been developed by symbolic reasoning in § 10·31.

To see how the formula is applied, consider the table of differences in the following form.

Argument	*Function*	Δ	Δ^2	Δ^3	.
$a-3\omega$	$f(a-3\omega)$		.		.
		$\Delta f(a-3\omega)$		.	
$a-2\omega$	$f(a-2\omega)$		$\Delta^2 f(a-3\omega)$		.
		$\Delta f(a-2\omega)$		$\Delta^3 f(a-3\omega)$	
$a-\omega$	$f(a-\omega)$		$\Delta^2 f(a-2\omega)$		
		$\Delta f(a-\omega)$			
a	$f(a)$				

The table shows that the differences to be used lie on a line sloping upwards from $f(a)$. The formula is useful in dealing with interpolation near the end of a table.

Example.—*Given that the values of* sinh x *for* $x = 0{\cdot}20$, $0{\cdot}24$, *and* $0{\cdot}28$ *are* $0{\cdot}20134$, $0{\cdot}24231$, *and* $0{\cdot}28367$ *respectively, find an approximate value for* sinh x *when* $x = 0{\cdot}27$.

x	sinh x	Δ	Δ^2
0·20	0·20134		
		0·04097	
0·24	0·24231		0·00049
		0·04136	
0·28	0·28367		

Take $x = 0{\cdot}27$, $a = 0{\cdot}28$, $\omega = 0{\cdot}04$. Applying the general formula,

$$\sinh 0{\cdot}27 = 0{\cdot}28367 + \frac{(-0{\cdot}01)}{0{\cdot}04} \times 0{\cdot}04136 + \frac{(-0{\cdot}01)(-0{\cdot}01 + 0{\cdot}04)}{2(0{\cdot}04)^2} \times 0{\cdot}00049$$

$$= 0{\cdot}27328.$$

The value of sinh 0·27 correct to 5 decimal places is 0·27329.

10·62. Interpolation without Differences

In principle this is solved by the use of Lagrange's formula

$$f(x) = \sum_{r=1}^{n} \frac{f(x_r)\,\psi(x)}{(x - x_r)\,\psi'(x_r)} + \psi(x)\,[x x_1 x_2 \ldots x_n],$$

where $\psi(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$.

Various special formulae* may be derived by considering suitable distributions of $x_1, x_2, \ldots, x_n$.

The practical objections to the use of the Lagrange formula are the amount of numerical calculation involved and the fact that the introduction of an additional point x_{n+1} implies that the calculations have to be done all over again. In the case of formulae using differences it is only necessary to add additional terms in order to obtain a more accurate result. The Lagrange formula has the advantage that it can be used directly where the values of the argument are not equidistant.

Examples.—(1) *Find the polynomial* $f(x)$ *of the third degree such that* $f(-1) = 5$, $f(1) = 1$, $f(2) = -1$, $f(5) = -79$.

Since $f(x)$ is of the third degree we have from Lagrange's formula

$$f(x) = f(a)\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + f(b)\frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)}$$
$$+ f(c)\frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)} + f(d)\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}.$$

Write $a = -1$, $b = 1$, $c = 2$, $d = 5$. Then

$$f(x) = (5)\frac{(x-1)(x-2)(x-5)}{-2.-3.-6} + (1)\frac{(x+1)(x-2)(x-5)}{2.-1.-4} +$$
$$(-1)\frac{(x+1)(x-1)(x-5)}{3.1.-3} + (-79)\frac{(x+1)(x-1)(x-2)}{6.4.3}$$
$$= -\tfrac{5}{36}(x-1)(x-2)(x-5) + \tfrac{1}{8}(x+1)(x-2)(x-5)$$
$$+ \tfrac{1}{9}(x+1)(x-1)(x-5) - \tfrac{79}{72}(x+1)(x-1)(x-2)$$
$$= 1 - x + 2x^2 - x^3.$$

Observe that in this example the arguments are not equidistant so that the formulae for differences do not apply.

* For these, reference may be made to Jordan: *Calculus of Finite Differences* and to Milne-Thomson: *The Calculus of Finite Differences.*

(2) *Given cosh* 0·30 = 1·04534, *cosh* 0·31 = 1·04844, *cosh* 0·33 = 1·05495, *find an approximate value of cosh* 0·32.

Apply the formula

$$f(x) = f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)},$$

where $x = 0{\cdot}32$, $a = 0{\cdot}30$, $b = 0{\cdot}31$, $c = 0{\cdot}33$, $f(x) = \cosh x$.

$$\cosh 0{\cdot}32 = 1{\cdot}04534 \times \frac{(0{\cdot}32 - 0{\cdot}31)(0{\cdot}32 - 0{\cdot}33)}{(0{\cdot}30 - 0{\cdot}31)(0{\cdot}30 - 0{\cdot}33)} +$$

$$1{\cdot}04844 \times \frac{(0{\cdot}32 - 0{\cdot}30)(0{\cdot}32 - 0{\cdot}33)}{(0{\cdot}31 - 0{\cdot}30)(0{\cdot}31 - 0{\cdot}33)} +$$

$$1{\cdot}05495 \times \frac{(0{\cdot}32 - 0{\cdot}30)(0{\cdot}32 - 0{\cdot}31)}{(0{\cdot}33 - 0{\cdot}31)(0{\cdot}33 - 0{\cdot}30)}$$

$$= -\frac{1{\cdot}04534}{3} + 1{\cdot}04844 + \tfrac{1}{3} \times 1{\cdot}05495 = 1{\cdot}05164.$$

10·7. Difference Equations

We have seen in Chap. IX, how difference equations arise in connection with the study of recurring series. The equations were of the form

$$u_n + p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_m u_{n-m} = 0$$

where n is a variable positive integer, m a fixed positive integer, $p_1, p_2, \ldots, p_m$ constants.

With changed notation we can represent the equation in other equivalent forms. Thus

$$u_{x+m} + p_1 u_{x+m-1} + p_2 u_{x+m-2} + \ldots + p_m u_x = 0$$

$$f(x+m) + p_1 f(x+m-1) + p_2 f(x+m-2) + \ldots + p_m f(x) = 0$$

where x represents a variable positive integer and u_x, $f(x)$ are functions defined for positive integral values.

Since $f(x+r) = E^r f(x)$, $u_{x+r} = E^r u_x$, the equations can be written in the symbolic forms

$$(E^m + p_1 E^{m-1} + p_2 E^{m-2} + \ldots + p_m)\, u_x = 0;$$

$$(E^m + p_1 E^{m-1} + p_2 E^{m-2} + \ldots + p_m)\, f(x) = 0.$$

Again, since $E \doteq 1 + \Delta$ we can express the equation as

$$(\Delta^m + a_1 \Delta^{m-1} + a_2 \Delta^{m-2} + \ldots + a_m)\, f(x) = 0.$$

where $a_1, a_2, \ldots, a_m$ are constants depending on $p_1, p_2, \ldots, p_m$. Thus difference equations can be represented in a variety of equivalent forms.

In general, we can regard an ordinary difference equation as a relation between an independent variable x which takes integral values, a dependent variable y, and any successive differences of y, i.e. Δy, $\Delta^2 y$, ... The order of the equation is determined by the order of the highest difference and the equation is said to be linear if y, Δy, $\Delta^2 y$, ... in the equation are all of the first degree. Thus the original equation can be described as a linear mth order difference equation with constant coefficients, since p_1, p_2, ..., p_m are independent of x.

If we are given a function $F(z)$ defined for $z = z_0$, $z_0 + \omega$, $z_0 + 2\omega$, ..., $z_0 + r\omega$, ..., r being a positive integer we can reduce the function to one of the type considered above by the substitution $x = (z - z_0)/\omega$. If $F(z)$ becomes $f(x)$, then $f(x)$ is defined for positive integral values of x, ω being replaced by unity.

Consider any function $\phi(x, y, c) = 0$ where c is a variable parameter, which is constant as far as the operations E, Δ are concerned. Thus c can either be an absolute constant or a periodic function of period unity.* If we operate on $\phi(x, y, c) = 0$ with E, we have $\phi(x + 1, Ey, c) = 0$. Eliminating c between the two equations we obtain an equation of the form $\psi(x, y, Ey) = 0$. This is a difference equation of the first order.

If we begin with $\phi(x, y, c_1, c_2) = 0$, where c_1 and c_2 are parameters, operate twice with E and then eliminate c_1, c_2 between the three equations, we obtain an equation of the form

$$\psi(x, y, Ey, E^2y) = 0$$

which is a difference equation of the second order.

Similarly, if we start with m arbitrary parameters, operate with E successively m times, and then eliminate the m parameters we obtain a difference equation of the mth order.

$$\psi(x, y, Ey, \ldots, E^m y) = 0.$$

Thus if we are given a difference equation of the mth order we may expect to find in its general solution m arbitrary constants which may be, as mentioned above, periodic functions of period unity.

Examples.—(1) *Given $y = c$, then $Ey = c = y$ and the difference equation is*

$$Ey - y = 0, \text{ or } (E - 1)y = 0.$$

* A function $f(x)$ is periodic and of period ω if $f(x + r\omega) = f(x)$, where r denotes an integer. If $\omega = 1$, $f(x + r) = f(x)$.

(2) *Given* $y = c_1 + c_2 x$, *then*

$$Ey = c_1 + c_2(x+1) = y + c_2$$
$$E^2 y = c_1 + c_2(x+2) = y + 2c_2.$$

Eliminating c_1 and c_2,

$$E^2 y - 2Ey + y = 0, \text{ or } (E-1)^2 y = 0.$$

(3) *Given* $y = c_1\alpha^x + c_2\beta^x$, *then*

$$Ey = c_1\alpha^{x+1} + c_2\beta^{x+1}$$
$$E^2 y = c_1\alpha^{x+2} + c_2\beta^{x+2}.$$

Eliminating c_1 and c_2,

$$E^2 y - (\alpha + \beta) Ey + \alpha\beta y = 0, \text{ or } (E-\alpha)(E-\beta) y = 0.$$

(4) *Given* $y = (c_1 + c_2 x)\alpha^x$, then

$$Ey = \{c_1 + c_2(x+1)\}\alpha^{x+1}$$
$$E^2 y = \{c_1 + c_2(x+2)\}\alpha^{x+2}.$$

On elimination we obtain

$$E^2 y - 2\alpha Ey + \alpha^2 y = 0, \text{ or } (E-\alpha)^2 y = 0.$$

10·71. Linear First Order Difference Equations

The general form of a first order equation is

$$Ey - y\phi(x) = \psi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

where $\phi(x)$ and $\psi(x)$ are given functions of x, the variable x being defined over a suitable set of $\{x\}$ values.

Equivalent forms are

$$u_{x+1} - u_x\phi(x) = \psi(x)$$
$$f(x+1) - f(x)\phi(x) = \psi(x)$$
$$(1+\Delta) y - y\phi(x) = \psi(x)$$

The general solution of (i) is of the form $y = v(x) + \lambda(x)$ where $\lambda(x)$ is any *particular* solution of (i) and $z = v(x)$ is the *general* solution of $Ez - z\phi(x) = 0$ . . . . (ii)

This result may be proved immediately by substitution in (i) when the form (ii) is obtained.

Consider then the problem of finding a solution to the equation

$$Ey - y\phi(x) = 0$$

subject to the following conditions:—

(*a*) A system of values of x of the form $x = a + n$ is given, where n denotes a variable positive integer;

(*b*) k is an arbitrary constant and the solution is to be such that $y = k$ when $x = a$.

Write $y = u_x = u_{a+n}$ so that we require a solution of the equation

$$\frac{u_{a+n+1}}{u_{a+n}} = f(a+n) \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

such that $u_a = k$.

In (iii) take the values 0, 1, 2, ..., $n - 1$ for n in succession and multiply corresponding equations. Then

$$\frac{u_{a+n}}{u_a} = f(a)\,f(a+1)\ldots f(a+n-1).$$

Writing $u_a = k$ and returning to the original notation,

$$y = u_x = kf(a)\,f(a+1)\ldots f(x-1).$$

Examples.—(1) *Solve the equation* $Ey - (x+1)\,y = 0$ *where* x *takes positive integral values and* $y = 1$ *when* $x = 1$.

The general solution is

$$y = k.1.2\ldots x = kx!$$

Since $y = 1$ when $x = 1$, $k = 1$.

(2) *Find the general solution of* $Ey + y = 2x + 3$ *where* x *takes positive integral values.*

A particular solution is seen to be $x + 1$. For

$$E(x+1) + x + 1 = x + 2 + x + 1 = 2x + 3.$$

The general solution of

$$Ey + y = 0$$

is $y = k(-1)^x$ so that the complete solution is $y = k(-1)^x + x + 1$, where k denotes an arbitrary constant or a periodic function of period unity.

Note.—If in the first order difference equation $Ey - y\,\phi(x) = \psi(x)$, $\phi x = \lambda$, where λ is a constant, $\neq 1$, and $\psi(x) = ax + b$, a and b, being constants then a particular solution of the given equation is $a'x + b'$ where a' and b' are constants depending on λ, a, b. Thus substituting $y = a'x + b'$ in $Ey - \lambda y = ax + b$ we have

$$a'(x+1) + b' - \lambda(a'x + b') = ax + b.$$

Equating corresponding coefficients,

$$a'(1-\lambda) = a,\ a' + b'(1-\lambda) = b$$

giving

$$a' = a/(1-\lambda),\ b' = \{(1-\lambda)\,b - a\}/(1-\lambda)^2.$$

Thus if $\lambda = -1$, $a = 2$, $b = 3$ then $a' = 1$, $b' = 1$ as in Ex. 2.

10·72. Homogeneous Linear Difference Equations and Constant Coefficients

We shall consider only formal methods of solution. The equation is of the form

$$f(x+m) + p_1 f(x+m-1) + p_2 f(x+m-2) + \ldots + p_m f(x) = 0,$$

or $$E^m y + p_1 E^{m-1} y + p_2 E^{m-2} y + \ldots + p_m y = 0$$
where $y = f(x)$, and $p_1, p_2, \ldots, p_m$ are constants.

In symbolic form the equation may be written

$$(E^m + p_1 E^{m-1} + p_2 E^{m-2} + \ldots + p_m) y = 0 \text{ or } \phi(E) y = 0$$

where $\phi(E)$ denotes a polynomial in E.

The equation

$$\rho^m + p_1 \rho^{m-1} + p_2 \rho^{m-2} + \ldots + p_m = 0 \quad \text{or} \quad \phi(\rho) = 0$$

is called the *characteristic* or *auxiliary* equation of the difference equation. It has m roots which may be real or complex, simple or multiple.

CASE I. Suppose first that the m roots, $\rho_1, \rho_2, \ldots, \rho_m$, are all distinct but may be real or complex.

If c_r denote an arbitrary constant or a periodic function of period unity then $y_r = c_r \rho_r^x$ is one solution of $\phi(E) y = 0$.
For

$$\begin{aligned} E^m c_r \rho_r^x + p_1 E^{m-1} c_r \rho_r^x + p_2 E^{m-2} c_r \rho_r^x + \ldots + p_m c_r \rho_r^x \\ = c_r \rho_r^{x+m} + p_1 c_r \rho_r^{x+m-1} + p_2 c_r \rho_r^{x+m-2} + \ldots + p_m c_r \rho_r^x \\ = c_r \rho_r^x \phi(\rho_r) = 0, \end{aligned}$$

since ρ_r is a root of $\phi(\rho) = 0$.

The functions $y_1, y_2, \ldots, y_m$ are thus m particular different solutions of the equation each containing an arbitrary periodic function. Hence $y_1 + y_2 + \ldots + y_m$ is also a solution, and since it contains m arbitrary periodic functions it is the general solution. Thus

$$y = c_1 \rho_1^x + c_2 \rho_2^x + \ldots + c_m \rho_m^x$$

gives the general solution, provided $\rho_1, \rho_2, \ldots, \rho_m$ are distinct.

Examples.—(1) *Solve the difference equation* $E^3 y - 6E^2 y + 11 Ey - 6y = 0$.
The characteristic equation is

$$\rho^3 - 6\rho^2 + 11\rho - 6 = 0$$

and the roots are $\rho = 1, 2, 3.$

Hence the general solution is

$$y = c_1 + c_2 2^x + c_3 3^x.$$

Note.—The given equation could have been expressed as

$$f(x+3) - 6f(x+2) + 11 f(x+1) - 6f(x) = 0$$

or as $$u_{x+3} - 6u_{x+2} + 11u_{x+1} - 6u_x = 0.$$

(2) *Solve the equation* $2\,\Delta^2 u_x + 2\,\Delta u_x - u_x = 0$.

Using $\Delta \doteq E - 1$ the equation becomes

$$2\,(E \qquad\qquad + 2\,(E-1)\,u_x - u_x = 0,$$

or

$$(2\,E^2 - 2E - 1)\,u_x = 0.$$

The characteristic equation is $2\rho^2 - 2\rho - 1 = 0$ and its roots are

$$\rho = \tfrac{1}{2}\,(1 \pm \sqrt{3}).$$

Hence the general solution is

$$u_x = c_1\left(\frac{1+\sqrt{3}}{2}\right)^x + c_2\left(\frac{1-\sqrt{3}}{2}\right)^x.$$

Note.—It is not necessary to transform the Δ form of the equation into the E form in order to solve it. The given equation is

$$(2\,\Delta^2 + 2\,\Delta - 1)\,u_x = 0$$

and the roots of $2\,\Delta^2 + 2\,\Delta - 1 = 0$ considered as a quadratic equation in Δ are $\tfrac{1}{2}\,(-1 \pm \sqrt{3})$.

Since $E \doteq 1 + \Delta$ the roots of the characteristic equation are

$$1 + \tfrac{1}{2}\,(-1 \pm \sqrt{3}) = \tfrac{1}{2}\,(1 \pm \sqrt{3})$$

as already proved.

Suppose now that one or more of the roots of the characteristic equation are complex. These complex roots will occur in pairs of the form $\alpha \pm i\beta$ where α and β are real. Since the roots are all distinct it is sufficient to consider one pair $\alpha \pm i\beta$. The corresponding terms in the general solution will be of the form

$$k_1\,(\alpha + i\beta)^x + k_2\,(\alpha - i\beta)^x$$

where k_1, k_2 are the arbitrary periodic functions of period unity.

Write $r = |\alpha + i\beta| = +\sqrt{(\alpha^2 + \beta^2)}$, $\phi =$ principal value of amp. $(\alpha + i\beta)$, then $r = |\alpha - i\beta|$, $-\phi =$ amp. $(\alpha - i\beta)$. Then

$$\begin{aligned} k_1\,(\alpha + i\beta)^x + k_2\,(\alpha - i\beta)^x &= k_1 r^x e^{xi\phi} + k_2 r^x e^{-xi\phi} \\ &= k_1 r^x\,(\cos x\phi + i\sin x\phi) + k_2 r^x\,(\cos x\phi - i\sin x\phi) \\ &= (k_1 + k_2)\,r^x\cos x\phi + i\,(k_1 - k_2)\,r^x\sin x\phi \\ &= K_1 r^x\cos x\phi + K_2 r^x\sin x\phi. \end{aligned}$$

The solution is now in real form.

Examples.—(3) *Solve the equation* $y_{n+3} + 2y_{n+2} + 2y_{n+1} + y_n = 0$.

The characteristic equation is

$$\rho^3 + 2\rho^2 + 2\rho + 1 = 0 = (\rho + 1)\,(\rho^2 + \rho + 1).$$

The roots are -1, $\tfrac{1}{2}\,(-1 \pm i\sqrt{3})$.

Now $|\tfrac{1}{2}\,(-1 + i\sqrt{3})| = 1$, amp. $\tfrac{1}{2}\,(-1 + i\sqrt{3}) = \tfrac{2}{3}\pi$.

Hence the general solution is

$$y_n = c_1\,(-1)^n + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}.$$

(4) *Find the general solution of the difference equation*

$$y_{n+3} - 9y_{n+2} + 43y_{n+1} - 75y_n = 0$$

given that $y = 3^n$ *is a particular solution.*

Find y_n *for all positive integral values of* n, *where* $y_1 = 0$, $y_2 = 1$, $y_3 = 2$ *also.* [*Lond. B.Sc.*]

The characteristic equation is $\rho^3 - 9\rho^2 + 43\rho - 75 = 0$. Since $y = 3^n$ is a particular solution of the given equation, $\rho = 3$ is a root of the characteristic equation. Thus $\rho^3 - 9\rho^2 + 43\rho - 75 = (\rho - 3)(\rho^2 - 6\rho + 25)$ and the roots are $3, 3 \pm 4i$.

Since $|3 + 4i| = 5$, amp. $(3 + 4i) = \phi = \tan^{-1}\frac{4}{3}$, the general solution may be written as

$$u_n = A.3^n + B.5^n \cos n\phi + C.5^n \sin n\phi.$$

Substituting $y_1 = 0$, $y_2 = 1$, $y_3 = 2$, we have

$$0 = 5A \cos\phi + 5B \sin\phi + 3C$$
$$1 = 25A \cos 2\phi + 25B \sin 2\phi + 9C$$
$$2 = 125A \cos 3\phi + 125B \sin 3\phi + 27C.$$

Now $\cos\phi = \frac{3}{5}$, $\sin\phi = \frac{4}{5}$, $\cos 2\phi = -\frac{7}{25}$, $\sin 2\phi = \frac{24}{25}$,

$$\cos 3\phi = -\tfrac{117}{125}, \ \sin 3\phi = \tfrac{44}{125}.$$

Substituting these values,

$$3A + 4B + 3C = 0$$
$$-7A + 24B + 9C = 1$$
$$-117A + 44B + 27C = 0$$

giving $A = -\frac{1}{100}$, $B = \frac{7}{100}$, $C = -\frac{1}{12}$.

CASE II. Suppose now that the characteristic equation $\phi(\rho) = 0$ has multiple roots.

In this case the number of distinct values of ρ will be less than m. Hence the form written down above for the general solution will now contain less than m arbitrary constants or periodic functions. Thus it will not be the general solution and it is necessary to find further different particular solutions of the given difference equation. Suppose first that $\rho = a$ is a double root corresponding to the factor $(\rho - a)^2$ of $\phi(\rho)$. We have shown in § 10·7, Ex. 4, that $y = (c_1 + c_2x)a^x$ gives rise to the equation $(E - a)^2 y = 0$. The term c_2xa^x provides the other particular solution required. Thus corresponding to each double root a we have terms in the general solution of the form $(c_1 + c_2x)a^x$.

Note.—If the double root is complex corresponding to $\rho = a + i\beta$, a, β real, then the terms in the general solution (*four* in number) will be of the form

$$(c_1 + c_2x) r^x \cos x\phi + (c_3 + c_4x) r^x \sin x\phi$$

where $$r = |a + i\beta|, \ \phi = \text{amp.}\ (a + i\beta).$$

Suppose now that $\rho = \alpha$ is a root of multiplicity s of the equation $\phi(\rho) = 0$. Then the corresponding terms in the general solution will be

$$y = \left[c_1 + c_2\binom{x}{1} + c_3\binom{x}{2} + \ldots + c_s\binom{x}{s-1}\right]\alpha^x \quad \ldots \text{(i)}$$

The formula holds for both real and complex values of α. We can verify directly that (i) is the general solution of

$$(E - \alpha)^s y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

by showing that $c_t\binom{x}{t-1}\alpha^x$, $t = 1, 2, \ldots, s$ are particular solutions of (ii). Their sum, which is (i), will contain s arbitrary constantss or periodic functions and so will be the general solution.

Consider $$u = c_t\binom{x}{t-1}\alpha^x.$$

$$Eu = c_t\binom{x+1}{t-1}\alpha^{x+1},\ Eu - \alpha u = c_t\alpha^{x+1}\left\{\binom{x+1}{t-1} - \binom{x}{t-1}\right\}$$
$$= c_t\binom{x}{t-2}\alpha^{x+1}.$$

Hence $$(E - \alpha)\,u = c_t\binom{x}{t-2}\alpha^{x+1}$$

Applying the operator $E - \alpha$ in succession we have

$$(E - \alpha)^2 u = c_t\binom{x}{t-3}\alpha^{x+2}$$

$$(E - \alpha)^3 u = c_t\binom{x}{t-4}\alpha^{x+3}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$(E - \alpha)^{t-2} u = c_t\binom{x}{1}\alpha^{x+t-2}$$

$$(E - \alpha)^{t-1} u = c_t\alpha^{x+t-1}$$

$$(E - \alpha)^t u = 0.$$

Thus, provided $1 \leq t \leq s$, $c_t\binom{x}{t-1}\alpha^x$ are all particular solutions of $(E - \alpha)^s y = 0$, which proves the result.

Example.—(5) *Solve the difference equation*

$$f(x+4) - 9f(x+3) + 30f(x+2) - 44f(x+1) + 24f(x) = 0$$

given that a particular solution is 3^x.

The characteristic equation is

$$f(\rho) = \rho^4 - 9\rho^3 + 30\rho^2 - 44\rho + 24 = 0.$$

Since 3^x is a particular solution, $\rho - 3$ is a factor of $\phi(\rho)$. It is easily seen that

$$\phi(\rho) = (\rho - 3)(\rho^3 - 6\rho^2 + 12\rho - 8) = (\rho - 3)(\rho - 2)^3.$$

Hence the general solution of the difference equation is

$$f(x) = c_1 . 3^x + \left[c_2 + c_3\binom{x}{1} + c_4\binom{x}{2}\right]2^x$$

$$= c_1 . 3^x + \left[c_2 + c_3 x + c_4 x\,\frac{(x-1)}{2!}\right]2^x.$$

10·73. Complete Linear Equation with Constant Coefficients

The equations to be considered are of the form

$$f(x+m) + p_1 f(x+m-1) + p_2 f(x+m-2) + \ldots + p_m f(x) = V(x) \qquad \text{(i)}$$

where $V(x)$ is a given function of x.

To find the general solution of the complete equation it is sufficient to find a particular solution and add to this the general solution of

$$f(x+m) + p_1 f(x+m-1) + p_2 f(x+m-2) + \ldots + p_m f(x) = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

Thus if $F(x)$ is a particular solution of (i), and $G(x)$ is the general solution of (ii) then the general solution of (i) is

$$f(x) = F(x) + G(x).$$

In this section we shall consider a symbolic method of finding the particular integral when $V(x)$ is of the form ca^x where a, c are constants.

The given difference equation may be written in the form

$$\phi(E) f(x) = ca^x$$

and symbolically, $$f(x) = \frac{1}{\phi(E)}\, ca^x.$$

We now consider how to attach a meaning to the expression $\left[\dfrac{\phi(E)}{\psi(E)}\right]a^x$ where $\phi(E)$ and $\psi(E)$ are polynomials in E.

Since $E(a^x) = a^{x+1} = a.a^x$, $E^r(a^x) = a^r.a^x$, $E^{-r}(a^x) = a^{-r}.a^r$ it follows that $\phi(E)\,a^x = \phi(a)\,a^x$.

If we write $\left[\dfrac{\phi(E)}{\psi(E)}\right]a^x = v(x)$, we can regard $v(x)$ as a function satisfying the difference equation

$$\phi(E)\,a^x = \psi(E)\,v(x).$$

A solution of this equation is $v(x) = \dfrac{\phi(a)}{\psi(a)}\,a^x$, provided $\psi(a) \neq 0$.

For $\phi(E)\,a^x = \phi(a)\,a^x$ and $\psi(E)\left[\dfrac{\phi(a)}{\psi(a)}\,a^x\right] = \dfrac{\phi(a)}{\psi(a)}\,\psi(E)\,a^x$
$= \phi(a)\,a^x$.

Hence returning to the original equation a solution of

$$f(x) = \frac{1}{\phi(E)}\,ca^x$$

will be $f(x) = \dfrac{1}{\phi(a)}\,ca^x$, provided $\phi(a) \neq 0$, *i.e.* $\rho = a$ is not a root of the characteristic equation.

Thus if $F(x)$ is the general solution of (ii) then the complete solution of (i) is

$$f(x) = F(x) + \frac{ca^x}{\phi(a)}.$$

In the particular case in which $V(x) = c$ corresponding to $a = 1$, the complete solution will be $f(x) = F(x) + \dfrac{c}{\phi(1)}$, $\phi(1) \neq 0$.

Examples.—(1) *Solve the equation*

$$2f(x+2) - 2f(x+1) - f(x) = 1.$$

The characteristic equation is $2\rho^2 - 2\rho - 1 = 0$, the roots being

$$\tfrac{1}{2}(1 \pm \sqrt{3}).$$

Thus the general solution of the given equation is

$$f(x) = c_1\left(\frac{1+\sqrt{3}}{3}\right)^x + c_2\left(\frac{1-\sqrt{3}}{2}\right)^x - 1.$$

(2) *Solve the equation*

$$f(x+4) - 9f(x+3) + 30f(x+2) - 44f(x+1) + 24f(x) = 3.4^x.$$

The characteristic equation is

$$\phi(\rho) = \rho^4 - 9\rho^3 + 30\rho^2 - 44\rho + 24 = 0 = (\rho - 3)(\rho - 2)^3.$$

The particular solution is

$$\frac{1}{\phi(4)} 3 \cdot 4^x = \tfrac{3}{8} \cdot 4^x.$$

The complete solution is

$$f(x) = c_1 \cdot 3^x + \left\{ c_2 + c_3 x + c_4 x \frac{(x-1)}{2!} \right\} 2^x + \tfrac{3}{8} \cdot 4^x.$$

If $\rho = a$ is a simple root of $\phi(\rho) = 0$ we may proceed as follows. Replace the equation $\phi(E) f(x) = ca^x$ by $\phi(E) f(x) = c(a+h)^x$ where in the limit $h \to 0$. The particular solution will be $c(a+h)^x/\phi(a+h)$. The general solution will now take the form

$$f(x) = \frac{c(a+h)^x}{\phi(a+h)} + c_1 a^x + \text{terms corresponding to other roots of } f(\rho) = 0$$

$$= \frac{c(a+h)^x - ca^x}{\phi(a+h)} + \left\{ c_1 + \frac{c}{\phi(a+h)} \right\} a^x + \ldots$$

Replace $c_1 + \dfrac{c}{\phi(a+h)}$ by a new constant C_1 and consider the limit as $h \to 0$ of $\{c(a+h)^x - ca^x\}/\phi(a+h)$. Since a is a simple root of $\phi(\rho)$, the first differential coefficient $\phi'(a) \neq 0$.

Hence
$$\lim_{h \to 0} c \frac{(a+h)^x - a^x}{\phi(a+h)} = \frac{cxa^{x-1}}{\phi'(a)}$$

and the general solution is

$$f(x) = \frac{cxa^{x-1}}{\phi'(a)} + c_1 a^x + \ldots$$

In the special case in which $V(x) = c$, i.e. $a = 1$ and $\phi(1) = 0$ the particular integral takes the form $cx/\phi'(1)$.

Example.—(3) *Find the general solution of*

$$f(x+2) - 3f(x+1) + 2f(x) = 2^x.$$

The characteristic equation is

$$\rho^2 - 3\rho + 2 = 0$$

and the roots are $\rho = 1, 2$. Since $\phi(\rho) = \rho^2 - 3\rho + 2$, $\phi'(\rho) = 2\rho - 3$, $\phi'(2) = 1$. Thus the general solution is

$$f(x) = x 2^{x-1} + c_1 + c_2 2^x.$$

EXERCISES X

1. Given two functions $\phi(x)$, $\psi(x)$, defined for $x = x_0, x_1, \ldots, x_n$ and $f(x) = \phi(x) + \psi(x)$, prove that the divided differences of $f(x)$ are equal to the sum of the corresponding divided differences of $\phi(x)$ and $\psi(x)$.

Show also that if c is a constant, the divided differences of $c\phi(x)$ are equal to the product of c and the corresponding divided differences of $\phi(x)$.

2. Prove

(i) $\Delta^m x^m = m!$.

(ii) $\Delta^m x^n = (x+m)^n - \binom{m}{1}(x+m-1)^n$
$$+ \binom{m}{2}(x+m-2)^n - \ldots + (-1)^m x^n.$$

Hence show that

$$m! = m^m - \binom{m}{1}(m-1)^m + \binom{m}{2}(m-2)^m - \ldots + (-1)^{m-1}\binom{m}{m-1}.$$

3. If $x_0, x_1, x_2, \ldots, x_n$ are values which are equally spaced, the distance between neighbouring values being h, prove that

$$[x_r x_{r-1} \ldots x_0] = \frac{1}{r!}\frac{1}{h^r}\Delta^r u(x_0),$$

where $r = 1, 2, \ldots, n$.

4. If $\phi(E)$ denote a polynomial in E, prove that

$$\phi(E)\,u(x) = a^x\phi(a^\omega E)\{a^{-x}u(x)\}$$

where ω denotes the constant increment of x.

5. If $\Delta^m 0^n$ denote the value of $\Delta^m x^n$ when $x = 0$, and $E^m 0^n$ the value of $E^m x^n$ when $x = 0$, show that

(i) $E^m 0^n = mE\ E^{m-1}0^{n-1}$.

(ii) $\Delta^m 0^n = mE.\Delta^{m-1}0^{n-1} = m(\Delta^{m-1}0^{n-1} + \Delta^m 0^{n-1})$.

Write down the values of

$$\Delta\, 0^n,\ \Delta^2 0^n,\ \Delta^3 0^n.$$

6. Show that $E \doteqdot 1 + \Delta$ and use the result to obtain the formula

$$\sum_{n=0}^{\infty}\frac{u_n x^n}{n!} = e^x \sum^{\infty} \frac{x^n}{n!}\Delta^n u_0.$$

Hence find the sum of the series

$$1^3.x + 2^3.\frac{x^2}{2!} + 3^3.\frac{x^3}{3!} + 4^3.\frac{x^4}{4!} + \ldots.$$

7. Prove that

$$\Delta\left\{\frac{f(x)}{\phi(x)}\right\} = \frac{\phi(x)\,\Delta f(x) - f(x)\,\Delta\phi(x)}{\phi(x)\ \phi(x+1)}.$$

8. If $x^{(n)} = x(x-1)(x-2)\ldots(x-n+1)$ where $n > 1$ and is an integer, show that $\sum_{x=1}^{n} x^{(r)} = \frac{(n+1)^{(r+1)}}{r+1}$.

The nth term of a series u_n is a polynomial of degree 3 in n, and $u_1 = 0$, $u_2 = -16$, $u_3 = -24$, $u_4 = -12$, $u_5 = 32$. Determine u_n in terms of $n^{(1)}$, $n^{(2)}$, $n^{(3)}$, and evaluate $\sum_{r=1}^{n} u_r$. [*Lond. B.Sc.*]

9. If $f(x)$ is a polynomial of degree n, show that

$$f(x) = f(0) + x\,\Delta f(0) + \frac{1}{2!}x(x-1)\,\Delta^2 f(0) + \dots$$
$$+ \frac{1}{n!}x(x-1)\dots(x-n+1)\,\Delta^n f(0).$$

A polynomial of degree 3 takes the values -1, 4, 15, and 20 when $x = 2$, 3, 4, and 5. Find the polynomial. [*Lond. B.Sc.*]

10. A function is tabulated at constant intervals. Consider the effect in successive columns of differences of one incorrect entry.

Show that if one entry in the following table is corrected, the function tabulated is of degree 3 in x, and find $f(12\cdot72)$ correct to 4 places of decimals.

x	$f(x)$	x	$f(x)$	x	$f(x)$
12·0	0·00059	12·4	0·00950	12·8	0·02043
12·1	0·00237	12·5	0·01219	12·9	0·02297
12·2	0·00449	12·6	0·01497	13·0	0·02529
12·3	0·00688	12·7	0·01774		

[*Lond. B.Sc.*]

11. If $(x)_{r,\omega} = x(x-\omega)(x-2\omega)\dots(x-r\omega+\omega)$, express the polynomial $f(x) = x^3 - 5x^2 + 9x + 2$ in terms of $(x)_{1,2}$, $(x)_{2,2}$, $(x)_{3,2}$.

12. Solve the difference equation $u_{n+2} - (k+2)\,u_{n+1} + 2ku_n = 0$, where k is a positive constant, distinguishing between the cases $k = 2$ and $k \neq 2$.

Find the particular solution, when $k = 2$, $u_1 = 0$, $u_2 = 4$.

[*Lond. B.Sc.*]

13. Solve the difference equation

$$\Delta^3 y_x - 3\,\Delta^2 y_x - 24\,\Delta y_x + 80 y_x = 0,$$

given that the auxiliary equation has a repeated root. Find the constants in the solution if $y_0 = y_1 = 0$ and $y_2 = 45$. [*Lond. B.Sc.*]

14. Find the general solution of the equation

$$u_{x+2} - 4u_{x+1} + 4 = x.$$

15. Find the general solution of the equation

$$\Delta^2 y - \Delta y - 6y = a^x.$$

What is the solution when $a = 4$ and $a = -1$?

16. If $f(t)$ is the generating function of u_x, prove

(i) $G u_{x+n} = \dfrac{f(t)}{t^n}$, (ii) $G\,\Delta u_x = \left(\dfrac{1}{t} - 1\right) f(t)$,

(iii) $G\,\Delta^n u_x = \left(\dfrac{1}{t} - 1\right)^n f(t)$.

By writing $\dfrac{1}{t} = \left(1 + \dfrac{1}{t} - 1\right)$ and using these results, show that

$$u_{x+n} = u_x + \binom{n}{1}\Delta u_x + \binom{n}{2}\Delta^2 u_x + \dots.$$

17. If p and n denote positive integers or zero and

(i) $a_{np} \to 0$ for every fixed $p \geqslant 0$ as $n \to \infty$,

(ii) $\sum_{p=0}^{\infty} |a_{np}| < k$ where k is a constant, every $n > 0$,

(iii) $x_n \to 0$ as $n \to \infty$,

show that $\sigma_n = a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n \to 0$ as $n \to \infty$.

[Choose $\epsilon > 0$ arbitrarily; then there exists a positive integer M such that for $n > M$, $|x_n| < \epsilon/2k$, so that

$$|\sigma_n| < |a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nM}x_M| + \tfrac{1}{2}\epsilon.$$

Since $a_{np} \to 0$, there exists a positive integer N such that for all $n > N$,

$$|a_{np}x_p| < \epsilon/2M.$$

Writing $p = 0, 1, 2, \ldots M$ in the above inequality for $|\sigma_n|$ it follows that

$$|\sigma_n| \leqslant |\sum_{p=0}^{M} a_{np}x_p| < \frac{M\epsilon}{2M} + \frac{\epsilon}{2} < \epsilon,$$

from which the result follows.

18. Use Ex. 17 to show that if $\sum_{n=0} a_n$ converges to L, then

$$\sum_{n=0}^{\infty} \frac{(1 + E)^n a_0}{2^{n+1}}$$

also converges to L.

[*Hint.*—In Ex. 17 choose $a_{np} = \frac{1}{2^n}\binom{n}{p}$, $x_n = a_{k+n}$, and prove

$$\lim_{n \to \infty} \frac{(1 + E)^n a_k}{2} = 0.]$$

CHAPTER XI

SUMMATION OF SERIES

IN earlier chapters various methods of summation have been considered. Before elaborating further methods we give examples of the application of those given earlier. We shall consider examples on (i) geometrical progressions, (ii) sum of powers of integers, (iii) induction, (iv) binomial theorem, (v) exponential series, (vi) logarithmic series and (vii) recurring series.

11·11. Geometric Series

Example.—*Sum the series* $1 + x - x^2 - x^3 + x^4 + x^5 - \ldots$ *to infinity* $(-1 < x < 1)$. [*Camb. Sch.*]

Since the series converges absolutely for $|x| < 1$, the order of the terms may be deranged without affecting the sum. Hence if S denote the sum of the series

$$\begin{aligned} S &= 1 + x^4 + x^8 + x^{12} + \ldots \\ &\quad + x + x^5 + x^9 + x^{13} + \ldots \\ &\quad - x^2 - x^6 - x^{10} - x^{14} - \ldots \\ &\quad - x^3 - x^7 - x^{11} - x^{15} - \ldots \\ &= \frac{1}{1-x^4} + \frac{x}{1-x^4} - \frac{x^2}{1-x^4} - \frac{x^3}{1-x^4} \\ &= (1 + x - x^2 - x^3)/(1 - x^4) = (1 + x)/(1 + x^2), \quad |x| < 1. \end{aligned}$$

11·12. Sums of Powers of Integers

Examples.—(1) *Sum to* n *terms the series*

$$1 + 2^2 + 3^3 + 4 + 5^2 + 6^3 + 7 + 8^2 + 9^3 + \ldots \quad [\textit{Camb. Sch.}]$$

The sum depends on the form of n, i.e. whether it has the form $3p$, $3p + 1$, $3p + 2$, where p is a positive integer. Let S_n denote the sum to n terms. Then

$$\begin{aligned} S_{3p} &= \sum_{r=1}^{p} (3r - 2) + \sum_{r=1}^{p} (3r - 1)^2 + \sum_{r=1}^{p} 3^3 r^3 \\ &= 3\sum_{r=1}^{p} r - 2p + \sum_{r=1}^{p} (9r^2 - 6r + 1) + 27\sum_{r=1}^{p} r^3 \\ &= 3\sum_{r=1}^{p} r - 2p + 9\sum_{r=1}^{p} r^2 - 6\sum_{r=1}^{p} r + p + 27\sum_{r=1}^{p} r^3 \\ &= -p - 3\sum_{r=1}^{p} r + 9\sum_{r=1}^{p} r^2 + 27\sum_{r=1}^{p} r^3 . \end{aligned}$$

$$= -p - 3\cdot\tfrac{1}{2}p\,(p+1) + 9\cdot\tfrac{1}{6}p\,(p+1)\,(2p+1) + 27\cdot\tfrac{1}{4}p^2\,(p+1)^2$$
$= \frac{27}{4}p^4 + \frac{33}{2}p^3 + \frac{39}{4}p^2 - p$, on reduction,
$$= \tfrac{1}{12}n^4 + \tfrac{11}{18}n^3 + \tfrac{13}{12}n^2 - \tfrac{1}{3}n.$$
If n is of the form $3p+1$, we have $S_{3p+1} = S_{3p} + 3p + 1$

i.e. $S_{3p+1} = \frac{27}{4}p^4 + \frac{33}{2}p^3 + \frac{39}{4}p^2 + 2p + 1.$

Substituting $p = \frac{1}{3}(n-1)$,
$$S_n = \tfrac{1}{12}(n-1)^4 + \tfrac{11}{18}(n-1)^3 + \tfrac{13}{12}(n-1)^2 + \tfrac{2}{3}(n-1) + 1.$$
If n is of the form $3p+2$,
$$S_{3p+2} = S_{3p+1} + (3p+2)^2 = \tfrac{27}{4}p^4 + \tfrac{33}{2}p^3 + \tfrac{75}{4}p^2 + \tfrac{14}{3}p + 5.$$
Writing $p = \frac{1}{3}(n-2)$,
$$S_n = \tfrac{1}{12}(n-2)^4 + \tfrac{11}{18}(n-2)^3 + \tfrac{25}{12}(n-2)^2 + \tfrac{14}{3}(n-2) + 5.$$

(2) *Express*
$$(x+n)^3 + (2x + \overline{n-1})^3 + (2^2x + \overline{n-2})^3$$
$$+ (2^3x + \overline{n-3})^3 + \ldots + (2^{n-1}x + 1)^3$$
in the form $px^3 + qx^2 + rx + s$ *and verify that*
$$r = 18\,(7p+1)^{\frac{1}{3}} - 6s^{\frac{1}{2}} - 6 - 12\,(7p+1)^{\frac{2}{3}} + 9q.$$
[*Camb. Sch.*]

The sum may be written in the form
$$\sum_{t=0}^{n-1} (2^tx + n - t)^3$$
$$= \sum_{t=0}^{n-1} \{2^{3t}x^3 + 3.2^{2t}x^2(n-t) + 3.2^tx\,(n-t)^2 + (n-t)^3\}.$$
Hence $p = \sum_{t=0}^{n-1} 2^{3t}$, $q = 3\sum_{t=0}^{n-1} 2^{2t}(n-t)$, $r = 3\sum_{t=0}^{n-1} 2^t(n-t)^2$,
$$s = \sum_{t=0}^{n-1} (n-t)^3.$$
Hence $p = \{(2^3)^n - 1\}/(2^3 - 1) = \frac{1}{7}(2^{3n} - 1)$,

and $s = \sum_{t=1}^{n} t^3 = \frac{1}{4}n^2(n+1)^2.$

In order to find q write
$$s_n = \tfrac{1}{3}q = \sum_{t=0}^{n-1} 2^{2t}(n-t)$$
$$= n + (n-1).2^2 + (n-2).2^4 + \ldots + 2.2^{2n-4} + 1.2^{2n-2}$$
Then $s_{n-1} = n - 1 + (n-2).2^2 + (n-3).2^4 + \ldots + 1.2^{2n-4}$.

Thus $s_n - s_{n-1} = 1 + 2^2 + 2^4 + \ldots + 2^{2n-4} + 2^{2n-2} = (2^{2n} - 1)/3$.

Hence $s_n = (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + (s_{n-2} - s_{n-3}) + \ldots + (s_2 - s_1) + s_1$
$$= \tfrac{1}{3}(2^{2n} - 1) + \tfrac{1}{3}(2^{2n-2} - 1) + \tfrac{1}{3}(2^{2n-4} - 1) + \ldots + \tfrac{1}{3}(2^4 - 1) + 1$$
$$= \tfrac{1}{3}.2^4.\tfrac{1}{3}\{2^{2(n-1)} - 1\} - \tfrac{1}{3}(n-1) + 1 = \tfrac{1}{9}\{2^{2n+2} - 3n - 4\}.$$
Hence $q = \frac{1}{3}\{2^{2n+2} - 3n - 4\}$.

In order to find r consider $\sigma_n = \sum_{t=0}^{n-1} 2^t (n-t)^2$. Then

$$\sigma_n = n^2 + (n-1)^2.2 + (n-2)^2.2^2 + \ldots + 2^2.2^{n-2} + 1^2.2^{n-1}$$
$$\sigma_{n-1} = (n-1)^2 + (n-2)^2.2 + (n-3)^2.2^2 + \ldots + 1^2.2^{n-2}$$
$$\sigma_n - \sigma_{n-1} = 2n - 1 + (2n-3).2 + (2n-5).2^2 + \ldots + 3.2^{n-2} + 1.2^{n-1}.$$

Write $\rho_n = \sigma_n - \sigma_{n-1}$ so that

$$\rho_{n-1} = 2n - 3 + (2n-5).2 + (2n-7).2^2 + \ldots + 1.2^{n-2}.$$

Thus $\rho_n - \rho_{n-1} = 2 + 2.2 + 2.2^2 + \ldots + 2.2^{n-2} + 2^{n-1}$

$$= 2.(2^{n-1} - 1) + 2^{n-1} = 3.2^{n-1} - 2;$$

$$\begin{aligned} \therefore\ \rho_n &= (\rho_n - \rho_{n-1}) + (\rho_{n-1} - \rho_{n-2}) + (\rho_{n-2} - \rho_{n-3}) + \ldots + (\rho_2 - \rho_1) + \rho_1 \\ &= (3.2^{n-1} - 2) + (3.2^{n-2} - 2) + (3.2^{n-3} - 2) + \ldots + (3.2^1 - 2) + 1 \\ &= 3\,(2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2) - 2\,(n-1) + 1 \\ &= 6\,(2^{n-1} - 1) - 2n + 3 = 6.2^{n-1} - 2n - 3. \end{aligned}$$

Again
$$\begin{aligned} \sigma_n &= (\sigma_n - \sigma_{n-1}) + (\sigma_{n-1} - \sigma_{n-2}) \\ &\qquad + (\sigma_{n-2} - \sigma_{n-3}) + \ldots + (\sigma_2 - \sigma_1) + \sigma_1 \\ &= \rho_n + \rho_{n-1} + \rho_{n-2} + \ldots + \rho_2 + \sigma_1 \\ &= \{6.2^{n-1} - 2n - 3\} + \{6.2^{n-2} - 2\,(n-1) - 3\} \\ &\qquad + \{6.2^{n-3} - 2\,(n-2) - 3\} + \ldots + \{6.2 - 2.2 - 3\} + 1 \\ &= 6\,\{2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2\} \\ &\qquad - 2\,\{n + (n-1) + (n-2) + \ldots 2\} - 3\,(n-1) + 1 \\ &= 12\,(2^{n-1} - 1) - (n-1)(n+2) - 3\,(n-1) + 1 \\ &= 12.2^{n-1} - n^2 - 4n - 6. \end{aligned}$$

Hence $r = 36.2^{n-1} - 3n^2 - 12n - 18.$

In order to complete the question it is necessary to evaluate

$$18\,(7p+1)^{\frac{1}{3}} - 6s^{\frac{1}{2}} - 6 - 12\,(7p+1)^{\frac{2}{3}} + 9q.$$

Now $7p + 1 = 2^{3n}$, $(7p+1)^{\frac{1}{3}} = 2^n$, $(7p+1)^{\frac{2}{3}} = 2^{2n}$. Also

$$s^{\frac{1}{2}} = \tfrac{1}{2}n\,(n+1).$$

Substituting in the expression we obtain

$$\begin{aligned} &18.2^n - 3n\,(n+1) - 6 - 12.2^{2n} + 3\,(2^{2n+2} - 3n - 4) \\ &= 36.2^{n-1} - 3n^2 - 3n - 6 - 12.2^{2n} + 12.2^{2n} - 9n - 12 \\ &= 36.2^{n-1} - 3n^2 - 12n - 18 = r. \end{aligned}$$

11·13. Induction

Examples.—(1) *By induction, show that the sum to n terms of the series whose rth term is $\tan^{-1}(1/2r^2)$ is $\tan^{-1}(2n+1) - \frac{1}{4}\pi$.* [*Lond. B.Sc.*]

When $n = 1$, $\tan^{-1}(2n+1) - \frac{1}{4}\pi = \tan^{-1}3 - \frac{1}{4}\pi$. This is equal to $\tan^{-1}(1/2)$ provided

$$\tan^{-1}3 - \tan^{-1}\tfrac{1}{2} = \tfrac{1}{4}\pi, \text{ i.e. } \frac{3 - \frac{1}{2}}{1 + 3\cdot\frac{1}{2}} = \tan\tfrac{1}{4}\pi = 1.$$

This condition is obviously satisfied. Thus the result is true when $n = 1$.

Now assume that it is true for $n = p$, i.e.

$$\sum_{r=1}^{p} \tan^{-1}(1/2r^2) = \tan^{-1}(2p+1) - \tfrac{1}{4}\pi.$$

Then $\displaystyle\sum_{r=1}^{p+1} \tan^{-1}(1/2r^2) = \tan^{-1}(2p+1) - \tfrac{1}{4}\pi + \tan^{-1}\{1/2\,(p+1)^2\}.$

This is equal to $\tan^{-1}(2p+3) - \frac{1}{4}\pi$ provided

$$\tan^{-1}(2p+3) = \tan^{-1}(2p+1) + \tan^{-1}\frac{1}{2\,(p+1)^2},$$

i.e. $$2p+3 = \left\{2p+1+\frac{1}{2\,(p+1)^2}\right\}\Big/\left\{1-\frac{2p+1}{2\,(p+1)^2}\right\}.$$

The right-hand side is equal to $\dfrac{4p^3+10p^2+8p+3}{2p^2+2p+1} = 2p+3$. Thus the result is true for $n = p+1$ if it is true for $n = p$, and so it is true in general.

(2) *Prove that, if S_r is the sum of the rth powers of the numbers*

$a + d,\ a + 2d,\ a + 3d,\ \ldots,\ (a + nd)$, *then*

$nS_2 - S_1^2 = \frac{1}{12}n^2(n^2-1)d^2$, and $n^2S_3 - 3nS_1S_2 + 2S_1^3 = 0$. [*Camb. Sch.*]

$$S_1 = (a+d) + (a+2d) + \ldots + (a+nd)$$
$$= \frac{n}{2}\{2a + (n+1)\,d\}$$
$$S_2 = (a+d)^2 + (a+2d)^2 + \ldots + (a+nd)^2$$
$$S_3 = (a+d)^3 + (a+2d)^3 + \ldots + (a+nd)^3.$$

The method of induction is used. The results are easily verified in the case $n = 1$. Consider first

$$nS_2 - S_1^2 = \frac{1}{12}n^2(n^2-1)\,d^2,$$

i.e. $$nS_2 = \frac{n^2}{4}\{2a+(n+1)\,d\}^2 + \frac{1}{12}n^2(n^2-1)d^2$$
$$= \frac{n^2}{6}\{6a^2 + 6a\,(n+1)\,d + (2n^2+3n+1)\,d^2\}$$

i.e. $S_2 = \frac{1}{6}n\,\{6a^2 + 6a\,(n+1)\,d + (2n^2+3n+1)\,d^2\}$.

Assume the result is true for $n = r$ and denote the sum in this case by $S_2^{(r)}$. Then

$$S_2^{(r+1)} = S_2^{(r)} + \{a + (r+1)\,d\}^2$$
$$= \tfrac{1}{6}r\,\{6a^2 + 6a\,(r+1)\,d + (2r^2+3r+1)\,d^2\} + \{a^2 + 2a\,(r+1)\,d + (r+1)d^2\}$$
$$= \tfrac{1}{6}\{6a^2 + 12a\,(r+1)\,d + 6\,(r^2+2r+1)\,d^2 + 6a^2r + 6a\,(r^2+r)\,d + (2r^3+3r^2+r)\,d^2\}$$
$$= \tfrac{1}{6}\{(r+1)\,6a^2 + 12a\,(r+1)\,d + 6\,(r^2+2r+1)\,d^2 + 6a\,(r^2+r)\,d + r\,(r+1)\,(2r+1)\,d^2\}$$
$$= \tfrac{1}{6}(r+1)\,\{6a^2 + 12ad + 6ard + 6\,(r+1)\,d^2 + r\,(2r+1)\,d^2\}$$
$$= \tfrac{1}{6}(r+1)\,\{6a^2 + 6ad\,(r+2) + [2(r+1)^2 + 3\,(r+1) + 1]\,d^2\}.$$

Hence the result is true for $n = r+1$ and so is true in general.

Next consider $n^2S_3 = 3nS_1S_2 - 2S_1^3$

$$= S_1(3nS_2 - 2S_1^2)$$

$$= \tfrac{1}{2}n\,\{2a + (n + 1)\,d\}\,\{3nS_2 - 2S_1^2\}.$$

Now $3nS_2 - 2S_1^2 = 3\,(nS_2 - S_1^2) + S_1^2$

$$= \tfrac{1}{4}n^2(n^2 - 1)\,d^2 + \tfrac{1}{4}n^2\{2a + (n + 1)\,d\}^2$$

$$= \tfrac{1}{2}n^2\{2a^2 + 2a\,(n + 1)\,d + (n^2 + n)\,d^2\}.$$

i.e. $S_3 = \tfrac{1}{4}n\,\{2a + (n + 1)\,d\}\,\{2a^2 + 2a\,(n + 1)\,d + (n^2 + n)\,d^2\}.$

Assume that this result is true for $n = r$ and denote the corresponding sum by $S_3^{(r)}$. Then

$$\begin{aligned} S_3^{(r+1)} &= S_3^{(r)} + \{a + (r + 1)\,d\}^3 \\ &= \tfrac{1}{4}r\,\{2a + (r + 1)\,d\}\,\{2a^2 + 2a\,(r + 1)\,d + (r^2 + r)\,d^2\} \\ &\qquad\qquad + \{a + (r + 1)\,d\}^3 \\ &= \tfrac{1}{4}\,\{4a^3 + 12a^2(r + 1)\,d + 12a\,(r + 1)^2d^2 + 4\,(r + 1)^3d^3 \\ &\qquad + 4a^3r + 4a^2r\,(r + 1)\,d + 2ar\,(r^2 + r)\,d^2 + 2a^2(r + 1)\,rd \\ &\qquad\qquad + 2a\,(r + 1)^2rd^2 + r\,(r + 1)\,(r^2 + r)\,d^3\} \\ &= \tfrac{1}{4}\,(r + 1)\,\{4a^3 + 12a^2d + 12a\,(r + 1)\,d^2 + 4d^3 + 4a^2rd \\ &\qquad + 2ar^2d^2 + 2a^2rd + 2ar\,(r + 1)\,d^2 + r\,(r^2 + r)\,d^3\} \\ &= \tfrac{1}{4}\,(r + 1)\,\{2a + (r + 2)\,d\} \\ &\qquad\qquad \{2a^2 + 2a\,(r + 2)\,d + [(r + 1)^2 + (r + 1)]\,d^2\}. \end{aligned}$$

Hence the result is true for $n = r + 1$ and the result follows.

11·14. The Binomial Series

Examples.—(1) *Sum the series, n being a positive integer:*

$$\frac{1}{(2n)!\,(2n)!} + \frac{1}{(2n - 2)!\,(2n + 2)!} + \frac{1}{(2n - 4)!\,(2n + 4)!} + \ldots$$

$$+ \frac{1}{2!\,(4n - 2)!} + \frac{1}{(4n)!}. \qquad [\textit{Camb. Sch.}]$$

The given series is

$$\frac{1}{(4n)!}\left\{1 + \frac{4n\,(4n - 1)}{2!} + \frac{4n\,(4n - 1)\,(4n - 2)\,(4n - 3)}{4!} + \ldots \right.$$

$$\left. + \frac{4n\,(4n - 1)\ldots(4n - 2n + 1)}{(2n)!}\right\}$$

$$= \frac{1}{(4n)!}\,\{1 + {}_{4n}C_2 + {}_{4n}C_4 + \ldots + {}_{4n}C_{2n}\}.$$

From the properties of the binomial coefficients,

$$1 + {}_{4n}C_2 + {}_{4n}C_4 + \ldots + {}_{4n}C_{2n-2} + {}_{4n}C_{2n} + {}_{4n}C_{2n+2} + \ldots + {}_{4n}C_{4n}$$

$$= \tfrac{1}{2}.2^{4n} = 2^{4n-1}.$$

Also ${}_{4n}C_r = {}_{4n}C_{4n-r}$. Hence

$$1 + {}_{4n}C_2 + {}_{4n}C_4 + \ldots + {}_{4n}C_{2n} = {}_{4n}C_{4n} + {}_{4n}C_{4n-2} + \ldots + {}_{4n}C_{2n}.$$

$$\therefore\ 2\,\{1 + {}_{4n}C_2 + {}_{4n}C_4 + \ldots + {}_{4n}C_{2n}\} - {}_{4n}C_{2n} = 2^{4n-1},$$

i.e. $1 + {}_{4n}C_2 + {}_{4n}C_4 + \ldots + {}_{4n}C_{2n} = 2^{4n-2} + \tfrac{1}{2}.\dfrac{(4n)!}{(2n)!\,(2n)!}.$

Hence the sum of the given series is $\dfrac{2^{4n-2}}{(4n)!} + \dfrac{1}{2\,(2n)!\,(2n)!}.$

(2) *Prove that*

(i) $\dfrac{1}{2^3 . 3\,!} - \dfrac{1 . 3}{2^4 . 4\,!} + \dfrac{1 . 3 . 5}{2^5 . 5\,!} - \ldots$ *to infinity* $= \frac{23}{24} - \frac{2}{3}\sqrt{2}$;

(ii) $1 + \dfrac{n}{m} + \dfrac{n(n-1)}{m(m-1)} + \dfrac{n(n-1)(n-2)}{m(m-1)(m-2)} + \ldots$

to $n + 1$ *terms* $= \dfrac{m+1}{m-n+1}$,

provided m *is not less than* n. [*Camb. Sch.*]

(i) $\dfrac{1}{2^3 . 3\,!} - \dfrac{1 . 3}{2^4 . 4\,!} + \dfrac{1 . 3 . 5}{2^5 . 5\,!} - \ldots$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{3\,!} - \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{4\,!} + \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{5\,!} - \ldots$$

$$= \frac{1}{3}\left\{-\frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)}{3\,!} - \frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{4\,!} - \frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{5\,!} - \ldots\right\}$$

Now $(1 + x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \dfrac{\frac{3}{2}\cdot\frac{1}{2}}{2\,!}x^2 + \dfrac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)}{3\,!}x^3 + \dfrac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{4\,!}x^4 + \ldots$

When $x = 1$ the expansion becomes

$$2^{\frac{3}{2}} = 1 + \tfrac{3}{2} + \tfrac{3}{8} - \left\{-\frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)}{3\,!} - \frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{4\,!} - \ldots\right\}.$$

Hence if s denote the sum of the given series

$$2^{\frac{3}{2}} = 1 + \tfrac{3}{2} + \tfrac{3}{8} - 3s \text{ giving } s = \tfrac{23}{24} - \tfrac{2}{3}\sqrt{2}.$$

(ii) The $(r + 1)$th term of the series is $\dfrac{n(n-1)\ldots(n-r+1)}{m(m-1)\ldots(m-r+1)}$.

Expressing $1/m(m-1)(m-2)\ldots(m-r+1)$ as a sum of partial fractions

$$\frac{1}{m(m-1)(m-2)\ldots(m-r+1)} = \frac{1}{m}\frac{(-1)^{r-1}}{(r-1)\,!} + \frac{1}{(m-1)}\cdot\frac{(-1)^{r-2}}{1\,!\,(r-2)\,!} + \frac{1}{(m-2)}\cdot\frac{(-1)^{r-3}}{2\,!\,(r-3)\,!} + \ldots$$
$$+ \frac{1}{m-s}\cdot\frac{(-1)^{r-s-1}}{(s-1)\,!\,(r-s-1)\,!} + \ldots + \frac{1}{m-r+1}\cdot\frac{1}{(r-1)\,!}.$$

Writing ${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$ the sum of the series is

$$1 + \sum_{r=1}^{n}\frac{{}_nP_r}{m(m-1)(m-2)\ldots(m-r+1)}$$
$$= 1 + \left\{\frac{{}_nP_1}{m}\right\} + \left\{\frac{{}_nP_2}{m}\cdot\frac{-1}{1\,!} + \frac{{}_nP_2}{m-1}\cdot\frac{1}{1\,!}\right\}$$
$$+ \left\{\frac{{}_nP_3}{m}\cdot\frac{1}{2\,!} + \frac{{}_nP_3}{m-1}\cdot\frac{-1}{1\,!\,1\,!} + \frac{{}_nP_3}{m-2}\cdot\frac{1}{2\,!}\right\}$$
$$+ \left\{\frac{{}_nP_4}{m}\cdot\frac{-1}{3\,!} + \frac{{}_nP_4}{m-1}\cdot\frac{1}{1\,!\,2\,!} + \frac{{}_nP_4}{m-2}\cdot\frac{-1}{2\,!\,1\,!} + \frac{{}_nP_4}{m-3}\cdot\frac{1}{3\,!}\right\}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left\{\frac{{}_nP_r}{m}\cdot\frac{(-1)^{r-1}}{(r-1)!}+\frac{{}_nP_r}{m-1}\cdot\frac{(-1)^{r-2}}{1!\,(r-2)!}+\frac{{}_nP_r}{m-2}\cdot\frac{(-1)^{r-3}}{2!\,(r-3)}+\ldots\right.$$
$$\left.+\frac{{}_nP_r}{m-r+1}\cdot\frac{1}{(r-1)!}\right\}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\left\{\frac{{}_nP_n}{m}\cdot\frac{(-1)^{n-1}}{(n-1)!}+\frac{{}_nP_n}{m-1}\cdot\frac{(-1)^{n-2}}{1!\,(n-2)!}+\frac{{}_nP_n}{m-2}\cdot\frac{(-1)^{n-3}}{2!\,(n-3)!}+\ldots\right.$$
$$\left.+\frac{{}_nP_n}{m-n+1}\cdot\frac{1}{(n-1)!}\right\}$$
$$=1+\frac{1}{m}\left\{{}_nP_1-\frac{{}_nP_2}{1!}+\frac{{}_nP_3}{2!}-\frac{{}_nP_4}{3!}+\ldots+(-1)^{n-1}\frac{{}_nP_n}{(n-1)!}\right\}$$
$$+\frac{1}{m-1}\left\{\frac{{}_nP_2}{1!}-\frac{{}_nP_3}{1!\,1!}+\frac{{}_nP_4}{1!\,2!}-\frac{{}_nP_5}{1!\,3!}+\ldots\right.$$
$$\left.+(-1)^{n-2}\frac{{}_nP_n}{1!\,(n-2)!}\right\}$$
$$+\frac{1}{m-2}\left\{\frac{{}_nP_3}{2!}-\frac{{}_nP_4}{2!\,1!}+\frac{{}_nP_5}{2!\,2!}-\frac{{}_nP_6}{2!\,3!}+\ldots\right.$$
$$\left.+(-1)^{n-3}\frac{{}_nP_n}{2!\,(n-3)!}\right\}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{1}{m-r+1}\left\{\frac{{}_nP_r}{(r-1)!}-\frac{{}_nP_{r+1}}{(r-1)!\,1!}+\frac{{}_nP_{r+2}}{(r-1)!\,2!}-\ldots\right.$$
$$\left.+(-1)^{n-r}\frac{{}_nP_n}{(r-1)!\,(n-r)!}\right\}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{1}{m-n+2}\left\{\frac{{}_nP_{n-1}}{(n-2)!}-\frac{{}_nP_n}{(n-2)!\,1!}\right\}$$
$$+\frac{1}{m-n+1}\cdot\frac{{}_nP_n}{(n-1)!}$$
$$=1+\frac{n}{m}\{1-{}_{n-1}C_1+{}_{n-1}C_2-{}_{n-1}C_3+\ldots+(-1)^{n-1}\,{}_{n-1}C_{n-1}\}$$
$$+\frac{n(n-1)}{1!\,(m-1)}\{1-{}_{n-2}C_1+{}_{n-2}C_2-{}_{n-2}C_3+\ldots$$
$$+(-1)^{n-2}\,{}_{n-2}C_{n-2}\}$$
$$+\frac{n(n-1)(n-2)}{2!\,(m-2)}\{1-{}_{n-3}C_1+{}_{n-3}C_2-{}_{n-3}C_3+\ldots$$
$$+(-1)^{n-3}\,{}_{n-3}C_{n-3}\}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{n(n-1)\ldots(n-r+1)}{(r-1)!\,(m-r+1)}\{1-{}_{n-r}C_1+{}_{n-r}C_2-{}_{n-r}C_3+\ldots$$
$$+(-1)^{n-r}\,{}_{n-r}C_{n-r}\}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{n(n-1)\ldots 2}{(n-2)!\,(m-n+2)}\{1-{}_1C_1\}+\frac{n}{m-n+1}.$$

Now if p is a positive integer,

$$(1-x)^p = 1 - {}_pC_1x + {}_pC_2x^2 - \ldots + (-1)^r {}_pC_r x^r + \ldots + (-1)^p {}_pC_p,$$

for all values of x. Writing $x = 1$,

$$0 = 1 - {}_pC_1 + {}_pC_2 - \ldots r\,(-1)^r {}_pC_r + \ldots + (-1)^p {}_pC_p.$$

Substituting $p = n-1,\ n-2,\ n-3,\ \ldots 1$ in succession it follows that all the expressions in { } are 0. Hence the required sum is

$$1 + \frac{n}{m-n+1} = \frac{m+1}{m-n+1}.$$

11·15. The Exponential Series

Examples.—(1) *Sum to infinity the series of which the rth term is*

$$(3r^2 - 4r - 2)/r\,!.$$ [*Lond. B.Sc.*]

Write $u_r = (3r^2 - 4r - 2)/r\,!$. Then

$$u_r = \frac{3r}{(r-1)\,!} - \frac{4}{(r-1)\,!} - \frac{2}{r\,!} = \frac{3(r-1)+3}{(r-1)\,!} - \frac{4}{(r-1)\,!} - \frac{2}{r\,!}$$

$$= \frac{3}{(r-2)\,!} - \frac{1}{(r-1)\,!} - \frac{2}{r\,!},\quad r \geqslant 2.$$

Also $u_1 = -\dfrac{1}{0\,!} - \dfrac{2}{1\,!}$.

Hence $$\sum_{r=1}^{\infty} \frac{3r^2-4r-2}{r\,!} = 3\sum_{r=2}^{\infty}\frac{1}{(r-2)\,!} - \sum_{r=1}^{\infty}\frac{1}{(r-1)\,!} - 2\sum_{r=1}^{\infty}\frac{1}{r\,!}, \quad \ldots \text{(i)}$$

provided each of the series converges.

Now $$\sum_{r=2}^{\infty}\frac{1}{(r-2)\,!} = \sum_{s=0}^{\infty}\frac{1}{s\,!},\quad \sum_{r=1}^{\infty}\frac{1}{(r-1)\,!} = \sum_{s=0}^{\infty}\frac{1}{s\,!}.$$

Also the series $\displaystyle\sum_{s=0}^{\infty}\frac{1}{s\,!}$ converges to e.

Thus the series on the right hand side of (i) are convergent. Hence

$$\sum_{r=1}^{\infty}\frac{3r^2-4r-2}{r\,!} = 3e - e - 2e + 2 = 2.$$

Alternatively the series may be summed as follows. Let S_n denote the sum to n terms, S the sum to infinity. Then $S = \lim\limits_{n\to\infty} S_n$, if this limit exists.

$$\begin{array}{rcccc} S_n = & & -\dfrac{1}{0\,!} & - & \dfrac{2}{1\,!} \\ & +\dfrac{3}{0\,!} & -\dfrac{1}{1\,!} & - & \dfrac{2}{2\,!} \\ & +\dfrac{3}{1\,!} & -\dfrac{1}{2\,!} & - & \dfrac{2}{3\,!} \\ & +\dfrac{3}{2\,!} & -\dfrac{1}{3\,!} & - & \dfrac{2}{4\,!} \\ & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots \end{array}$$

$$\begin{array}{lll} +\dfrac{3}{(n-4)!} & -\dfrac{1}{(n-3)!} & -\dfrac{2}{(n-2)!} \\ +\dfrac{3}{(n-3)!} & -\dfrac{1}{(n-2)!} & -\dfrac{2}{(n-1)!} \\ +\dfrac{3}{(n-2)!} & -\dfrac{1}{(n-1)!} & -\dfrac{2}{n!} \end{array}$$

Adding by the method indicated by the dotted lines,

$$S_n = \frac{3}{0!} - \frac{1}{0!} - \frac{1}{(n-1)!} - \frac{2}{(n-1)!} - \frac{2}{n!} = 2 - \frac{3}{(n-1)!} - \frac{2}{n!}$$

Letting $n \to \infty$, $S = 2$.

(2) *Sum to infinity the series whose nth term is* $(n^2 + n)\, x^n/n\,!$.

Let u_n denote the nth term. Then

$$u_n = \frac{n(n+1)x^n}{n!} = \frac{(n+1)x^n}{(n-1)!}$$

$$= \frac{x^n}{(n-2)!} + \frac{2x^n}{(n-1)!}, \quad n > 2.$$

Hence $\displaystyle\sum_{n=1}^{\infty} u_n = 2x + \sum_{n=2}^{\infty} u_n = 2x + \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} + 2\sum_{n=2}^{\infty} \frac{x^n}{(n-1)!}$,

since both the series on the right obviously converge for finite x,

$$\text{i.e.} \quad \sum_{n=1}^{\infty} u_n = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} + 2x \sum_{n=0}^{\infty} \frac{x^n}{n!} = x^2 e^x + 2xe^x.$$

11·16. The Logarithmic Series

Examples.—(1) *Express* $x/(3x-1)(3x-2)$ *in partial fractions. Prove that* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{(3n-1)(3n-2)} \frac{1}{2^{3n-3}} = \tfrac{8}{9}\log_e 3.$ [*Lond. B.Sc.*]

$$\text{Write} \quad \frac{x}{(3x-1)(3x-2)} \equiv \frac{A}{(3x-1)} + \frac{B}{(3x-2)},$$

$$\text{i.e. } x \equiv A(3x-2) + B(3x-1).$$

Put $x = \frac{1}{3}, \frac{2}{3}$ in succession, giving $A = -\frac{1}{3}$, $B = \frac{2}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{(3n-1)(3n-2)} \cdot \frac{1}{2^{3n-3}}$$

$$= \tfrac{8}{3} \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{-1}{3n-1} \cdot \frac{1}{2^{3n-2}} + \frac{1}{3n-2} \cdot \frac{1}{2^{3n-2}} \right\}$$

$$= \tfrac{8}{3} \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{1}{3n} \cdot \frac{1}{2^{3n-1}} + \frac{-1}{3n-1} \cdot \frac{1}{2^{3n-2}} + \frac{1}{3n-2} \cdot \frac{1}{2^{3n-3}} \right\}$$

$$- \tfrac{8}{3} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{3n} \cdot \frac{1}{2^{3n-1}},$$

provided each of the series converges. The first series on the right may be written in the form

$$\tfrac{4}{3} \sum_{r=1}^{\infty} (-)^{r-1} \frac{1}{r} \cdot \frac{1}{2^r}$$

and the second as $-\tfrac{2}{3} \sum_{r=1}^{\infty} (-1)^{r-1} \cdot \frac{1}{r} \cdot \frac{1}{8^r}$.

Now $\log_e(1 + x) = \sum_{r=1}^{\infty} (-1)^{r-1} x^r/r$, provided $|x| < 1$.

Putting $x = \frac{1}{2}$, $x = \frac{1}{8}$ the two series are obtained. Hence the sum of the given series is

$$\tfrac{2}{3} \log (1 + \tfrac{1}{2}) - \tfrac{2}{3} \log (1 + \tfrac{1}{8}),$$

$$\text{i.e. } \tfrac{2}{3} \log 3 - \tfrac{2}{3} \log 2 - \tfrac{4}{3} \log 3 + \tfrac{6}{3} \log 2, \text{ i.e. } \tfrac{2}{3} \log 3.$$

(2) *Show that* $\log_2 e - \log_4 e + \log_8 e - \log_{16} e + \ldots = 1$.

[*S.T., Prelim.*]

The series may be written in the form

$$\frac{1}{\log_e 2} - \frac{1}{\log_e 4} + \frac{1}{\log_e 8} - \frac{1}{\log_e 16} + \ldots$$

$$= \frac{1}{\log_e 2} - \frac{1}{2 \log_e 2} + \frac{1}{3 \log_e 2} - \frac{1}{4 \log_e 2} + \ldots$$

$$= \{1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\}/\log_e 2 = \log_e 2/\log_e 2 = 1.$$

11·17. Recurring Series

Example.—*Sum to n terms the series whose rth term is* $(2r + 1)\, 3^r$.

[*Lond. B.Sc.*]

From Chapter IX., § 9·8, it follows that the series $\Sigma(2r + 1)\, x^r$ is a recurring one whose scale of relation is $(1 - x)^2$.

Let s_n be the sum of the series to n terms. Then if x is 3

$$s_n = 3x + 5x^2 + 7x^3 + \ldots + (2n + 1)\, x^n.$$

$$(1 - x)\, s_n = 3x + 2x^2 + 2x^3 + \ldots + 2x^n - (2n + 1)\, x^{n+1}$$

$$= 3x + 2x^2 \cdot \frac{x^{n-1} - 1}{x - 1} - (2n + 1)\, x^{n+1},$$

Hence $s_n = \{3x - x^2 - (2n + 3)\, x^{n+1} + (2n + 1) x^{n+2}\}/(x - 1)^2$.

Substituting $x = 3$, $s_n = n \cdot 3^{n+1}$.

11·2. Application of Partial Fractions to Summation Series

Let u_n be the nth term of the given series, u the sum of the series. Then it may be possible, by the method of partial fractions to express u_n as the sum or difference of other terms. Then suppose, *e.g.* that

$$u_n = v_n + w_n + x_n + \ldots$$

Then if the series Σv_n, Σw_n, Σx_n, ... can be summed separately to v, w, x, ..., it follows that

$$u = v + w + x + \ldots$$

It should be observed that if the given series contains an infinite number of terms it is necessary that the series Σv_n, Σw_n, Σx_n, ... all converge. In such a case it is frequently convenient to consider first the sum to n terms and then let $n \to \infty$.

Examples.—(1) *Show that*

$$\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \ldots \textit{ to infinity } = 3 \log_e 2 - 1.$$

Let u_r denote the rth term of the series. Then

$$u_r = \frac{2r+3}{(2r-1)(2r)(2r+1)} \equiv \frac{A}{2r-1} + \frac{B}{2r} + \frac{C}{2r+1}.$$

Thus $2r + 3 \equiv A2r(2r+1) + B(2r-1)(2r+1) + C2r(2r-1)$.

Putting $2r = 1$, 0, -1 in succession we obtain, $A = 2$, $B = -3$, $C = 1$. Hence u_r may be written in the form,

$$u_r = \left(\frac{2}{2r-1} - \frac{2}{2r}\right) - \left(\frac{1}{2r} - \frac{1}{2r+1}\right).$$

Consider the sum to n terms of the given series

$$\sum_{r=1}^{n} u_r = 2\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r}\right) - \sum_{r=1}^{n}\left(\frac{1}{2r} - \frac{1}{2r+1}\right)$$

$$= 2\sum_{r=1}^{2n}(-1)^{r-1}\frac{1}{r} - \sum_{r=2}^{2n+1}(-1)^{r}\frac{1}{r}$$

$$= 2\sum_{r=1}^{2n}(-1)^{r-1}\frac{1}{r} + \sum_{r=1}^{2n}(-1)^{r-1}\frac{1}{r} - 1 + \frac{1}{2n+1}$$

$$= 3\sum_{r=1}^{2n}(-1)^{r-1}\frac{1}{r} - 1 + \frac{1}{2n+1}.$$

Letting $n \to \infty$,

$$\sum_{r=1}^{\infty} u_r = 3\sum_{r=1}^{\infty}(-1)^{r-1}\frac{1}{r} - 1 + \lim_{n\to\infty}\frac{1}{2n+1},$$

provided the series on the right converges.

Now $\log_e 2 = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{1}{r}$ Hence $\sum_{r=1}^{\infty} u_r = 3\log_e 2 - 1$.

In the example just considered we were able to sum the series because the method of partial fractions enabled us to obtain a series of standard type. Sometimes the method enables us to obtain a series of terms such that cancelling occurs leaving only a small number of terms. Consider the following examples.

(2) *Sum to n terms the series whose nth term is* $1/n(n+1)(n+3)$. [*Lond. B.A.*]

Write $\dfrac{1}{n(n+1)(n+3)} \equiv \dfrac{A}{n} + \dfrac{B}{n+1} + \dfrac{C}{n+3}$,

i.e. $1 \equiv A(n+1)(n+3) + Bn(n+3) + Cn(n+1)$.

In the identity write $n = 0, \ -1, \ -3$ in succession. It follows that $A = \frac{1}{3}, \ B = -\frac{1}{2}, \ C = \frac{1}{6}$. Thus

$$\frac{1}{n(n+1)(n+3)} = \frac{\frac{1}{3}}{n} - \frac{\frac{1}{2}}{n+1} + \frac{\frac{1}{6}}{n+3}$$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+3)} = \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r} - \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r+1} + \frac{1}{6}\sum_{r=1}^{n}\frac{1}{r+3}$$

$$= \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r} - \frac{1}{2}\sum_{s=2}^{n+1}\frac{1}{s} + \frac{1}{6}\sum_{t=4}^{n+3}\frac{1}{t},$$

where $s = r + 1, \ t = r + 3$.

This sum may be written in the form

$$\frac{1}{3}\sum_{r=4}^{n}\frac{1}{r} + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{2}\sum_{r=4}^{n}\frac{1}{r} - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{n+1}\right)$$

$$+ \frac{1}{6}\sum_{r=4}^{n}\frac{1}{r} + \frac{1}{6}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)$$

$$= \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right)\sum_{r=4}^{n}\frac{1}{r} + \frac{11}{18} - \frac{5}{12} - \frac{1}{2(n+1)}$$

$$+ \frac{1}{6}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)$$

$$= \frac{7}{36} - \frac{1}{3(n+1)} + \frac{1}{6(n+2)} + \frac{1}{6(n+3)}.$$

(3) *Sum to n terms* $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + \dfrac{1}{4.5.6} + \ldots$

Write $u_r = 1/r(r+1)(r+2)$ so that the required sum is $\sum_{r=1}^{n} u_r$

Expressing u_r as the sum of three partial fractions,

$$u_r = \frac{\frac{1}{2}}{r} - \frac{1}{r+1} + \frac{\frac{1}{2}}{r+2}.$$

$$\sum_{r=1}^{n} u_r = \begin{array}{lll}
\frac{1}{2}.1 & -\frac{1}{2} & +\frac{1}{2}.\frac{1}{3} \\
+\frac{1}{2}.\frac{1}{2} & -\frac{1}{3} & +\frac{1}{2}.\frac{1}{4} \\
+\frac{1}{2}.\frac{1}{3} & -\frac{1}{4} & +\frac{1}{2}.\frac{1}{5} \\
+\frac{1}{2}.\frac{1}{4} & -\frac{1}{5} & +\frac{1}{2}.\frac{1}{6} \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
+\frac{1}{2}.\frac{1}{n-2} & -\frac{1}{n-1} & +\frac{1}{2}.\frac{1}{n} \\
+\frac{1}{2}.\frac{1}{n-1} & -\frac{1}{n} & +\frac{1}{2}.\frac{1}{n+1} \\
+\frac{1}{2}.\frac{1}{n} & -\frac{1}{n+1} & +\frac{1}{2}.\frac{1}{n+2}
\end{array}$$

Adding the terms in accordance with the scheme indicated by the lines,

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \tfrac{1}{2}(1+\tfrac{1}{2}) - \tfrac{1}{2} + \frac{1}{n+1}(\tfrac{1}{2}-1) + \tfrac{1}{2}\cdot\frac{1}{n+2} \\ &= \tfrac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}.\end{aligned}$$

(4) *If* $u_n = \dfrac{n^2+9n+5}{(n+1)(2n+3)(2n+5)(n+4)}$, *prove that* $\sum\limits_{n=1}^{\infty} u_n = \frac{5}{36}$.

[*Lond. B.Sc.*]

We observe that the factors in the denominator of u_n split into two groups, $(n+1)(n+4)$ and $(2n+3)(2n+5)$. In each group the second factor may be obtained from the first by increasing the value of n. Thus $n+1$ becomes $n+4$ when n is changed into $n+3$ and $2n+3$ becomes $2n+5$ when n is changed into $n+1$. This suggests that u_n should be expressed in partial fractions and the terms grouped in accordance with the two sets of factors.

Write
$$\begin{aligned}u_n &= \frac{n^2+9n+5}{(n+1)(2n+3)(2n+5)(n+4)} \\ &\equiv \frac{A}{n+1} + \frac{B}{n+4} + \frac{C}{2n+3} + \frac{D}{2n+5}.\end{aligned}$$

Then, in accordance with the usual methods.

$$A = -\tfrac{1}{3},\ B = \tfrac{1}{3},\ C = \tfrac{5}{2},\ D = -\tfrac{5}{2}, \text{ giving}$$

$$u_n = \frac{1}{3}\left(\frac{1}{n+4} - \frac{1}{n+1}\right) + \frac{5}{2}\left(\frac{1}{2n+3} - \frac{1}{2n+5}\right).$$

Write $v_n = \dfrac{1}{n+4} - \dfrac{1}{n+1}$, $w_n = \dfrac{1}{2n+3} - \dfrac{1}{2n+5}$.

Then

$$\begin{array}{ll}
v_1 = \tfrac{1}{5} - \tfrac{1}{2} & w_1 = \tfrac{1}{5} - \tfrac{1}{7} \\
v_2 = \tfrac{1}{6} - \tfrac{1}{3} & w_2 = \tfrac{1}{7} - \tfrac{1}{9} \\
v_3 = \tfrac{1}{7} - \tfrac{1}{4} & w_3 = \tfrac{1}{9} - \tfrac{1}{11} \\
v_4 = \tfrac{1}{8} - \tfrac{1}{5} & w_4 = \tfrac{1}{11} - \tfrac{1}{13} \\
\cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
v_{n-3} = \dfrac{1}{n+1} - \dfrac{1}{n-2} & \cdot\ \cdot\ \cdot\ \cdot \\
v_{n-2} = \dfrac{1}{n+2} - \dfrac{1}{n-1} & \cdot\ \cdot\ \cdot\ \cdot \\
v_{n-1} = \dfrac{1}{n+3} - \dfrac{1}{n} & w_{n-1} = \dfrac{1}{2n+1} - \dfrac{1}{2n+3} \\
v_n = \dfrac{1}{n+4} - \dfrac{1}{n+1} & w_n = \dfrac{1}{2n+3} - \dfrac{1}{2n+5}
\end{array}$$

Hence
$$\begin{aligned}\sum_{r=1}^{n} v_r &= -\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} \\ &= -\frac{13}{12} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4}.\end{aligned}$$

$$\sum_{r=1}^{n} w_r = \frac{1}{5} - \frac{1}{2n+5}.$$

$$\sum_{r=1}^{n} u_r = \frac{1}{3}\left(-\frac{13}{12} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4}\right) + \frac{5}{2}\left(\frac{1}{5} - \frac{1}{2n+5}\right).$$

$$\sum_{r=1}^{\infty} u_r = \lim_{n \to \infty} \sum_{r=1}^{n} u_r = -\tfrac{13}{36} + \tfrac{1}{2} = \tfrac{5}{36}.$$

When the general term contains a factorial in the denominator it is frequently best to express it as the sum of partial fractions with denominators which are also factorials. The following examples illustrate the method:

(5) *Find the sum of the infinite series*

$$1 + 3x + \frac{5x^2}{2!} + \frac{7x^3}{3!} + \ldots \qquad \text{[Camb. Sch.]}$$

If u_n denote the nth term of the series,

$$u_n = (2n - 1)\,x^{n-1}/(n - 1)\,!.$$

Write $\dfrac{2n-1}{(n-1)!} = \dfrac{a}{(n-1)!} + \dfrac{b}{(n-2)!}$. Then $2n - 1 = a - b + bn$.

Equating corresponding coefficients, $a - b = -1$, $b = 2$, so that $a = 1$.

Hence if $n > 2$, $\dfrac{(2n-1)}{(n-1)!}\,x^{n-1} = \dfrac{x^{n-1}}{(n-1!)} + \dfrac{2x^{n-1}}{(n-2)!}$, and

$$\sum_{n=1}^{\infty} u_n = 1 + \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)!} + 2x \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!}$$

$$= e^x + 2xe^x = e^x(1 + 2x).$$

The two series involved are clearly convergent for all finite values of x.

(6) *Find the values of A, B, and C when*

$$\frac{n^3}{n!} = \frac{A}{(n-1)!} + \frac{B}{(n-2)!} + \frac{C}{(n-3)!}.$$

Use the result or any other method to prove that

$$1 + \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \ldots \textit{ to infinity is } 5e + 1.$$

$$\frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{A}{(n-1)!} + \frac{B}{(n-2)!} + \frac{C}{(n-3)!}$$

$$\text{Hence } n^2 = A + B\,(n-1) + C\,(n-1)\,(n-2)$$

$$= A - B + 2C + n\,(B - 3C) + Cn^2.$$

Thus $C = 1$, $B - 3C = 0$, $A - B + 2C = 0$. Hence $A = 1$, $B = 3$, $C = 1$ and

$$\frac{n^3}{n!} = \frac{1}{(n-1)!} + \frac{3}{(n-2)!} + \frac{1}{(n-3)!}.$$

Now $1 + \frac{1^3}{1!} + \frac{2^3}{2!} + \sum_{n=3}^{\infty} \frac{n^3}{n!}$

$$= 1 + 1 + 4 + \sum_{n=3}^{\infty} \frac{1}{(n-1)!} + 3\sum_{n=3}^{\infty} \frac{1}{(n-2)!} + \sum_{n=3}^{\infty} \frac{1}{(n-3)!}$$

$$= 6 + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - 1 - 1 + 3\sum_{n=2}^{\infty} \frac{1}{(n-2)!} - 3 + \sum_{n=3}^{\infty} \frac{1}{(n-3)!}$$

$$= 1 + e + 3e + e = 1 + 5e.$$

(7) *Sum to infinity the series whose nth term is*

$$(n^2 + 4n - 1)\, x^n/(n + 4)\,(n\,!). \qquad \text{[Lond. B.Sc.]}$$

Write $u_n = \frac{n^2 + 4n - 1}{(n + 4)\,(n\,!)} = \frac{n\,(n + 4) - 1}{(n + 4)\,(n\,!)} = \frac{1}{(n - 1)\,!} - \frac{1}{(n + 4)\,(n\,!)}$.

Suppose now that

$$\frac{1}{n + 4} \equiv \frac{A}{(n + 4)\,(n + 3)\,(n + 2)\,(n + 1)} + \frac{B}{(n + 3)\,(n + 2)\,(n + 1)} + \frac{C}{(n + 2)\,(n + 1)} + \frac{D}{n + 1}.$$

Thus $(n + 3)\,(n + 2)\,(n + 1)$

$$\equiv A + B\,(n + 4) + C\,(n + 4)\,(n + 3) + D\,(n + 4)\,(n + 3)\,(n + 2)$$

Put $n = -4, -3, -2, -1$ in succession. Then $A = -6$, $B = 6$, $C = -3$, $D = 1$. Hence

$$u_n = \frac{1}{(n - 1)\,!} + \frac{6}{(n + 4)\,!} - \frac{6}{(n + 3)\,!} + \frac{3}{(n + 2)\,!} - \frac{1}{(n + 1)\,!},$$

$$\therefore \sum_{n=1}^{\infty} u_n x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n - 1)\,!} + 6\sum_{n=1}^{\infty} \frac{x^n}{(n + 4)\,!} - 6\sum_{n=1}^{\infty} \frac{x^n}{(n + 3)\,!} + 3\sum_{n=1}^{\infty} \frac{x^n}{(n + 2)\,!} - \sum_{n=1}^{\infty} \frac{x^n}{(n + 1)\,!}.$$

provided that each of the series on the right, converges.

By comparison with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n\,!}$ it is clear that each of the series converges for all finite values of x. Hence provided $x \neq 0$.

$$\sum_{n=1}^{\infty} u_n x^n = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n - 1)\,!} + \frac{6}{x^4}\sum_{n=1}^{\infty} \frac{x^{n+4}}{(n + 4)\,!} - \frac{6}{x^3}\sum_{n=1}^{\infty} \frac{x^{n+3}}{(n + 3)\,!} + \frac{3}{x^2}\sum_{n=1}^{\infty} \frac{x^{n+2}}{(n + 2)\,!} - \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n + 1)\,!},\; x \neq 0$$

$$= x\sum_{m=0}^{\infty} \frac{x^m}{m\,!} + \frac{6}{x^4}\sum_{m=5}^{\infty} \frac{x^m}{m\,!} - \frac{6}{x^3}\sum_{m=4}^{\infty} \frac{x^m}{m\,!} + \frac{3}{x^2}\sum_{m=3}^{\infty} \frac{x^m}{m\,!} - \frac{1}{x}\sum_{m=2}^{\infty} \frac{x^m}{m\,!}$$

$$= xe^x + \frac{6}{x^4}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}\right)$$
$$- \frac{6}{x^3}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}\right)$$
$$+ \frac{3}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right) - \frac{1}{x}(e^x - 1 - x)$$
$$= e^x\left(x + \frac{6}{x^4} - \frac{6}{x^3} + \frac{3}{x^2} - \frac{1}{x}\right) - \frac{6}{x^4} + \frac{1}{4}.$$

If $x = 0$ it is clear that the sum of the series is zero.

11·3. Application of Identities

The method of partial fractions explained in § 10·2 is a particular application of identities. We now consider some further examples.

Examples.—(1) *If $f(x) \equiv Ax^3 + Bx^2 + Cx + D$, determine A, B, C, D, so that $f(x) + f(x-1) \equiv x^3$. Hence, or otherwise, show that*

$$1^3 - 2^3 + 3^3 - 4^3 + \ldots + (-1)^{n-1}n^3 = (-1)^{n-1}\{\tfrac{1}{2}n^3 + \tfrac{3}{4}n^2 - \tfrac{1}{8}\} - \tfrac{1}{8}.$$

[*Lond. B.Sc.*]

$$x^3 \equiv f(x) + f(x-1) \equiv A[x^3 + (x-1)^3] + B[x^2 + (x-1)^2]$$
$$+ C[x + (x-1)] + D + D$$
$$\equiv 2Ax^3 + x^2(2B - 3A) + x(3A - 2B + 2C)$$
$$- A + B - C + 2D.$$

Equating coefficients of corresponding powers of x,

$$2A = 1, \quad 2B - 3A = 0, \quad 3A - 2B + 2C = 0, \quad -A + B - C + 2D = 0.$$

Hence $A = \frac{1}{2}$, $B = \frac{3}{4}$, $C = 0$, $D = -\frac{1}{8}$.

Thus $f(x) = \frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{8}$.

Write $f(n) + f(n-1)$

$= n^3$ so that $(-1)^{n-1}f(n) + (-1)^{n-1}f(n-1) = (-1)^{n-1}.n^3$.

Change n into $n-1, n-2, \ldots, 2, 1$ in succession and add the results. Then

$$(-1)^{n-1}f(n) + f(0) = \sum_{r=1}^{n}(-1)^{r-1}r^3.$$

Hence $\displaystyle\sum_{r=1}^{n}(-1)^{r-1}r^3 = (-1)^{n-1}\{\tfrac{1}{2}n^3 + \tfrac{3}{4}n^2 - \tfrac{1}{8}\} - \tfrac{1}{8}.$

(2) *Sum to n terms $\sin^3\theta + \frac{1}{3}\sin^3 3\theta + \frac{1}{9}\sin^3 9\theta + \ldots$* [*Camb. Sch.*]

Since $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, it follows that

$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta.$$

The $(r+1)$th term of the given series is

$$\frac{1}{3^r}\sin^3(3^r\theta) = \tfrac{3}{4}.\frac{1}{3^r}\sin(3^r\theta) - \tfrac{1}{4}.\frac{1}{3^r}\sin(3^{r+1}\theta)$$
$$= \tfrac{1}{4}\frac{1}{3^{r-1}}\sin(3^r\theta) - \tfrac{1}{4}.\frac{1}{3^r}\sin(3^{r+1}\theta).$$

Hence $\sum_{r=0}^{n-1} \frac{1}{3^r}\sin^3(3^r\theta) = \frac{3}{4}\sum_{r=0}^{n-1} \frac{1}{3^{r-1}}\sin(3^r\theta) - \frac{1}{4}\sum_{r=0}^{n-1}\frac{1}{3^r}\sin(3^{r+1}\theta)$

$$= \frac{9}{4}\sin\theta + \frac{3}{4}\sum_{r=1}^{n-1}\frac{1}{3^{r-1}}\sin(3^r\theta) - \frac{3}{4}\sum_{r=1}^{n}\frac{1}{3^{r-1}}\sin(3^r\theta)$$

$$= \frac{9}{4}\sin\theta - \frac{3}{4}\cdot\frac{1}{3^{n-1}}\sin(3^n\theta).$$

(3) *Express* $(2n+1)^2$ *in the form* $2An(2n-1) + 2Bn + C$. *Show that the sum of the series*

$$\frac{3^2}{2!} - \frac{5^2}{4!} + \frac{7^2}{6!} - \ldots + (-1)^{n-1}\frac{(2n+1)^2}{(2n)!} + \ldots$$

is $1 + 3\sin 1$, *where the angle is measured in radians.* [*Lond. B.A.*]

$$(2n+1)^2 = 2An(2n-1) + 2Bn + C,$$

i.e. $4n^2 + 4n + 1 = 4An^2 + 2n(B-A) + C$. $\therefore A = 1,\ B - A = 2, C = 1,$

$$\text{Thus } (2n+1)^2 = 2n(2n-1) + 6n + 1.$$

The general term of the series is

$$(-1)^{n-1}\frac{(2n+1)^2}{(2n)!} = (-1)^{n-1}\frac{(2n)(2n-1)}{(2n)!} + (-1)^{n-1}\frac{6n}{(2n)!} + (-1)^{n-1}\frac{1}{(2n)!}$$

$$= (-1)^{n-1}\frac{1}{(2n-2)!} + (-1)^{n-1}\frac{3}{(2n-1)!} + (-1)^{n-1}\frac{1}{(2n)!}.$$

Now $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\theta^{2n-1}}{(2n-1)!}$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\theta^{2n-2}}{(2n-2)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}.$$

Hence the sum of the given series is

$$\cos 1 + 3\sin 1 - \cos 1 + 1 = 1 + 3\sin 1.$$

11·4. Method of Differences

Let u_r denote the rth term of a series. Then if u_r can be written in the form $k(v_r - v_{r-1})$ where k is independent of r then

$$\sum_{r=1}^{n} u_r = k(v_n - v_0).$$

This result may be seen immediately by writing down the terms of $\sum_{r=1}^{n} u_r$ and adding. We now give two important types of series to which the method applies.

Let ρ be a fixed positive integer and consider a series whose rth term is the product of ρ factors in arithmetical progression, the first factors of successive terms being in the same arithmetical progression.

Hence $u_r = \{a + rd\}\{a + (r+1)d\}\{a + (r+2)d\} \ldots \{a + (r + \rho - 1)d\}$,

where a, d are independent of r. Write $v_r = \{a + (r+\rho)d\}u_r$. Then it is easily seen that

$$v_r - v_{r-1} = (\rho + 1)\,du_r.$$

$$\text{Hence } v_n - v_0 = (\rho+1)\,d\sum_{r=1}^{n} u_r.$$

In particular if $a = 0$, $d = 1$,

$$1.2.3\ldots\rho + 2.3.4\ldots(\rho+1) + \ldots + n(n+1)(n+2)\ldots(n+\rho-1)$$
$$= \{n(n+1)(n+2)\ldots(n+\rho)\}/(\rho+1).$$

Next consider a series each of whose terms is the reciprocal of the product of ρ factors in arithmetical progression, the first factors of successive terms being in the same arithmetical progression.

In this case

$$\frac{1}{u_r} = \{a + rd\}\{a + (r+1)d\}\{a + (r+2)d\} \ldots \{a + (r+\rho-1)d\}.$$

Write $v_r = u_r(a + rd)$ so that

$$\frac{1}{v_r} = \{a + (r+1)d\}\{a + (r+2)d\} \ldots \{a + (r+\rho-1)d\}.$$

Then it is easily verified that

$$v_{r-1} - v_r = (\rho - 1)\,du_r.$$

$$\text{Hence } v_0 - v_n = (\rho - 1)\,d\sum_{r=1}^{n} u_r.$$

We now consider some examples.

Examples.—(1) *Find the sum to n terms of the series*

$$1.2.3.4 + 2.3.4.5 + 3.4.5.6 + \ldots$$

Write $u_r = r(r+1)(r+2)(r+3)$,

$v_r = r(r+1)(r+2)(r+3)(r+4)$. Then

$$v_r - v_{r-1} = r(r+1)(r+2)(r+3)(r+4) - (r-1)r(r+1)(r+2)(r+3)$$
$$= r(r+1)(r+2)(r+3)(r+4-r+1) = 5u_r.$$

$$\therefore 5 \sum_{r=1}^{n} u_r = (v_n - v_{n-1}) + (v_{n-1} - v_{n-2}) + \ldots + (v_1 - v_0)$$

$$= v_n - v_0 = n(n+1)(n+2)(n+3)(n+4),$$

since $v_0 = 0$. Thus the sum to n terms is

$$\tfrac{1}{5}n(n+1)(n+2)(n+3)(n+4).$$

(2) *Sum to n terms and to infinity the series*

$$\frac{1}{1.4.7.10} + \frac{1}{4.7.10.13} + \frac{1}{7.10.13.16} + \ldots$$

Write $1/u_r = (3r-2)(3r+1)(3r+4)(3r+7)$,

$1/v_r = (3r+1)(3r+4)(3r+7)$.

Then $$v_{r-1} - v_r = \frac{1}{(3r-2)(3r+1)(3r+4)} - \frac{1}{(3r+1)(3r+4)(3r+7)}$$

$$= \frac{3r+7-3r+2}{(3r-2)(3r+1)(3r+4)(3r+7)} = 9u_r.$$

Hence $$9 \sum_{r=1}^{n} u_r = (v_0 - v_1) + (v_1 - v_2) + (v_2 - v_3) + \ldots + (v_{n-1} - v_n)$$

$$= v_0 - v_n = \frac{1}{28} - \frac{1}{(3n+1)(3n+4)(3n+7)}.$$

Hence the sum to n terms is $\dfrac{1}{252} - \dfrac{1}{9(3n+1)(3n+4)(3n+7)}$.

Letting $n \to \infty$ it follows that the sum to infinity is $1/252$.

(3) *Prove that* $\tan\theta = \cot\theta - 2\cot 2\theta$ *and deduce the sum to n terms of the series*

$$\tan a + \tfrac{1}{2}\tan\tfrac{1}{2}a + \frac{1}{2^2}\tan\frac{1}{2^2}a + \ldots$$ [*Camb. Sch.*]

$$\cot\theta - 2\cot 2\theta = \frac{1}{\tan\theta} - \frac{2}{\tan 2\theta}$$

$$= \frac{1}{\tan\theta} - \frac{2(1-\tan^2\theta)}{2\tan\theta} = \tan\theta.$$

Write $$u_r = \frac{1}{2^{r-1}}\tan\frac{a}{2^{r-1}} = \frac{1}{2^{r-1}}\cot\frac{a}{2^{r-1}} - \frac{1}{2^{r-2}}\cot\frac{a}{2^{r-2}}$$

$= v_r - v_{r-1}$, say.

Then $$\sum_{r=1}^{n} u_r = v_n - v_0 = \frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}} - 2\cot 2a.$$

(4) *Sum to n terms* $\dfrac{1}{2} + \dfrac{1.3}{2.4} + \dfrac{1.3.5}{2.4.6} + \dfrac{1.3.5.7}{2.4.6.8} + \ldots$

Let u_r denote the rth term of the series. Then

$$u_r = \frac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}.$$

The problem is to find an expression v_r such that $u_r = k(v_r - v_{r-1})$, where k is independent of r.

Write $v_r = \dfrac{1.3.5\ldots(2r+1)}{2.4.6\ldots2r}$. Then

$$v_r - v_{r-1} = \frac{1.3.5\ldots(2r+1)}{2.4.6\ldots2r} - \frac{1.3.5\ldots(2r-1)}{2.4.6\ldots(2r-2)}$$

$$= \frac{1.3.5\ldots(2r-1)}{2.4.6\ldots2r}\{2r+1-2r\} = u_r.$$

$$\text{Hence}\quad \sum_{r=1}^{n} u_r = v_n - v_0.$$

Now $v_1 = \dfrac{1.3}{2}$. Also $\frac{1}{2} = u_1 = v_1 - v_0 = \frac{3}{2} - v_0$. Thus $v_0 = 1$ and the sum of the series is

$$\frac{1.3.5\ldots(2n+1)}{2.4.6\ldots2n} - 1.$$

(5) *Sum to n terms the series* $\dfrac{1}{a} + \dfrac{1.2}{a(a+1)} + \dfrac{1.2.3}{a(a+1)(a+2)} + \ldots$

If this series converges, find also the sum to infinity, and investigate the condition for convergence. [*Camb, Sch.*]

Let u_r denote the rth term of the series. Then

$$u_r = r!/\{a(a+1)(a+2)\ldots(a+r-1)\},$$

$$\frac{u_r}{u_{r-1}} = \frac{r}{a+r-1},\ \text{ i.e. }\ (a-1)u_r = ru_{r-1} - ru_r.$$

This last equation may be written in the form

$$(a-2)u_r = ru_{r-1} - (r+1)u_r.$$

$$\text{Thus}\quad (a-2)u_n = nu_{n-1} - (n+1)u_n$$

$$(a-2)u_{n-1} = (n-1)u_{n-2} - nu_{n-1}$$

. . . .

. . . .

$$(a-2)u_2 = 2u_1 - 3u_2.$$

$$\text{Adding}\quad (a-2)\sum_{r=2}^{n} u_r = 2u_1 - (n+1)u_n,$$

$$\text{i.e.}\quad \sum_{r=1}^{n} u_r = \{au_1 - (n+1)u_n\}/(a-2).$$

$$= \left\{1 - \frac{(n+1)!}{a(a+1)\ldots(a+n-1)}\right\}\Big/(a-2).$$

The series will converge provided

$$\lim_{n\to\infty} \frac{(n+1)!}{a(a+1)(a+2)\ldots(a+n-1)}$$

exists, and $a \neq 2$. Suppose that $a > 2$ and write $a = 2 + h$, where $h > 0$.

Then $\dfrac{a(a+1)\dots(a+n-1)}{(n+1)!} = \dfrac{(2+h)(3+h)\dots(n+1+h)}{(n+1)!}$

$$= (1+\tfrac{1}{2}h)(1+\tfrac{1}{3}h) + \dots \left(1+\frac{1}{n+1}h\right)$$

$$> 1 + h\left(\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{n+1}\right).$$

Since $\Sigma\dfrac{1}{n}$ diverges it follows that

$$\frac{a(a+1)\dots(a+n-1)}{(n+1)!} \to \infty \text{ as } n \to \infty.$$

Hence $\lim\limits_{n\to\infty} \dfrac{(n+1)!}{a(a+1)\dots(a+n-1)} = 0$ and the series converges to $1/(a-2)$, $a > 2$.

If $a = 2$, the series reduces to $\Sigma\dfrac{1}{r+1}$, which diverges.

If $a < 2$, the comparison test shows that the series diverges.

(6) *Prove that if* $nu_n = u_{n-2} + u_{n-3} + \dots + u_1$ *for all integral values of* n *greater than* 2, *and* $u_1 = 1$, $u_2 = \frac{1}{2}$, then

$$u_n = \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \qquad \text{[\textit{Camb. Sch.}]}$$

$$nu_n = u_{n-2} + u_{n-3} + \dots + u_1$$

Hence $nu_n - (n-1)u_{n-1} = u_{n-2}$, i.e. $n(u_n - u_{n-1}) = -(u_{n-1} - u_{n-2})$,

$$\frac{u_n - u_{n-1}}{u_{n-1} - u_{n-2}} = -\frac{1}{n}$$

Changing n into $n-1$, $n-2$, ... we have

$$\frac{u_{n-1} - u_{n-2}}{u_{n-2} - u_{n-3}} = -\frac{1}{(n-1)}$$

$$\frac{u_4 - u_3}{u_3 - u_2} = -\tfrac{1}{4}$$

Multiplying corresponding sides of the equations,

$$\frac{u_n - u_{n-1}}{u_3 - u_2} = (-1)^{n-3}\frac{3}{n!}$$

Now $3u_3 = u_1 = 1$, $u_3 = \frac{1}{3}$, $u_2 = \frac{1}{2}$. Hence

$$u_n - u_{n-1} = (-1)^n\frac{1}{n!},$$

$$u_{n-1} - u_{n-2} = (-1)^{n-1}\frac{1}{(n-1)!}$$

$$u_3 - u_2 = (-1)^3\frac{1}{3!}$$

Thus $u_n - u_2 = (-1)^n\dfrac{1}{n!} + (-1)^{n-1}\dfrac{1}{(n-1)!} + \dots + (-1)^3\dfrac{1}{3!}$,

or $$u_n = \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n\frac{1}{n!}$$

11·5. Series whose rth Term is a Polynomial in r

The assumption is that u_r has the form

$$a_0 + a_1 r + a_2 r^2 + \ldots + a_p r^p$$

where $a_0, a_1, a_2, \ldots, a_p$ are constants independent of r and p is a fixed positive integer.

Since u_r is a polynomial of degree p it may be written in the form

$$u_r = b_0 + b_1 r + b_2 r(r+1) + \ldots + b_s r(r+1)\ldots(r+s-1) + \ldots + b_p r(r+1)\ldots(r+p-1)$$

$$= b_0 + \sum_{s=1}^{p} b_s r(r+1)\ldots(r+s-1).$$

Hence $\displaystyle\sum_{r=1}^{n} u_r = nb_0 + \sum_{s=1}^{p} b_s \left\{ \sum_{r=1}^{n} r(r+1)\ldots(r+s-1) \right\}.$

Now each of the series $\displaystyle\sum_{r=1}^{n} r(r+1)\ldots(r+s-1)$, $s = 1, 2, \ldots, p$ is one whose general term is the product of s factors in arithmetical progression, the first factors of successive terms being in the same arithmetical progression. Thus § 10·4 applies and each sum can be written down.

It follows that the sum to n terms of a series whose rth term is a polynomial of degree p in r is a polynomial in n of degree $p + 1$.

In practical cases we require a method of determining the coefficients $b_0, b_1, b_2, \ldots, b_p$. These may be calculated in succession as follows: write $u_r = \phi(r)$ so that

$$\phi(r) = b_0 + \sum_{s=1}^{p} b_s r(r+1)(r+2)\ldots(r+s-1).$$

Put $r = 0, -1, -2, \ldots, -(p-1), -p$ in succession. Then

$$\phi(0) = b_0,$$
$$\phi(-1) = b_0 - b_1,$$
$$\phi(-2) = b_0 - 2b_1 + 2b_2,$$
$$\cdots\cdots\cdots\cdots$$
$$\phi(-p) = b_0 - b_1 p + b_2 p(p-1) - \ldots + (-1)^p b_p p!$$

As an example we consider the application of the method to the series $\displaystyle\sum_{r=1}^{n} r^4$. Since r^4 is a polynomial of the fourth degree we write it in the form

$$r^4 = b_0 + b_1 r + b_2 r(r+1) + b_3 r(r+1)(r+2) + b_4 r(r+1)(r+2)(r+3).$$

Now writing $r = 0, -1, -2, -3, -4$ in succession,

$$0 = b_0,$$
$$1 = b_0 - b_1,$$
$$16 = b_0 - 2b_1 + 2b_2,$$
$$81 = b_0 - 3b_1 + 6b_2 - 6b_3,$$
$$256 = b_0 - 4b_1 + 12b_2 - 24b_3 + 24b_4.$$

Solving these equations, $b_0 = 0$, $b_1 = -1$, $b_2 = 7$, $b_3 = -6$, $b_4 = 1$. Hence

$$r^4 = -r + 7r(r+1) - 6r(r+1)(r+2) + r(r+1)(r+2)(r+3).$$

$$\sum_{r=1}^{n} r^4 = -\sum_{r=1}^{n} r + 7\sum_{r=1}^{n} r(r+1) - 6\sum_{r=1}^{n} r(r+1)(r+2) + \sum_{r=1}^{n} r(r+1)(r+2)(r+3)$$

$$= -s_1 + 7s_2 - 6s_3 + s_4 \text{ (say)}.$$

Now $s_1 = \frac{1}{2}n(n+1)$.

To find s_2 write $u_r = r(r+1)$, $v_r = r(r+1)(r+2)$:

$$v_r - v_{r-1} = r(r+1)\{(r+2) - (r-1)\} = 3u_r.$$

Hence $3s_2 = v_n - v_0 = n(n+1)(n+2)$,

i.e. $s_2 = \frac{1}{3}n(n+1)(n+2)$.

To find s_3 write $w_r = r(r+1)(r+2)(r+3)$. Then

$$w_r - w_{r-1} = r(r+1)(r+2)\{(r+3) - (r-1)\} = 4v_r.$$

Hence $4s_3 = w_n - w_0 = n(n+1)(n+2)(n+3)$,

i.e. $s_3 = \frac{1}{4}n(n+1)(n+2)(n+3)$.

To find s_4 write $x_r = r(r+1)(r+2)(r+3)(r+4)$. Then

$$x_r - x_{r-1} = r(r+1)(r+2)(r+3)\{(r+4) - (r-1)\} = 5w_r.$$

Hence $5s_4 = x_n - x_0 = n(n+1)(n+2)(n+3)(n+4)$,

i.e. $s_4 = \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$.

Collecting results:

$$\sum_{r=1}^{n} r^4 = n(n+1)\{-\tfrac{1}{2} + \tfrac{7}{3}(n+2) - \tfrac{3}{2}(n+2)(n+3) + \tfrac{1}{5}(n+2)(n+3)(n+4)\},$$

$$= \tfrac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1).$$

11·51. Alternative Methods when rth Term is a Polynomial in r

If the degree of the polynomial is p we know from § 10·5 that the sum to n terms is a polynomial in n of degree $p + 1$.

Thus if $u_r = a_0 + a_1 r + a_2 r^2 + \ldots + a_p r^p$, then

$$\sum_{r=1}^{n} u_r = c_0 + c_1 n + c_2 n^2 + \ldots + c_{p+1} n^{p+1},$$

where $c_0, c_1, c_2, \ldots, c_{p+1}$ are constants which are independent of n. It follows that these $p + 2$ constants can be determined by giving n any $p + 2$ values, in particular, $1, 2, 3, \ldots, p + 2$.

Another method of finding the constants is to use the fact that $u_n = \sum\limits_{r=1}^{n} u_r - \sum\limits_{r=1}^{n-1} u_r$ and equate coefficients of corresponding powers of n.

A third method is to use differences and apply Newton's interpolation formula.

Examples.—(1) *Find the sum to n terms of the series for which $u_r = r^4$.* (See § 11·5.)

METHOD I.

Write $s_n = \sum\limits_{r=1}^{n} r^4$ so that $s_1 = 1$, $s_2 = 17$, $s_3 = 98$, $s_4 = 354$, $s_5 = 979$, $s_6 = 2275$, ... Since s_n is a polynomial of the fifth degree

$$s_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4 + c_5 n^5.$$

First it is clear that $c_0 = 0$, for if $n = 0$ the sum must be zero. Now write $n = 1, 2, 3, 4, 5$ in succession.

$$1 = c_1 + c_2 + c_3 + c_4 + c_5 \quad \ldots\ldots \quad \text{(i)}$$

$$17 = 2c_1 + 4c_2 + 8c_3 + 16c_4 + 32c_5 \quad \ldots\ldots \quad \text{(ii)}$$

$$98 = 3c_1 + 9c_2 + 27c_3 + 81c_4 + 243c_5 \quad \ldots\ldots \quad \text{(iii)}$$

$$345 = 4c_1 + 15c_2 + 64c_3 + 256c_4 + 1024c_5 \quad \ldots\ldots \quad \text{(iv)}$$

$$979 = 5c_1 + 25c_2 + 125c_3 + 625c_4 + 3125c_5 \quad \ldots\ldots \quad \text{(v)}$$

Solving the equations in the usual way we find

$$c_5 = \tfrac{1}{5},\ c_4 = \tfrac{1}{2},\ c_3 = \tfrac{1}{3},\ c_2 = 0,\ c_1 = -\tfrac{1}{30}.$$

Thus $$\sum_{r=1}^{n} r^4 = -\frac{n}{30} + \frac{n^3}{3} + \frac{n^4}{2} + \frac{n^5}{5}$$

$$= \tfrac{1}{30} n (n + 1) (6n^3 + 9n^2 + n - 1).$$

METHOD 2.

$$u_n = \sum_{r=1}^{n} u_r - \sum_{r=1}^{n-1} u_r$$

$$= c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4 + c_5 n^5 - c_1 (n-1) - c_2 (n-1)^2 - c_3 (n-1)^3 - c_4 (n-1)^4 - c_5 (n-1)^5,$$

i.e. $n^4 = 5c_5 n^4 + (4c_4 - 10c_5) n^3 + (3c_3 - 6c_4 + 10c_5) n^2 + (2c_2 - 3c_3 + 4c_4 - 5c_5) n + c_1 - c_2 + c_3 - c_4 + c_5.$

Equating coefficients,

$1 = 5c_5$, $4c_4 - 10c_5 = 0$, $3c_3 - 6c_4 + 10c_5 = 0$, $2c_2 - 3c_3 + 4c_4 - 5c_5 = 0$, $c_1 - c_2 + c_3 - c_4 + c_5 = 0$.

Solving these equations we obtain as before

$$c_5 = \tfrac{1}{5},\ c_4 = \tfrac{1}{2},\ c_3 = \tfrac{1}{3},\ c_2 = 0,\ c_1 = -\tfrac{1}{30}.$$

METHOD 3.

Write $f(n) = \sum_{r=1}^{n} r^4$ and form a table of differences.

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1	1					
		16				
2	17		65			
		81		110		
3	98		175		84	
		256		194		24
4	354		369		108	
		625		302		
5	979		671			
		1296				
6	2275					

Since $f(n)$ is a polynomial in n of the fifth degree Newton's formula (§ 10·6) gives

$$f(n) = f(1) + \binom{n-1}{1}\Delta f(1) + \binom{n-1}{2}\Delta^2 f(1) + \binom{n-1}{3}\Delta^3 f(1) + \binom{n-1}{4}\Delta^4 f(1) + \binom{n-1}{5}\Delta^5 f(1)$$

$$= 1 + 16(n-1) + \tfrac{65}{2}(n-1)(n-2) + \tfrac{110}{6}(n-1)(n-2)(n-3) + \tfrac{84}{24}(n-1)(n-2)(n-3)(n-4) + \tfrac{24}{120}(n-1)(n-2)(n-3)(n-4)(n-5)$$

$$= -\frac{n}{30} + \frac{n^3}{3} + \frac{n^4}{2} + \frac{n^5}{5}, \text{ on reduction.}$$

(2) *Find the sum to n terms of the series*

$$1\cdot3\cdot4 + 4\cdot5\cdot5 + 7\cdot7\cdot6 + 10\cdot9\cdot7 + \ldots$$

In this case $u_r = (3r - 2)(2r + 1)(r + 3) = 6r^3 + 17r^2 - 5r - 6$. Thus u_r is a polynomial of the third degree in r. Hence if $s_n = \sum_{r=1}^{n} u_r$,

$$s_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4.$$

Since $s_n = 0$ when $n = 0$, it is clear that $c_0 = 0$. Again

$$\begin{aligned}6n^3 + 17n^2 - 5n - 6 = u_n &= s_n - s_{n-1}\\ &= c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4\\ &\quad - c_1(n-1) - c_2(n-1)^2 - c_3(n-1)^3 - c_4(n-1)^4\\ &= 4c_4 n^3 + (3c_3 - 6c_4)n^2 + (2c_2 - 3c_3 + 4c_4)n\\ &\qquad + c_1 - c_2 + c_3 - c_4.\end{aligned}$$

Equating corresponding coefficients,

$$4c_4 = 6,\ 3c_3 - 6c_4 = 17,\ 2c_2 - 3c_3 + 4c_4 = -5,\ c_1 - c_2 + c_3 - c_4 = -6.$$

Solving these equations, $c_4 = \frac{3}{2}$, $c_3 = \frac{26}{3}$, $c_2 = \frac{15}{2}$, $c_1 = -\frac{17}{3}$.
Substituting the values and simplifying

$$s_n = \tfrac{1}{6}n(9n^3 + 52n^2 + 45n - 34).$$

Using the method of differences the working is as follows:—

Write $f(n) = \sum_{r=1}^{n} u_r$. Then $u_1 = 12$, $u_2 = 100$, $u_3 = 294$, $u_4 = 630$, $u_5 = 1144$.

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4
1	12				
		100			
2	112		194		
		294		142	
3	406		336		36
		630		178	
4	1036		514		
		1144			
5	2180				

Since $f(n)$ is a polynomial of the fourth degree in n,

$$f(n) = f(1) + \binom{n-1}{1}\Delta f(1) + \binom{n-1}{2}\Delta^2 f(1) + \binom{n-1}{3}\Delta^3 f(1) + \binom{n-1}{4}\Delta^4 f(1).$$

$$\begin{aligned}&= 12 + 100(n-1) + \tfrac{194}{2}(n-1)(n-2)\\ &\qquad + \tfrac{142}{6}(n-1)(n-2)(n-3) + \tfrac{36}{24}(n-1)(n-2)(n-3)(n-4)\\ &= \tfrac{3}{2}n^4 + \tfrac{26}{3}n^3 + \tfrac{15}{2}n^2 - \tfrac{17}{3}n.\end{aligned}$$

11·52. The Series $1^p + 2^p + 3^p + \ldots + n^p$, where p is a Positive Integer

This series is important and we indicate a special method of summation for it.

Write $S_p = \sum_{r=1}^{n} r^p$. Then

$$(x+1)^{p+1} \equiv x^{p+1} + {}_{p+1}C_1 x^p + {}_{p+1}C_2 x^{p-1} + \ldots + {}_{p+1}C_p x + 1.$$

In this identity replace x by n, $(n-1)$, $(n-2)$, . . . , 2, 1 in succession. Thus

$$(n+1)^{p+1} = n^{p+1} + {}_{p+1}C_1 n^p + {}_{p+1}C_2 n^{p-1} + \ldots + {}_{p+1}C_p n + 1$$

$$n^{p+1} = (n-1)^{p+1} + {}_{p+1}C_1(n-1)^p + {}_{p+1}C_2(n-1)^{p-1} + \ldots + {}_{p+1}C_p(n-1) + 1.$$

$$(n-1)^{p+1} = (n-2)^{p+1} + {}_{p+1}C_1(n-2)^p + {}_{p+1}C_2(n-2)^{p-1} + \ldots + {}_{p+1}C_p(n-2) + 1.$$

. . . .

. . . .

$$3^{p+1} = 2^{p+1} + {}_{p+1}C_1 . 2^p + {}_{p+1}C_2 . 2^{p-1} + \ldots + {}_{p+1}C_p . 2 + 1,$$

$$2^{p+1} = 1^{p+1} + {}_{p+1}C_1 . 1^p + {}_{p+1}C_2 . 1^{p-1} + \ldots + {}_{p+1}C_p . 1 + 1.$$

Adding $(n+1)^{p+1} = 1 + {}_{p+1}C_1 . S_p + {}_{p+1}C_2 . S_{p-1} + \ldots + {}_{p+1}C_p . S_1 + n.$

This equation determines S_p in terms of S_{p-1}, S_{p-2}, . . ., S_1.

Example.—*Find the value of* S_2.

$$S_1 = 1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n\,(n+1).$$

To find S_2 we have $(n+1)^3 = 1 + 3S_2 + 3S_1 + n$,

i.e. $S_2 = \frac{1}{3}\{(n+1)^3 - 1 - n - \frac{3}{2}n\,(n+1)\} = \frac{1}{6}n\,(n+1)\,(2n+1)$.

11·6. Series whose rth term is a Rational Function of r

Series whose rth term has the form

$$\phi\,(r)/\{r(r+1)\,(r+2)\ldots(r+p-1)\},$$

where $\phi\,(r)$ *is a polynomial in* r *of degree* q *where* $q < p - 2$.

Since $\phi\,(r)$ is of degree q in r we may write

$$\phi\,(r) = a_0 + a_1 r + a_2 r\,(r+1) + \ldots + a_q r\,(r+1)\,(r+2)\ldots(r+q-1),$$

where $a_0, a_1, a_2, \ldots, a_q$ are constants which are independent of r. Then if u_r denote the rth term of the series,

$$u_r = \frac{a_0}{r(r+1)(r+2)\ldots(r+p-1)}$$
$$+\frac{a_1}{(r+1)(r+2)\ldots(r+p-1)}$$
$$+\frac{a_2}{(r+2)(r+3)\ldots(r+p-1)}+\ldots$$
$$+\frac{a_q}{(r+q)\ldots(r+p-1)}$$
$$= \sum_{s=0}^{q} a_s/\{(r+s)(r+s+1)\ldots(r+p-1)\}.$$

Hence

$$\sum_{r=1}^{n} u_r = \sum_{s=0}^{q} a_s \left\{ \sum_{r=1}^{n} 1/(r+s)(r+s+1)\ldots(r+p-1) \right\}.$$

Now each term of the series

$$\sum_{r=1}^{n} 1/(r+s)(r+s+1)\ldots(r+p-1)$$

is the reciprocal of the product of $p-s$ factors in arithmetical progression, the first factors of successive terms being in the same arithmetical progression. Hence the series may be summed by the method of differences for each value of s.

Examples.—(1) *Find the sum to n terms and to infinity of the series whose rth term is $(r^2+1)/r(r+1)(r+2)(r+3)$.*

Let u_r denote the rth term. Since $r^2+1 = 1 - r + r(r+1)$.

$$u_r = \frac{1-r+r(r+1)}{r(r+1)(r+2)(r+3)} = \frac{1}{r(r+1)(r+2)(r+3)}$$
$$- \frac{1}{(r+1)(r+2)(r+3)} + \frac{1}{(r+2)(r+3)}.$$

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$$
$$- \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)} + \sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$$
$$= s_n^{(3)} - s_n^{(2)} + s_n^{(1)}, \text{ say.}$$

Write $v_r = 1/(r+3)$, $w_r = 1/(r+2)(r+3)$,

$x_r = 1/(r+1)(r+2)(r+3)$, $y_r = 1/r(r+1)(r+2)(r+3)$.

$$\text{Then } v_{r-1} - v_r = \frac{1}{r+2} - \frac{1}{r+3} = w_r.$$

$$\text{Hence } s_n^{(1)} = \sum_{r=1}^{n} w_r = v_0 - v_n = \tfrac{1}{3} - \frac{1}{n+3}.$$

$$\text{Again } w_{r-1} - w_r = \frac{1}{(r+1)(r+2)} - \frac{1}{(r+2)(r+3)} = 2x_r.$$

$$\text{Hence } s_n^{(2)} = \sum_{r=1}^{n} x_r = \tfrac{1}{2}(w_0 - w_n) = \tfrac{1}{12} - \frac{1}{2(n+2)(n+3)}.$$

$$\text{Finally, } x_{r-1} - x_r = \frac{1}{r(r+1)(r+2)} - \frac{1}{(r+1)(r+2)(r+3)}.$$

$$\text{Hence } s_n^{(3)} = \sum y_r = \tfrac{1}{3}(x_0 - x_n) = \tfrac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}.$$

It follows that

$$\sum_{r=1} u_r = \tfrac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)} - \tfrac{1}{12} + \frac{1}{2(n+2)(n+3)} + \tfrac{1}{3} - \frac{1}{n+3}$$

$$= \tfrac{11}{36} - \frac{1}{n+3} + \frac{1}{2(n+2)(n+3)} - \frac{1}{3(n+1)(n+2)(n+3)}.$$

Letting $n \to \infty$, $\sum\limits_{r=1}^{\infty} u_r = \tfrac{11}{36}$.

(2) *Find the sum of the infinite series*

$$\frac{3}{1.2.4} + \frac{5}{2.3.5} + \frac{7}{3.4.6} + \ldots$$

If u_r denote the rth term of the series $u_r = \dfrac{2r+1}{r(r+1)(r+3)}$.

It will be observed that u_r is not of the form considered above, but it may be expressed in this form by introducing the factor $(r+2)$. Thus

$$u_r = \frac{(2r+1)(r+2)}{r(r+1)(r+2)(r+3)}.$$

Now $(2r+1)(r+2) = a_0 + a_1 r + a_2 r(r+1)$. We find in the usual way that $a_0 = 2$, $a_1 = 3$, $a_2 = 2$. Hence

$$u_r = \frac{2}{r(r+1)(r+2)(r+3)} + \frac{3}{(r+1)(r+2)(r+3)} + \frac{2}{(r+2)(r+3)}.$$

Using the results of the previous example it follows that

$$\sum_{r=1}^{\infty} u_r = 2.\tfrac{1}{18} + 3.\tfrac{1}{12} + 2.\tfrac{1}{3} = \tfrac{37}{36}.$$

(3) *Find the sum to n terms of the series*

$$\frac{1}{1.5} + \frac{1}{3.7} + \frac{1}{5.9} + \ldots$$

If u_r denote the rth term of the series $u_r = \dfrac{1}{(2r-1)(2r+3)}$.

We may reduce the series to one of the type considered above in the following way. The factors in the denominator will be in arithmetical progression if we introduce the factor $2r + 1$. Thus

$$u_r = \frac{2r+1}{(2r-1)(2r+1)(2r+3)} = \frac{2+(2r-1)}{(2r-1)(2r+1)(2r+3)}$$

$$= \frac{2}{(2r-1)(2r+1)(2r+3)} + \frac{1}{(2r+1)(2r+3)}$$

$$\sum_{r=1}^{n} u_r = 2\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} + \sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}$$

$$= 2S_n'' + S_n', \text{ say.}$$

Write $v_r = 1/(2r+3)$, $w_r = 1/(2r+1)(2r+3)$,

$$x_r = 1/(2r-1)(2r+1)(2r+3).$$

$$\text{Then } v_{r-1} - v_r = \frac{1}{2r+1} - \frac{1}{2r+3} = 2w_r.$$

$$\text{Hence } S_n' = \sum_{r=1} w_r = \tfrac{1}{2}(v_0 - v_n) = \tfrac{1}{6} - \frac{1}{2(2n+3)}$$

$$\text{Again } w_{r-1} - w_r = \frac{1}{(2r-1)(2r+1)} - \frac{1}{(2r+1)(2r+3)} = 4x_r.$$

$$\text{Hence } S_n'' = \sum_{r=1} x_r = \tfrac{1}{4}(w_0 - w_n) = \tfrac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$$

$$\text{Thus } \sum_{r=1}^{n} u_r = \tfrac{1}{6} - \frac{1}{2(2n+1)(2n+3)} + \tfrac{1}{6} - \frac{1}{2(2n+3)}$$

$$= \tfrac{1}{3} - \frac{1}{2(2n+3)} - \frac{1}{2(2n+1)(2n+3)}.$$

11·7. Finite Differences and Series

Let $\{u_n\}$ denote a sequence of functions, n being a positive integer. Then the results listed below follow from the theory of finite differences.

1. *If u_n is polynomial in n of degree p then $\Delta^p u_n$ is constant. Conversely if $\Delta^p u_n$ is constant then u_n is a polynomial in n of degree p.*

2. *If m is a positive integer,*

$$u_{n+m} = u_n + \binom{m}{1}\Delta u_n + \binom{m}{2}\Delta^2 u_n + \ldots + \binom{m}{m}\Delta^m u_n.$$

3. *If $s_n = \sum_{r=1}^{n} u_r$ then*

$$s_n = \binom{n}{1}u_1 + \binom{n}{2}\Delta u_1 + \binom{n}{3}\Delta^2 u_1 + \ldots + \binom{n}{n}\Delta^{n-1} u_1.$$

1. See § 10·22.

The converse may be proved by induction. The theorem is clearly true for $p = 1$. For if $\Delta u_n = k$, where k is a constant, then $\{u_n\}$ forms an arithmetic progression whose common difference is k. Thus u_n is a linear function of n, or u_n is a polynomial of the first degree in n.

Suppose the theorem is true for $p = m$, *i.e.* if $\Delta^m u_n$ is constant then u_n is a polynomial of degree m. Now let $\{u_n\}$ be a sequence such that $\Delta^{m+1} u_n = k$ and write $v_n = \Delta u_n$. Then

$$k = \Delta^{m+1} u_n = \Delta^m (\Delta u_n) = \Delta^m v_n.$$

Hence v_n is a polynomial in n of degree m. Now

$$u_n - u_1 = (u_n - u_{n-1}) + (u_{n-1} + u_{n-2}) + \ldots + (u_2 - u_1)$$
$$= v_{n-1} + v_{n-2} + \ldots + v_1.$$

From § 11·5 we know that the sum to n terms of a series whose rth term is a polynomial of degree m in r is a polynomial in n of degree $m + 1$. Hence u_n is a polynomial of degree $m + 1$.

Thus if the theorem is true for $p = m$, it is true for $p = m + 1$. Since it has been shown to be true for $p = 1$, it is true in general.

2. Using symbolic reasoning we have

$$u_{n+m} = E^m u_n = (1 + \Delta)^m u_n$$
$$= \left\{1 + \binom{n}{1}\Delta + \binom{n}{2}\Delta^2 + \ldots + \binom{n}{n}\Delta^n\right\} u_n$$
$$= u_n + \binom{n}{1}\Delta u_n + \binom{n}{2}\Delta^2 u_n + \ldots + \binom{n}{n}\Delta^n u_n.$$

3. $$u_r = E^{r-1} u_1 = (1 + \Delta)^{r-1} u_1$$
$$= \left\{1 + \binom{r-1}{1}\Delta + \binom{r-1}{2}\Delta^2 + \ldots + \binom{r-1}{r-1}\Delta^{r-1}\right\} u_1$$
$$= u_1 + \binom{r-1}{1}\Delta u_1 + \binom{r-1}{2}\Delta^2 u_1 + \ldots$$
$$+ \binom{r-}{r-1}\Delta^{r-1} u_1$$

Hence $s_n = \sum u_r$

$$= nu_1 + \left\{\sum_{s=1}^{n-1}\binom{s}{1}\right\}\Delta u_1 + \left\{\sum_{s=2}^{n-1} \qquad \right. \Delta^2 u_1 + \ldots$$

$$+ \sum_{s=r}^{n-1}\left\{\binom{s}{r}\right\}\Delta^r u_1 + \ldots \binom{n-1}{n-1}\Delta^{n-1} u_1.$$

Now $\displaystyle\sum_{s=r}^{n-1}\binom{s}{r} = \frac{1}{r!}\sum_{s=r}^{n=1} v_s,$

where $v_s = s\,(s-1)\,(s-2)\ldots(s-r+1)$.

If $\qquad w_s = (s+1)\,s\,(s-1)\ldots(s-r+1)$, then

$$w_s - w_{s-1} = (s+1)\,s\,(s-1)\ldots(s-r+1)$$
$$- s\,(s-1)\,(s-2)\ldots(s-r)$$
$$= (r+1)\,v_s.$$

Hence $\displaystyle\sum_{s=r}^{n-1}\binom{s}{r} = \frac{1}{(r+1)!}\sum_{s=r}^{n-1}(w_s - w_{s-1})$

$$\frac{1}{(r+1)!}(w_{n-1} - w_{r-1}) = \frac{1}{(r+1)!}w_{n-1},$$

since $\qquad w_{r-1} = 0.$

Thus $\displaystyle\sum_{s=r}^{n-1}\binom{s}{r} = \frac{n\,(n-1)\,(n-2)\ldots(n-r)}{(r+1)!} = \binom{n}{r+1},$

and

$$s_n = nu_1 + \binom{n}{2}\Delta u_1 + \binom{n}{3}\Delta^2 u_1 + \ldots + \binom{n}{r+1}\Delta^r u_1 + \ldots$$
$$+ \Delta^{n-1} u_1$$

as required.

11·8. Application to Series

The results of the previous section may be used to obtain the general term and the sum to n terms of a series whose general term u_r is a polynomial in r of degree p, where p is a positive integer.

Thus to find u_r we can use the formula

$$u_r = u_1 + \binom{r-1}{1}\Delta u_1 + \binom{r-1}{2}\Delta^2 u_1 + \ldots \binom{r-1}{r-1}\Delta^{r-1} u_1.$$

Since u_r is a polynomial of degree p, $\Delta^s u_1 = 0$ for $s > p$, where s is a positive integer,

$$u_r = u_1 + \binom{r-1}{1}\Delta u_1 + \binom{r-1}{2}\Delta^2 u_1 + \dots$$
$$\binom{r-1}{p}\Delta^p u_1, \quad p < r.$$

The series on the right contains $(p + 1)$ terms. If $p > r$ the number of terms in the series for u_r will be r.

Examples.—(1) *Find the rth term of the series* 4, 10, 20, 35, 56, 84, 120, ...

r		Δ	Δ^2
1	4		
		6	
2	10		4
		10	
3	20		5
		15	
4	35		6
		21	
5	56		7
		28	
6	84		8
		36	
7	120		

The third order differences are constant so that u_r is a polynomial in r of the third degree. Using the above formula

$$u_r = 4 + \binom{r-1}{1}6 + \binom{r-1}{2}4 + \binom{r-1}{3}1$$
$$= 4 + 6(r-1) + \frac{4(r-1)(r-2)}{2!} + \frac{(r-1)(r-2)(r-3)}{3!}$$
$$= \tfrac{1}{6}(r^3 + 6r^2 + 11r + 6) = \tfrac{1}{6}(r+1)(r+2)(r+3).$$

(2) *Find a cubic function of n which has the values* 3, 7, 19, 45 *when* $n = 1, 2, 3, 4$ *respectively.*

Denote the function by u_n. Then $\Delta^4 u_n = 0$.

n	u_n	Δ	Δ^2	Δ^3
1	3			
		4		
2	7		8	
		12		6
3	19		14	
		26		
4	45			

Hence $u_n = u_1 + \binom{n-1}{1}\Delta u_1 + \binom{n-1}{2}\Delta^2 u_1 + \binom{n-1}{3}\Delta^3 u_1$

$$= 3 + 4(n-1) + \frac{8(n-1)(n-2)}{2!} + \frac{6(n-1)(n-2)(n-3)}{3!}$$

$$= n^3 - 2n^2 + 3n + 1.$$

(3) *Find the nth term and the sum to n terms of the series*

$$2.2 + 3.5 + 4.10 + 5.17 + 6.26 + \ldots.$$

The first factor of the nth term is obviously $n + 1$. To find the second factor, construct a table of differences for these factors u_n.

n	u_n	Δ	Δ^2	Δ^3
1	2			
		3		
2	5		2	
		5		0
3	10		2	
		7		0
4	17		2	
		9		
5	26			

Hence u_n is a quadratic in n and

$$u_n = 2 + (n-1)\,3 + \frac{(n-1)(n-2)}{2!}\,2 = n^2 + 1.$$

Thus the nth term is $(n+1)(n^2+1)$. If we denote this term by v_n we know that $\Delta^3 v_n$ is constant. Thus

n	v_n	Δ	Δ^2	Δ^3
1	4			
		11		
2	15		14	
		25		6
3	40		20	
		45		6
4	85		26	
		71		
5	156			

If s_n denote the sum of the series,

$$s_n = nv_1 + \binom{n}{2}\Delta v_1 + \binom{n}{3}\Delta^2 v_1 + \binom{n}{4}\Delta^3 v_1$$

$$= 4n + \frac{11n(n-1)}{2!} + \frac{14n(n-1)(n-2)}{3!} + \frac{6n(n-1)(n-2)(n-3)}{4!}$$

$$= \tfrac{1}{12}n(3n^3 + 10n^2 + 15n + 20).$$

(4) *Find by the method of differences or otherwise the nth term and the sum to n terms of the series*

$$1 + 4 + 11 + 26 + 57 + 120 + \ldots.$$

If u_n denote the nth term we have

n	u_n	Δ	Δ^2
1	1		
		3	
2	4		4
		7	
3	11		8
		15	
4	26		16
		31	
5	57		32
		63	
6	120		

Hence u_n satisfies the difference equation

$$\Delta^2 u_n = 2^{n+1} \text{ or } (E - 1)^2 u_n = 2^{n+1}.$$

This is a linear equation with constant coefficients and its general solution (§ 10·73) is

$$u_n = a + bn + 2^{n+1}.$$

Since $u_1 = 1$, $u_2 = 4$, we have

$$1 = a + b + 4$$

$$4 = a + 2b + 8$$

giving $a = -2$, $b = -1$, and $u_n = 2^{n+1} - n - 2$.

The sum to n terms of the series is

$$\sum_{r=1}^{n} 2^{r+1} - \sum_{r=1}^{n} r - 2n = 4(2^n - 1) - \tfrac{1}{2}n(n + 1) - 2n$$
$$= 2^{n+2} - \tfrac{1}{2}n(n + 5) - 4.$$

NOTE.—The value of u_n can easily be found without using the general technique for the solution of linear equations with constant coefficients Write $v_r = \Delta u_r$, $w_r = \Delta v_r = \Delta^2 u_r$. Then $w_r = 2^{r+1}$.

$$w_1 + w_2 + w_3 + \ldots + w_{n-2} = \sum_{r=1}^{n-2} \Delta v_r = v_{n-1} - v_1.$$

Since $$\sum_{r=1}^{n-1} 2^{r-1} = 2^n - 4 \text{ and } v_1 = \Delta u_1 = 3,$$

$$v_{n-1} = 2^n - 1.$$

Again $$v_1 + v_2 + v_3 + \ldots + v_{n-1} = \sum_{r=1}^{n-1} \Delta u_r = u_n - u_1.$$

Hence $$\sum_{r=1}^{n-1} 2^{n+1} - (n - 1) = u_n - 1, \text{ or}$$

$$u_n = 2^{n+1} - n - 2.$$

11·9. Summation of Series by Complex Methods

We now consider some methods involving the use of complex numbers. The fundamental properties involved are as follows.

(*a*) If θ is any real number, then $\cos\theta + i\sin\theta = e^{i\theta}$,

$\cos n\theta + i\sin n\theta = e^{ni\theta} = (e^{i\theta})^n$, $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$.

(*b*) If x, y, u, v are real and $f(x+iy) = u + iv$, $f(x+iy)$ denoting a function depending on the complex number $x + iy$, then u is equal to the real part of $f(x+iy)$ and iv is equal to the purely imaginary part.

(*c*) If $f(x+iy) = u + iv$, then $f(x-iy) = u - iv$.

(*d*) The product of $f(x+iy)$ by its conjugate function $f(x-iy)$ is real. This property is of frequent use for determining the real and imaginary parts of a function, e.g. if

$$f(x+iy) = \frac{\phi(x+iy)}{\psi(x+iy)}$$

then we can make the denominator real by multiplying numerator and denominator by $\psi(x-iy)$. Thus

$$f(x+iy) = \frac{\phi(x+iy)\,\psi(x-iy)}{\psi(x+iy)\,\psi(x-iy)} = \frac{\phi(x+iy)\,\psi(x-iy)}{\{|\psi(x+iy)|\}^2}$$

The methods of summation considered will depend ultimately on one or other of the following fundamental types of series: (i) geometrical progressions, (ii) binomial series, (iii) exponential and related series such as the sine and cosine series, (iv) logarithmic series, (v) recurring series.

The type of series to which the procedure applies are trigonometrical and involve sines or cosines of multiple angles such as $\sin n\theta$, $\cos n\theta$ where θ is a given angle and n is a positive integer. The functions $\sin n\theta$, $\cos n\theta$ are associated together to give $\cos n\theta + i\sin n\theta = e^{ni\theta}$. This implies, *e.g.*, that if we wish to sum a series involving sines of multiple angles by the method it will be necessary to introduce a second series involving the corresponding cosines.

11·91. Geometric Series

Examples.—(1) *Show that the sum to infinity of the series whose rth term is* $\cos^r\beta\sin(\alpha + r\beta)$ *is* $\cot\beta.\cos\alpha$. [*Lond. B.Sc.*]

Write $S = \cos\beta.\sin(\alpha+\beta) + \cos^2\beta.\sin(\alpha+2\beta) + \cos^3\beta.\sin(\alpha+3\beta) + \ldots$

$$= \sum_{r=1}^{\infty} \cos^r\beta.\sin(\alpha+r\beta).$$

We now introduce the cosine series corresponding to the multiple angle $\alpha + r\beta$ and write

$$C = \cos\beta.\cos(\alpha+\beta) + \cos^2\beta.\cos(\alpha+2\beta) + \cos^3\beta.\cos(\alpha+3\beta) + \ldots$$

$$= \sum_{r=1}^{\infty} \cos^r\beta.\cos(\alpha+r\beta).$$

Then $C + iS = \sum_{r=1}^{\infty} \cos^r\beta.\cos(\alpha+r\beta) + i\sum_{r=1}^{\infty} \cos^r\beta.\sin(\alpha+r\beta)$

$$= \sum_{r=1}^{\infty} \cos^r\beta.\{\cos(\alpha+r\beta) + i\sin(\alpha+r\beta)\}$$

$$= \sum_{r=1}^{\infty} \cos^r\beta.e^{i(\alpha+r\beta)} = e^{i\alpha}\sum_{r=1}^{\infty} (\cos\beta.e^{i\beta})^r.$$

The series is a geometrical progression whose first term is $e^{i(\alpha+\beta)}\cos\beta$ and whose common ratio is $e^{i\beta}\cos\beta$.

Now $|e^{i\beta}\cos\beta| = |\cos\beta| < 1$ provided $\beta \neq m\pi$, where m denotes zero or a positive or negative integer. Excluding the values $\beta = m\pi$ it follows that

$$C + iS = e^{i(\alpha+\beta)}\cos\beta/(1 - e^{i\beta}\cos\beta).$$

In order to make the denominator real, multiply numerator and denominator by the function conjugate to $1 - e^{i\beta}\cos\beta$, i.e. $1 - e^{-i\beta}\cos\beta$. Thus

$C + iS$

$$= e^{i(\alpha+\beta)}\cos\beta\,(1 - e^{-i\beta}\cos\beta)/\{(1 - e^{i\beta}\cos\beta)\,(1 - e^{-i\beta}\cos\beta)\}$$

$$= \cos\beta\,\{e^{i(\alpha+\beta)} - e^{i\alpha}\cos\beta\}/\{1 - (e^{i\beta} + e^{-i\beta})\cos\beta + \cos^2\beta\}$$

$$= \cos\beta\,\{e^{i(\alpha+\beta)} - e^{i\alpha}\cos\beta\}/(1 - 2\cos^2\beta + \cos^2\beta)$$

$$= \cos\beta\,\{\cos(\alpha+\beta) + i\sin(\alpha+\beta) - \cos\alpha\cos\beta - i\sin\alpha\cos\beta\}/\sin^2\beta.$$

Now iS is equal to the imaginary part of this expression.

Hence $S = \cos\beta\,\{\sin(\alpha+\beta) - \sin\alpha\cos\beta\}/\sin^2\beta.$

$$= \cos\beta\,\{\sin\alpha\cos\beta + \cos\alpha\sin\beta - \sin\alpha\cos\beta\}/\sin^2\beta$$

$$= \cos\beta\sin\beta\cos\alpha/\sin^2\beta = \cot\beta\cos\alpha.$$

(2) *Expand* $\sin\phi/(1 - 2x\cos\phi + x^2)$ *in ascending powers of* x *and prove that the remainder after* n *terms is*

$$x^n\left\{\frac{\sin(n+1)\phi - x\sin n\phi}{1 - 2x\cos\phi + x^2}\right\}$$ [*Camb. Sch.*]

Now $\dfrac{\sin\phi}{1 - 2x\cos\phi + x^2} = \dfrac{\sin\phi}{1 - xe^{i\phi} - xe^{-i\phi} + x^2}$

$$= \frac{1}{2i}\,\frac{e^{i\phi} - e^{-i\phi}}{(1 - xe^{i\phi})\,(1 - xe^{-i\phi})}.$$

Expressing as the sum of two partial fractions and expanding by the binomial theorem, we have if $|x| < 1$,

$$\frac{1}{2i}\cdot e^{i\phi}(1 - xe^{i\phi})^{-1} - \frac{1}{2i}\cdot e^{-i\phi}(1 - xe^{-i\phi})^{-1}$$

$$= \frac{1}{2i}\cdot e^{i\phi}\sum_{n=0}^{\infty} x^n e^{ni\phi} - \frac{1}{2i}\cdot e^{-i\phi}\sum_{n=0}^{\infty} x^n e^{-ni\phi}$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\{e^{(n+1)i\phi} - e^{-(n+1)i\phi}\}x^n = \sum_{n=0}^{\infty} x^n \sin(n+1)\phi.$$

The remainder after n terms is

$$\frac{1}{2i}\sum_{r=n}^{\infty} x^n e^{(n+1)i\phi} - \frac{1}{2i}\sum_{r=n}^{\infty} x^n e^{-(n+1)i\phi}.$$

Each of the series is a geometrical progression, the common ratios being $xe^{i\phi}$ and $xe^{-i\phi}$. Since $|e^{i\phi}| = |e^{-i\phi}| = 1$, it follows that if $|x| < 1$, the remainder is

$$\frac{1}{2i}\frac{x^n e^{(n+i)i\phi}}{1 - xe^{i\phi}} - \frac{1}{2i}\frac{x^n e^{-(n+1)i\phi}}{1 - xe^{-i\phi}}$$

$$= \frac{x^n}{2i}\frac{e^{(n+1)i\phi} - xe^{ni\phi} - e^{-(n+1)i\phi} + xe^{-ni\phi}}{1 - 2x\cos\phi + x^2}$$

$$= x^n\{\sin(n+1)\phi - x\sin n\phi\}/(1 - 2x\cos\phi + x^2).$$

11·92. The Binomial Series

Examples.—(1) *Find the sum to infinity of the series*

$$1 + 2\cos\alpha.\cos\alpha + 3\cos 2\alpha.\cos^2\alpha + 4\cos 3\alpha.\cos^3\alpha + \ldots$$

[*Camb. Sch.*]

Denote the given series by C and consider the corresponding sine series,

$$S = 2\sin\alpha.\cos\alpha + 3\sin 2\alpha.\cos^2\alpha + 4\sin 3\alpha.\cos^3\alpha + \ldots$$

Then $C + iS = \sum_{r=0}^{\infty}(r+1)\cos.r\alpha\cos^r\alpha + i\sum_{r=1}^{\infty}(r+1)\sin r\alpha.\cos^r\alpha$

$$= 1 + \sum_{r=1}^{\infty}(r+1)e^{ri\alpha}\cos^r\alpha = \sum_{r=0}^{\infty}(r+1)z^r,$$

where $z = e^{i\alpha}\cos\alpha$,

$$= (1 - z)^{-2}, \quad |z| < 1.$$

The condition $|z| < 1$ will be satisfied provided $|\cos\alpha| < 1$, i.e. $\alpha \neq m\pi$, m denoting 0 or a positive or negative integer. Excluding the cases $\alpha = m\pi$,

$$C + iS = (1 - e^{i\alpha}\cos\alpha)^{-2}$$

$$= (1 - e^{-i\alpha}\cos\alpha)^2/\{(1 - e^{i\alpha}\cos\alpha)(1 - e^{-i\alpha}\cos\alpha)\}^2$$

$$= (1 - \cos^2\alpha + i\sin\alpha\cos\alpha)^2/\{1 - \cos\alpha(e^{i\alpha} + e^{-i\alpha}) + \cos^2\alpha\}^2$$

$$= \sin^2\alpha(\sin\alpha + i\cos\alpha)^2/\sin^4\alpha$$

$$= (\sin^2\alpha + 2i\sin\alpha\cos\alpha - \cos^2\alpha)/\sin^2\alpha.$$

Equating real parts, $C = (\sin^2 a - \cos^2 a)/\sin^2 a = 1 - \cot^2 a$.

It should be observed that if $|\cos a| = 1$, i.e. $a = m\pi$ the given series obviously diverges.

(2) *Find the sum to infinity to the series*

$$1 + \tfrac{1}{2}\cos 2\theta - \frac{1}{2.4}\cos 4\theta + \frac{1.3}{2.4.6}\cos 6\theta - \ldots$$

Write $C = 1 + \tfrac{1}{2}\cos 2\theta - \dfrac{1}{2.4}\cos 4\theta + \dfrac{1.3}{2.4.6}\cos 6\theta - \ldots$

$$S = \tfrac{1}{2}\sin 2\theta - \frac{1}{2.4}\sin 4\theta + \frac{1.3}{2.4.6}\sin 6\theta - \ldots$$

$$C + iS = 1 + \tfrac{1}{2}e^{2i\theta} - \frac{1}{2.4}e^{4i\theta} + \frac{1.3}{2.4.6}e^{6i\theta} - \ldots$$

$$= 1 + \tfrac{1}{2}z - \frac{1}{2.4}z^2 + \frac{1.3}{2.4.6}z^3 - \ldots$$

where $z = e^{2i\theta}$,

$$= 1 + \frac{\frac{1}{2}}{1!}z + \frac{\frac{1}{2}.-\frac{1}{2}}{2!}z^2 + \frac{\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}}{3!}z^3 + \ldots$$

$$= (1 + z)^{\frac{1}{2}*}.$$

Hence $C + iS = \left(1 + e^{2i\theta}\right)^{\frac{1}{2}} = (1 + \cos 2\theta + i\sin 2\theta)^{\frac{1}{2}}$

$$= (2\cos^2\theta + 2i\sin\theta\cos\theta)^{\frac{1}{2}}$$

$$= (2\cos\theta)^{\frac{1}{2}}(\cos\theta + i\sin\theta)^{\frac{1}{2}}$$

$$= (2\cos\theta)^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta).$$

Equating real parts, $C = (2\cos\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta$.

11·93. The Exponential Series

Examples.—(1) *Prove that if x is real*

$$x\sin a + \frac{x^2}{2!}\sin 2a + \ldots + \frac{x^n}{n!}\sin na + \ldots = e^{x\cos a}\sin(x\sin a).$$

[*Camb. Sch.*]

Let S denote the given series, and write

$$C = 1 + x\cos a + \frac{x^2}{2!}\cos 2a + \ldots + \frac{x^n}{n!}\cos na + \ldots$$

Then $C + iS = 1 + x(\cos a + i\sin a) + \dfrac{x^2}{2!}(\cos 2a + i\sin 2a) + \ldots$

$$+ \frac{x^n}{n!}(\cos na + i\sin na) + \ldots$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}e^{nia} = \sum_{n=0}^{\infty}\frac{(xe^{ia})^n}{n!}$$

* The binomial series $1 + nz + \dfrac{n(n-1)}{2!}z^2 + \dfrac{n(n-1)(n-2)}{3!}z^3 + \ldots$ converges absolutely at all points of the circle of convergence $|z| = 1$ provided $n > 0$.

$$= e^{xe^{i\alpha}}, \text{ for all values of } x \text{ and } \alpha,$$
$$= e^{x(\cos\alpha + i\sin\alpha)} = e^{x\cos\alpha}.e^{ix\sin\alpha}$$
$$= e^{x\cos\alpha}\{\cos(x\sin\alpha) + i\sin(x\sin\alpha)\}.$$

Equating imaginary parts, $S = e^{x\cos\alpha}\sin(x\sin\alpha)$.

(2) *Show that the sum of the series*

$$1 - \frac{1}{2!}\cos 2\theta + \frac{1}{4!}\cos 4\theta - \frac{1}{6!}\cos 6\theta + \dots$$

is $\cos(\cos\theta).\cosh(\sin\theta)$. *Hence obtain the sum of the series*

$$1 - \frac{1}{4!} + \frac{1}{8!} - \frac{1}{12!} + \dots \textit{given } \cosh \tfrac{1}{2}\sqrt{2} = 1{\cdot}261.$$

[*N.Sc. Prelim.*]

Let C denote the given series, S the corresponding sine series. Thus

$$S = -\frac{1}{2!}\sin 2\theta + \frac{1}{4!}\sin 4\theta - \frac{1}{6!}\sin 6\theta + \dots$$

Then $C + iS = 1 - \dfrac{1}{2!}e^{2i\theta} + \dfrac{1}{4!}e^{4i\theta} - \dfrac{1}{6!}e^{6i\theta} + \dots$

$$= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \dots \quad \text{where } z = e^{i\theta},$$
$$= \cos z = \cos\left(e^{i\theta}\right).$$

Hence $C - iS = \cos\left(e^{-i\theta}\right) = \cos(1/z)$.

Adding $2C = \cos z + \cos(1/z) = 2\cos\frac{1}{2}\left(z + \frac{1}{z}\right).\cos\frac{1}{2}\left(z - \frac{1}{z}\right)$.

Now $\frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta$,

$$\tfrac{1}{2}\left(z - \frac{1}{z}\right) = \tfrac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i\sin\theta.$$

Hence $C = \cos(\cos\theta).\cos(i\sin\theta) = \cos(\cos\theta).\cosh(\sin\theta)$.
In given series write $\theta = \frac{1}{4}\pi$. Then

$$1 - \frac{1}{2!}\cos\tfrac{1}{2}\pi + \frac{1}{4!}\cos\pi - \frac{1}{6!}\cos\tfrac{3}{2}\pi + \frac{1}{8!}\cos 2\pi - \frac{1}{10!}\cos\tfrac{5}{2}\pi + \frac{1}{12!}\cos 3\pi - \dots$$

$$= \cos(\cos\tfrac{1}{4}\pi)\cosh(\sin\tfrac{1}{4}\pi),$$

i.e. $1 - \dfrac{1}{4!} + \dfrac{1}{8!} - \dfrac{1}{12!} + \dots = \cos(1/\sqrt{2}).\cosh(1/\sqrt{2})$.

Now $1/\sqrt{2} = 0{\cdot}7071\dots$, and $0{\cdot}7071$ radians $= 40° 31'$ approx. Thus $\cos(1/\sqrt{2}) = 0{\cdot}7602$, and the value of the series is

$$0{\cdot}7602 \times 1{\cdot}261 = 0{\cdot}959 \text{ approx.}$$

11·94. The Logarithmic Series

Examples.—(1) *If* $|x| < 1$, *find the sum to infinity of the series*

$$x\cos\theta - \tfrac{1}{2}x^2\cos 2\theta + \tfrac{1}{3}x^3\cos 3\theta - \dots$$ [*Camb. Sch.*]

Write $C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \cos n\theta$, $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \sin n\theta$. Then

$$C + iS = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} (\cos n\theta + i \sin n\theta)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (xe^{i\theta})^n = \log (1 + xe^{i\theta}).$$

Hence $C - iS = \log (1 + xe^{-i\theta})$.

Adding, $2C = \log (1 + xe^{i\theta}) + \log (1 + xe^{-i\theta})$

$= \log \{(1 + xe^{i\theta})(1 + xe^{-i\theta})\}$

$= \log \{1 + x(e^{i\theta} + e^{-i\theta}) + x^2\} = \log (1 + 2x \cos \theta + x^2)$.

(2) *If* $c = \cos^2\theta - \frac{1}{3}\cos^3\theta.\cos 3\theta + \frac{1}{5}\cos^5\theta.\cos 5\theta - \ldots$ *to infinity, prove that* $\tan 2c = 2 \cot^2\theta$. [*Camb. Sch.*]

Let s denote the series

$$\cos \theta.\sin \theta - \tfrac{1}{3}\cos^3\theta.\sin 3\theta + \tfrac{1}{5}\cos^5\theta.\sin 5\theta - \ldots$$

Then $c + is = e^{i\theta}\cos \theta - \frac{1}{3}(e^{i\theta}\cos \theta)^3 + \frac{1}{5}(e^{i\theta}\cos \theta)^5 - \ldots$

$= z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \ldots,$

where $z = e^{i\theta}\cos \theta$. Now if $|x| < 1$,

$$\tfrac{1}{2}\log \frac{1+x}{1-x} = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots + \frac{1}{2n-1}x^{2n-1} + \ldots$$

Write $x = iz$.* Then

$$\frac{1}{2i}\log \frac{1+iz}{1-iz} = z + \frac{i^2}{3}z^3 + \frac{i^4}{5}z^5 + \ldots + \frac{i^{2n-2}}{2n-1}z^{2n-1} + \ldots$$

$$= z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \ldots + \frac{(-1)^{n-1}}{2n-1}z^{2n-1} + \ldots$$

$$\text{Thus } c + is = \frac{1}{2i}\log \frac{1 + ie^{i\theta}\cos \theta}{1 - ie^{i\theta}\cos \theta}.$$

$$\text{Hence } c - is = \frac{1}{2(-i)}\log \frac{1 - ie^{-i\theta}\cos \theta}{1 + ie^{-i\theta}\cos \theta}.$$

$$\text{Adding, } 2c = \frac{1}{2i}\left\{\log \frac{1 + ie^{i\theta}\cos \theta}{1 - ie^{i\theta}\cos \theta} - \log \frac{1 - ie^{-i\theta}\cos \theta}{1 + ie^{-i\theta}\cos \theta}\right\}$$

$$= \frac{1}{2i}\log \frac{(1 + ie^{i\theta}\cos \theta)(1 + ie^{-i\theta}\cos \theta)}{(1 - ie^{i\theta}\cos \theta)(1 - ie^{-i\theta}\cos \theta)}$$

* The series converges absolutely if $|z| < 1$, i.e. $|\cos \theta| < 1$. Hence there is absolute convergence except for $\theta = m\pi$, whence m denotes 0 or a positive or negative integer. At points on the circle of convergence the series will converge except at the special points $iz = \pm 1$, i.e. $ie^{i\theta}\cos \theta = \pm 1$. But when $\theta = m\pi$, $e^{i\theta}$ and $\cos \theta$ are real, so that $ie^{i\theta}\cos \theta$ is purely imaginary and can never be equal to ± 1. It follows that the series will converge for all values of θ.

$$= \frac{1}{2i} \log \frac{1 + i\cos\theta\,(e^{i\theta} + e^{-i\theta}) - \cos^2\theta}{1 - i\cos\theta\,(e^{i\theta} + e^{-i\theta}) - \cos^2\theta}$$

$$= \frac{1}{2i} \log \frac{\sin^2\theta + 2i\cos^2\theta}{\sin^2\theta - 2i\cos^2\theta} = \frac{1}{2i} \log \frac{1 + 2i\cot^2\theta}{1 - 2i\cot^2\theta}.$$

$$\text{Hence } \frac{1 + 2i\cot^2\theta}{1 - 2i\cot^2\theta} = e^{4ic},$$

$$\text{i.e. } 2\cot^2\theta = \frac{1}{i}\frac{e^{4ic} - 1}{e^{4ic} + 1} = \frac{1}{i}\frac{e^{2ic} - e^{-2ic}}{e^{2ic} + e^{-2ic}} = \tan 2c.$$

11·95. Recurring Series

Examples.—(1) *Find the sum to n terms of the series:*

$$1.3\cos\theta + 3.5\cos 2\theta + 5.7\cos 3\theta + \ldots$$

If u_r denote the rth term of the series,

$$u_r = (2r - 1)(2r + 1)\cos r\theta = (4r^2 - 1)\cos r\theta.$$

Write $C = \sum_{r=1}^{n} (4r^2 - 1)\cos r\theta,\ S = \sum_{r=1}^{n} (4r^2 - 1)\sin r\theta.$ Then

$$C + iS = \sum_{r=1}^{n} (4r^2 - 1)\, e^{ri\theta} = \sum_{r=1}^{n} (4r^2 - 1)\, z^r,$$

where $z = e^{i\theta}$. For brevity we write $C + iS = s$. Then

$$s = 3z + 15z^2 + 35z^3 + \ldots + (4n^2 - 1)\, z^n$$

$$(1 - z)\, s = 3z + 12z^2 + 20z^3 + \ldots + (8n - 4)\, z^n - (4n^2 - 1)\, z^{n+1}$$

$$(1 - z)^2 s = 3z + 9z^2 + 8z^3 + \ldots + 8z^n$$
$$- (4n^2 + 8n - 5)z^{n+1} + (4n^2 - 1)\, z^{n+2}$$

$$= 3z + 9z^2 + \frac{8z^3(1 - \ )}{1 - z} - (4n^2 + 8n - 5)\, z^{n+1} + (4n^2 - 1)\, z^{n+2};$$

$$\therefore\ C + iS = \{3z + 6z^2 - z^3 - (4n^2 + 8n + 3)\, z^{n+1}$$
$$+ (8n^2 + 8n - 6)\, z^{n+2} - (4n^2 - 1)\, z^{n+3}\}/(1 - z)^3.$$

$$\text{Now } 1/(1 - z)^3 = 1/(1 - e^{i\theta})^3 = - e^{-\frac{3}{2}i\theta}/(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta})^3$$
$$= e^{-\frac{3}{2}i\theta}/8\,i\sin^3\tfrac{1}{2}\theta;$$

$$\therefore\ C + iS = \{3e^{-\frac{1}{2}i\theta} + 6e^{\frac{1}{2}i\theta} - e^{\frac{3}{2}i\theta} - (4n^2 + 8n + 3)\, e^{(n - \frac{1}{2})\, i\theta}$$
$$+ (8n^2 + 8n - 6)\, e^{(n + \frac{1}{2})i\theta} - (4n^2 - 1)\, e^{(n + \frac{3}{2})\, i\theta}\}/8i\sin^3\tfrac{1}{2}\theta.$$

Equating real parts, we have

$$C = \{- 3\sin\tfrac{1}{2}\theta + 6\sin\tfrac{1}{2}\theta - 3\sin\tfrac{3}{2}\theta - (4n^2 + 8n + 3)\sin(n - \tfrac{1}{2})\,\theta$$
$$+ (8n^2 + 8n - 6)\sin(n + \tfrac{1}{2})\,\theta - (4n^2 - 1)\sin(n + \tfrac{3}{2})\,\theta\}/8\sin^3\tfrac{1}{2}\theta.$$

$$= \{- 6\cos\theta\sin\tfrac{1}{2}\theta - (4n^2 + 8n + 3)\sin(n - \tfrac{1}{2})\theta$$
$$+ (8n^2 + 8n - 6)\sin(n + \tfrac{1}{2})\,\theta - (4n^2 - 1)\sin(n + \tfrac{3}{2})\,\theta\}/8\sin^3\tfrac{1}{2}\theta.$$

(2) *If* $C = x \cos\theta + 2x^2 \cos 2\theta + \ldots + nx^n \cos n\theta + \ldots$ *and*

$$S = x \sin\theta + 2x^2 \sin 2\theta + \ldots + nx^n \sin n\theta + \ldots \; |x| < 1,$$

show that $\dfrac{C}{S} = \dfrac{(1 + x^2)\cos\theta - 2x}{(1 - x^2)\sin\theta}$. [*Lond. B.Sc.*]

$$C + iS = xe^{i\theta} + 2x^2e^{2i\theta} + \ldots + nx^ne^{ni\theta} + \ldots$$

$$= \sum_{n=1}^{\infty} nz^n, \text{ where } z = xe^{i\theta}.$$

This is a recurring series whose scale of relation is $(1 - z)^2$. Summing in the usual way we see that the sum is

$$z/(1 - z)^2, \quad |z| < 1, \quad \text{i.e. } C + iS = xe^{i\theta}/(1 - xe^{i\theta})^2, \quad |x| < 1.$$

Hence $C - iS = xe^{-i\theta}/(1 - xe^{-i\theta})^2$ and $\dfrac{C + iS}{C - iS} = \dfrac{(e^{i\theta} - x)^2}{(1 - xe^{i\theta})^2}$.

Applying componendo and dividendo,

$$\frac{C}{iS} = \frac{(e^{i\theta} - x)^2 + (1 - xe^{i\theta})^2}{(e^{i\theta} - x)^2 - (1 - xe^{i\theta})^2} = \frac{(e^{2i\theta} + 1)(1 + x^2) - 4xe^{i\theta}}{(1 - x^2)(e^{2i\theta} - 1)}$$

$$= \frac{(e^{i\theta} + e^{-i\theta})(1 + x^2) - 4x}{(1 - x^2)(e^{i\theta} - e^{-i\theta})} = \frac{(1 + x^2)\cos\theta - 2x}{i(1 - x^2)\sin\theta},$$

giving the required form.

EXERCISES XI

1. The distance between two places is a, and on the first day $\dfrac{1}{m}$th of the journey from one to the other is performed; on the second day $\dfrac{1}{n}$ of the remainder; then $\dfrac{1}{m}$ and $\dfrac{1}{n}$ of the remainders alternately on succeeding days. Prove that the total distance travelled in $2p$ days is

$$a\left\{1 - \left(1 - \frac{1}{m}\right)^p\left(1 - \frac{1}{n}\right)^p\right\}.$$ [*Camb. Sch.*]

2. Evaluate $-1^3 + 3^3 - 5^3 + 7^3 - 9^3 + \ldots - 37^3 + 39^3$. [*Camb. Sch.*]

3. Prove that $x = 2/(n + 1)$ satisfies the equation

$$(x - 1)^3 + (2x - 1)^3 + (3x - 1)^3 + \ldots + (nx - 1)^3 = 0,$$

and find the quadratic factor satisfied by the other two roots. [*Camb. Sch.*]

4. Find the sum to the n terms of the series

$$1^2.2 + 3^2.3 + 5^2.4 + 7^2.5 + \ldots$$ [*Camb. Sch.*]

5. Find the sum to n terms of the series

$$1^2.3 + 2^2.5 + 3^2.7 + \ldots$$ [*Sc. T. Prelim.*]

6. Sum to n terms the series whose rth term is $r(r+1)(2r+1)$.
[*Lond. B.A.*]

7. Prove that $\sin x \sin 2x + \sin 2x \sin 3x + \ldots + \sin(n-1)x \sin nx = (n \sin 2x - \sin 2nx)/4 \sin x$. [*Lond. B.Sc.*]

8. Show that

$$1^2.n + 2^2.(n-1) + 3^2.(n-2) + \ldots + n^2.1 = \tfrac{1}{12}n(n+1)^2(n+2),$$

and prove independently that the latter expression is an integer.
[*Lond. B.Sc.*]

9. Prove that

$$1 + \frac{2}{3} + \frac{2.5}{3.6} + \frac{2.5.8}{3.6.9} + \ldots + \frac{2.5.8\ldots(3n-1)}{3.6.9\ldots(3n)} = \frac{5.8.11\ldots(3n+2)}{3.6.9\ldots(3n)}.$$

10. Prove that the sum to n terms of the series

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$$

is $n(n+3)/4(n+1)(n+2)$. [*Lond. B.Sc.*]

11. Prove that

$$1 + \frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30} + \ldots \text{ to infinity} = (10/7)^{\frac{1}{3}}.$$

12. Find the sum to infinity of the series

$$\frac{3}{4}.\frac{1}{2^4} + \frac{3.5}{4.6}.\frac{1}{2^6} + \frac{3.5.7}{4.6.8}.\frac{1}{2^8} + \ldots$$

13. (i) Sum to n terms the series $\dfrac{r}{(r+1)!} + \dfrac{2r^2}{(r+2)!} + \dfrac{3r^3}{(r+3)!} + \ldots$

(ii) Sum to infinity $1 + \dfrac{11}{14} + \dfrac{11.13}{14.16} + \dfrac{11.13.15}{14.16.18} + \ldots$ and establish its convergence. [*Camb. Sch.*]

14. (i) Sum to n terms the series whose rth term is $r(r+2)(r+4)$.

(ii) Find the sum to infinity of the series

$$1 + \frac{3}{10} + \frac{3.7}{10.20} + \ldots + \frac{3.7.11\ldots(4n-1)}{n!(10)^n} + \ldots$$

[*Lond. B.Sc.*]

15. Prove that

$$\frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + \frac{1.3.5.7}{4.6.8.10} + \ldots \text{ to } \infty = 1. \quad [\textit{Camb. Sch.}]$$

16. Sum to infinity the series

$$1 + \frac{1}{3} + \frac{1.3}{2!}.\frac{1}{3^2} + \frac{1.3.5}{3!}.\frac{1}{3^3} + \ldots \quad [\textit{Lond. B.A.}]$$

17. Sum to infinity the series

$$1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \ldots$$

18. Sum to infinity the series

$$\text{(i)}\ \frac{2\frac{1}{2}}{1!} - \frac{3\frac{1}{3}}{2!} + \frac{4\frac{1}{4}}{3!} - \frac{5\frac{1}{5}}{4!} + \ldots,$$

$$\text{(ii)}\ 1 + x(1 + y) + x^2(1 + y + y^2) + x^3(1 + y + y^2 + y^3) + \ldots$$

and state the condition for convergence of the second series. [*Camb. Sch.*]

19. Sum to infinity the series whose nth term is $(n + 3)/n\,!$. [*Lond. B.A.*]

20. Sum to infinity $\frac{1}{1!} + \frac{1 + 2}{2!} + \frac{1 + 2 + 3}{3!} + \frac{1 + 2 + 3 + 4}{4!}$

21. Show that $\sqrt{e} = 1 + \frac{1}{2!} + \frac{1.3}{4!} + \frac{1.3.5}{6!} + \ldots$ to ∞.

22. Find the sum of the series

$$\frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \frac{5^3}{4!} + \ldots \text{ to } \infty.$$

23. Sum to infinity $\frac{1}{1.2.3.4} + \frac{1}{3.4.5.6} + \frac{1}{5.6.7.8} + \ldots$

24. (i) Find the sum of the infinite series $1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \ldots$

(ii) Find the sum of n terms of the series

$$\frac{1}{3.7.11} + \frac{5}{7.11.15} + \frac{9}{11.15.19} + \ldots$$ [*Camb. Sch.*]

25. Sum to infinity the series whose rth term is $(r^2 - 3r + 5)/r\,!$. [*Lond. B.Sc.*]

26. Resolve $(12x^2 - 9x + 2)/\{(4x - 3)(4x - 2)(4x - 1)\}$ into partial fractions. Show that

$$\sum_{n=1}^{\infty} \frac{12n^2 - 9n + 2}{(4n - 3)(4n - 2)(4n - 1)} \cdot \frac{1}{2^{4n-3}} = \frac{1}{4} \log \frac{27}{5}.$$

[*Lond. B.Sc.*]

27. Prove that $\frac{1}{3}\left(1 - \frac{1}{2^3}\right) - \frac{1}{4}\left(1 + \frac{1}{2^4}\right) + \frac{1}{5}\left(1 - \frac{1}{2^5}\right) - \ldots = \frac{1}{8}$. [*Camb. Sch.*]

28. Prove that, if $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$,

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin^{2n} \alpha}{n} = \sum_{n=1}^{\infty} 2^n \sin^{2n} \tfrac{1}{2}\alpha$$ [*Lond. B.A.*]

29. Show that the series $\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{4}\left(\frac{1}{2}\right)^4 + \ldots$ and

$$\frac{2}{7} + \frac{2}{3 \cdot 7^3} + \frac{2}{5 \cdot 7^5} + \frac{2}{7 \cdot 7^7} + \ldots$$

are convergent infinite series and that the limit to which the sum of n terms of each tends is the same. What is the limit?

30. Prove that, when x lies between 0 and 1,

$$2\sum_{n}^{\infty} \frac{(1-x)^{2n+1}}{2n+1} + \sum_{n=1}^{\infty}\left[\left\{\frac{1}{3^n} + (-1)^{n-1}\right\}\frac{(2x-1)^n}{n}\right] = \log_e 3.$$

[*Lond. B.A.*]

31. Sum to infinity the series

$$\text{(i)}\quad 1 + \frac{3\cdot 5}{4\cdot 8} - \frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12} + \dots$$

$$\text{(ii)}\quad \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \frac{x^4}{4\cdot 5} + \dots$$

and state the condition of convergence of the second series. [*Camb. Sch.*]

32. Sum to infinity the series whose rth term is rx^r and find for what values of x the infinite series is convergent. Sum to infinity, the series whose rth term is $rx^r \cosh r\theta$, stating for what values of x, θ the series is convergent [*Camb. Sch*]

33. Prove that $1^2x + 2^2x^2 + 3^2x^2 + \dots = x(1-x)^{-2} + 2x^2(1-x)^{-3}$. [*Lond. B.A.*]

34. Sum the infinite series whose nth term is $(n^2+2)/n!$.

35. Express $\{n(n^2-1)^2\}^{-1}$ in partial fractions. Show that the sum to infinity of the series whose $(r-1)$th term is $\{r(r^2-1)^2\}^{-1}$ is $\frac{1}{16}$. [*Lond. B.Sc.*]

36. Sum to n terms and to infinity the series whose rth term is

$$1/(2r+1)(2r+7)(2r+9).$$ [*Lond. B.A.*]

37. Find the sum to infinity of the series

$$1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \dots$$

38. (i) Sum to n terms the series whose rth term is

$$\{r(r+1)(r+2)(r+3)\}^{-1}.$$

(ii) By expressing $r^2 + 2r + 3$ in the form

$$A + B(r+3) + C(r+2)(r+3),$$

or otherwise, find the sum to infinity of the series whose rth term is

$$(r^2+2r+3)/r(r+1)(r+2)(r+3).$$ [*Lond. B.Sc.*]

39. Prove that the series of which the nth term is

$$nx^n/\{(n+1)(n+3)(n+4)\}$$

converges if $-1 < x < 1$, and find the sum to infinity. [*Lond. B.Sc.*]

40. Sum to infinity $\dfrac{2.1^2+3}{2!} + \dfrac{2.2^2+4}{3!} + \dfrac{2.3^2+5}{4!} +$

41. Find when the infinite series $\frac{1}{1\cdot 3} + \frac{x^2}{3\cdot 5} + \frac{x^4}{5\cdot 7} + \ldots$ is convergent for real values of x. Find its sum when it converges. [*Lond. B.Sc.*]

42. (i) Sum to n terms the series $\frac{1}{1\cdot 3\cdot 5} + \frac{2}{3\cdot 5\cdot 7} + \frac{3}{5\cdot 7\cdot 9} + \ldots$

(ii) Prove that $\frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots$ to $\infty = 5e - 1$. [*Camb. Sch.*]

43. Sum to infinity the series whose nth term is $(n - 1)\,x^n/(n + 2)\,n!$

44. Sum to infinity the series whose nth term is $x^n/(n + 1)(n + 3)$, $x^2 < 1$. [*Lond. B.Sc.*]

45. Sum to infinity the series

(i) $\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \frac{1}{7}\left(\frac{1}{2}\right)^7 + \ldots$

(ii) $\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \frac{4^4}{4!} + \ldots$

46. Find the sum of the infinite series whose nth term is

$$(3n^2 + 3n + 1)/n^3(n + 1)^3.$$ [*Camb. Sch.*]

47. If $f(n) \equiv A + Bn + Cn^2 + Dn^3$, where A, B, C, D denote numerical quantities, find their values if $f(n + 1) - f(n) = n^2$, for all values of n, and $f(0) = 0$. Hence find the sum of the series $1^2 + 2^2 + 3^2 + \ldots + n^2$, where n is a positive integer.

48. (i) By using the identity

$$\frac{2}{a(a + 1)(a + 2)} = \frac{1}{a(a + 1)} - \frac{1}{(a + 1)(a + 2)}$$

or otherwise, find the sum of n terms of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \ldots$$

Show that, as n becomes large, this sum approaches the limit $\frac{1}{4}$.

(ii) Find the sum of n terms of the series

$$1\cdot 3\cdot 5 + 3\cdot 5\cdot 7 + 5\cdot 7\cdot 9 + \ldots$$ [*N.Sc. Prelim.*]

49. Find the sum of n terms of the series

$$\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \frac{1}{3\cdot 4\cdot 5\cdot 6} + \ldots$$

50. Determine the coefficients A, B, C, so that if $f(x)$ denote the polynomial $Ax^5 + Bx^3 + Cx$,

$$f(x) - f(x - 1) = (2x - 1)^4$$

for all values of x. Find the sum of the fourth powers of the first n odd positive integers. [*Lond. B.Sc.*]

51. Find the sum of n terms, and also to infinity, of the series

$$\frac{5}{1.2}\cdot\frac{1}{3}+\frac{7}{2.3}\cdot\frac{1}{3^2}+\frac{9}{3.4}\cdot\frac{1}{3^3}+\ldots$$ [*Camb. Sch.*]

52. Prove that if $b-1>a>0$, the series

$$1+\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\ldots$$

converges to the sum $(b-1)/(b-a-1)$.

53. Find the sum of the series

$$\sin\theta\sin 2\theta+2\sin 2\theta\sin 3\theta+\ldots+n\sin n\theta\sin(n+1)\theta.$$

[*Camb. Sch.*]

54. Find the nth term and the sum to n terms of the series

$$1.2+2.3+3.6+4.11+5.18+6.27+\ldots$$

55. Find the polynomial of the third degree in n which has the values $-3, -7, -5, 9$ when $n=1, 2, 3, 4$ respectively.

56. Find the value of $\sum\limits_{r=1}^{n} u_r$, when $u_r=\sum\limits_{s=1}^{r} s^2$.

57. Show that the sum to n terms of the series whose rth term is $r^2.2^r$ is $2^{n+1}(n^2-2n+3)-6$. [*Lond. B.Sc.*]

58. Find the sum to n terms of the series

$$\cos a\cos 2a+\cos 2a\cos 3a+\cos 3a\cos 4a+\ldots$$ [*Camb. Sch.*]

59. Prove that if x is real and numerically less than unity

$$1+2x\cos\theta+2x^2\cos 2\theta+\ldots+2x^n\cos n\theta+\ldots=\frac{1-x^2}{1-2x\cos\theta+x^2}.$$

[*Camb. Sch.*]

60. If $|x|<1$, sum to infinity the series

$$\cos\theta+x\cos 3\theta+x^2\cos 5\theta+\ldots+x^n\cos(2n+1)\theta+\ldots$$

[*Camb. Sch.*]

61. Sum to n terms the series whose rth term is $x^r\cos r\theta$. Prove that, if $x=\cos\theta$, the sum is $\sin n\theta\cos^{n+1}\theta/\sin\theta$. [*Lond. B.A.*]

62. (i) Sum to n terms the series whose rth term is $\cos r\alpha\sec^r\alpha$. (ii) Sum to infinity the series whose rth term is $(1/r!)\sin r\alpha$. [*Lond. B.A.*]

63. Sum to infinity

$$\cos\theta-\tfrac{1}{2}\cos 3\theta+\frac{1.3}{2.4}\cos 5\theta-\frac{1.3.5}{2.4.6}\cos 7\theta+\ldots$$

[*Camb. Sch.*]

64. Find the sum to infinity of the series

$$n\sin\theta+\frac{n(n+1)}{1.2}\sin 2\theta+\frac{n(n+1)(n+2)}{1.2.3}\sin 3\theta+\ldots$$

[*Camb. Sch.*]

65. Prove that

$$\sin\theta + \frac{1}{3!}\sin 3\theta + \frac{1}{5!}\sin 5\theta + \ldots \text{ to infinity} = \sin(\sin\theta).\cosh(\cos\theta).$$

[*Camb. Sch.*]

66. Show that the sum of the infinite series

$$1 + \frac{\cos\theta}{1!} + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \ldots \text{ is } e^{\cos\theta}\cos(\sin\theta).$$

67. Show that the sum of the infinite series

$$\tfrac{1}{2}\sin 2\theta + \tfrac{1}{4}\sin 4\theta + \tfrac{1}{6}\sin 6\theta + \ldots \text{ is } \tfrac{1}{4}\pi - \tfrac{1}{2}\theta.$$

[*N.Sc., Prelim.*]

68. Show that the sum of the infinite series

$$\tfrac{1}{2}\cos 2\theta + \tfrac{1}{4}\cos 4\theta + \tfrac{1}{6}\cos 6\theta + \ldots \text{ is } \tfrac{1}{2}\log\frac{1}{2\sin\theta}.$$

69. If $|x| < 1$, find the sum to infinity of the series

$$\sin\theta + x\sin 2\theta + x^2\sin 3\theta + \ldots$$

[*Camb. Sch.*]

70. Sum to infinity the series

$$x\sin\theta + 2x^2\sin 2\theta + 3x^3\sin 3\theta + \ldots$$

[*Lond. B.Sc.*]

71. Find the sum to n terms of the series whose $(r+1)$th term is $a_r\cos(\alpha + r\beta)$, where the coefficients a_r are in arithmetic progression.

72. Prove that the series $\sum\limits_{r=1}^{\infty}\dfrac{r\cos r\theta}{3^r}$ is convergent for all values of θ and that its sum to infinity is $\dfrac{3(5\cos\theta - 3)}{2(5 - 3\cos\theta)^2}$. [*Lond. B.Sc.*]

73. Sum to n terms the series whose rth term is $(2r-1)x^{r-1}$. If $\alpha = 2\pi/n$, where n is a positive integer, show that

$$1 + 3\cos\alpha + 5\cos 2\alpha + \ldots + (2n-1)\cos(n-1)\alpha = -n.$$

and

$$3\sin\alpha + 5\sin 2\alpha + \ldots + (2n-1)\sin(n-1)\alpha = -n\cot\tfrac{1}{2}\alpha.$$

[*Lond. B.Sc.*]

CHAPTER XII

DETERMINANTS

IN this chapter we discuss properties of a class of functions called determinants, which are of great importance in many branches of mathematics both pure and applied. With a knowledge of the properties of such functions many simplifications may be introduced and results represented in more compact forms.

Thus, *e.g.* consider the problem of eliminating x, y, z between the homogeneous linear equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0,$$

where we suppose that x, y, z are different from zero. From the last two equations we obtain, from the rule of cross-multiplication

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2} = k, \text{ say.}$$

Since x, y, z are different from 0, $k \neq 0$. Substituting these values of x, y, z in the first equation, it follows that

$$k\,\{a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)\} = 0.$$

Since $k \neq 0$, the required eliminant is

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0.$$

It will be shown later from the theory of determinants that this result may be written down *immediately* in *compact form.*

12·11. Formation of a Determinant

We now consider how a determinant is formed. Let us take the nine quantities a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 and arrange them in a square with three horizontal rows and three vertical columns as follows:

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3. \end{matrix}$$

This we may call a *square array* of the letters, and since there are *three* letters in each column and row, say that it is of the *third order.*

Now form all the products of the three letters a, b, c, keeping the alphabetical order and attaching suffixes corresponding to the permutations of the numbers 1, 2, 3. Since there are 3 ! permutations there will be 3 ! products. They are

$$a_1 b_2 c_3, \quad a_1 b_3 c_2, \quad a_2 b_3 c_1, \quad a_2 b_1 c_3, \quad a_3 b_1 c_2, \quad a_3 b_2 c_1.$$

It is now necessary to attach a sign to each of the products and this is determined by the *order of the suffixes*. We speak about the order 1, 2, 3 as the *natural order*, and the product with the suffixes attached in this order is given the positive sign. The signs of the other terms are determined by considering the number of changes in the new order of the suffixes necessary in order to obtain the natural order.

An *interchange of two adjacent suffixes* may be spoken of as an *inversion* of the suffixes. Thus 1 2 3 is obtained from 1 3 2 by one inversion, *viz.* by interchanging 2 and 3. Again, 1 2 3 may be obtained from 2 3 1 by two inversions. Thus: 2 3 1, 2 1 3, 1 2 3. We may look upon " inversion " from a slightly different point of view and say that an inversion occurs when any higher number precedes a lower one among the suffixes. Thus regarded from this angle there are two inversions in 2 3 1 because 2 precedes 1, and 3 precedes 1.

We have the following rule: *The sign of any term is positive or negative according as the term contains among its suffixes an even or odd number of inversions.*

With this convention the signs to be attached to the six products considered above are as follows:

$$+ a_1b_2c_3, \quad - a_1b_3c_2, \quad + a_2b_3c_1, \quad - a_2b_1c_3, \quad + a_3b_1c_2, \quad - a_3b_2c_1.$$

The algebraic sum formed by these six quantities, *viz.*

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

is ***defined*** to be the determinant formed by the nine quantities a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 and is written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Any one of the nine letters, a_1, a_2, a_3, b_1, . . . , may be called an **element** or **constituent** of the determinant, and any one of the six products (with the proper sign attached) used in defining the determinant, a **term** of the determinant.

12·12. General Definition of a Determinant

Let $a, b, c, \ldots k$ denote n numbers and consider the n^2 numbers obtained by attaching the suffixes $1, 2, 3, \ldots n$. Thus

$$\begin{array}{cccccc} a_1 & b_1 & c_1 & \cdots & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & \cdots & k_2 \\ a_3 & b_3 & c_3 & \cdots & \cdots & k_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & \cdots & k_n \end{array}$$

Then the determinant of the nth order *determined by the n^2 numbers is defined to be the sum of all the products, each with the proper sign attached in accordance with the rule of inversions, which can be formed by writing the letters $a\,b\,c \ldots k$ in their natural order and arranging the suffixes in all possible ways.* The determinant is represented as

$$\begin{array}{cccccc} a_1 & b_1 & c_1 & \cdots & \cdots & k_1 \\ a_2 & & c_2 & & & \\ a_3 & & c_3 & & & k_3 \end{array}$$

When the determinant is considered in this general form it may be conveniently represented by the notation:

$$(a_1 b_2 c_3 \ldots k_n) \text{ or } \Sigma \pm a_1 b_2 c_3 \ldots k_n.$$

Each of the n^2 numbers is called an **element** (or constituent) of the determinant. Any one of the products in the *expansion* of the determinant, with the proper sign attached is called a **term** of the determinant. The diagonal element $a_1 b_2 c_3 \ldots k_n$ is called **leading or principal term,** the diagonal itself being called the **leading or principal diagonal.**

The expanded form of the determinant has $n!$ terms for the n numbers $1, 2, 3, \ldots n$ can be permuted among themselves in $n!$ ways.

Further, half the terms have the $+$ sign attached, the other half having the $-$ sign.

For we can arrange the terms in pairs so that if one is positive the other is negative, and vice versa. This may be seen by considering the last two suffixes of any term. Keeping the first $n - 2$ suffixes fixed, interchange the last two. We obtain a new term

which is of opposite sign to the original term, for it is obtained by one inversion. Further, there are only two terms in which the order of the first $n - 2$ suffixes is fixed.

It should be noted that *every term of the determinant when expanded contains one, and only one, element from each row and from each column.*

Examples.—(1) *Find the sign of the term $a_5b_3c_4d_2e_7f_1g_6$ in the determinant of the 7th order.*

It is necessary to consider the number of inversions. The order of the suffixes is 5342716. Corresponding to 5 there are 4 inversions since 5 precedes 1, 2, 3, 4. Corresponding to 3 there are 2 inversions, since 3 precedes 2 and 1. Similarly to 4 corresponds 2 inversions, to 2 corresponds 1, while to 7 correspond 2 inversions. Hence the total number of inversions is

$$4 + 2 + 2 + 1 + 2 = 11,$$

and the sign to be attached is $-$.

The eleven inversions may be represented in full as follows. Interchange 1 and 7, then 1 and 2, then 1 and 4, 1 and 3, 1 and 5. Thus

5342176, 5341276, 5314276, 5134276, 1534276.

Now interchange 2 and 4, then 2 and 3, 2 and 5, giving

1532476, 1523476, 1253476.

Then interchange 5 and 3, 5 and 4, and 6 and 7, giving

1235476, 1234576, 1234567.

(2) *If the letters retain their original order and any two suffixes are interchanged then the sign of the term is altered.*

For suppose that the two suffixes involved are r and s, and that between the two elements considered there are p elements.

Keeping r fixed, then by $p + 1$ inversions we can obtain the order sr, s and r being adjacent; then by p more inversions we can move r into the position originally occupied by s. Thus the interchange of the two letters is equivalent to $2p + 1$ inversions. Since this is an odd number the sign of the term is changed.

The result of this example can frequently be used to determine more quickly the sign of any given term. Consider the term $a_5b_3c_4d_2e_7f_1g_6$ [Ex. 1.] The given order is 5342716. Interchange 1 and 5, 2 and 4, 2 and 3, 6 and 7, 5 and 6. Thus

1342756, 1324756, 1234756, 1234657, 1234567.

Since there are an odd number of interchanges the sign is $-$, as the sign corresponding to 1234567 is $+$. These five interchanges correspond to eleven inversions.

(3) *If the suffixes retain their original order and two letters are interchanged the sign of the term is altered.*

This is equivalent to the letters being retained in their original order and the suffixes interchanged, and the result follows from Ex. 2.

Thus, suppose the original term is $a_1b_3c_4d_2e_5$, and that b and e are interchanged. The new term is $a_1e_3c_4d_2b_5 = a_1b_5c_4d_2e_3$ and this is obtained from the original term by interchanging the suffixes 3 and 5. Thus the sign of the new term is opposite to that of the original term.

12·21. Interchange of Rows and Columns

We now prove some important properties of determinants.

THEOREM I.—**If two rows or two columns of a determinant are interchanged, the determinant is unaltered in absolute value but is changed in sign.**

The interchange of two rows is equivalent to the interchange of two suffixes. The result then follows immediately from § 12·12, Ex. 2. On the other hand, the interchange of two columns is equivalent to the interchange of two letters. Hence from § 12·12, Ex. 3, the determinant is change in sign.

12·22. Expansion of a Determinant

From § 12·11 it follows that the determinant of the third order, $\Sigma \pm a_1b_2c_3$ can be written in the form

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

Now $b_2c_3 - b_3c_2 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$,

$$b_3c_1 - b_1c_3 = \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}, \quad b_1c_2 - b_2c_1 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

$$\text{Hence } \Sigma \pm a_1b_2c_3 = a_1A_1 + a_2A_2 + a_3A_3,$$

where A_1, A_2, A_3 are determinants of the second order. This result is a particular case of a more general property which is as follows:

THEOREM II.—**If $\Delta = \Sigma \pm a_1b_2c_3 \ldots k_n$ is a determinant of the nth order, then Δ can be expanded in the form**

$$a_1A_1 + a_2A_2 + a_3A_3 + \ldots + a_nA_n$$

where $A_1, A_2, A_3, \ldots, A_n$ are determinants of order $(n - 1)$.

Consider first all those terms which have a_1 as a factor. We know that each term of a determinant contains one, and only one, element from each row and from each column. Thus all terms which have a_1 as a factor cannot have $a_2, a_3, \ldots, a_n, b_1, c_1, \ldots k_1$ as factors.

Regard a_1 as fixed. Then all the terms which have a_1 as a factor will be obtained by considering the letters $b, c, \ldots k$ and all the

permutations of the $(n-1)$ numbers $2, 3, 4, \ldots, n$. Thus there are $(n-1)!$ terms which have a_1 as a factor. Thus

$$a_1A_1 = a_1\,\Sigma \pm b_2c_3 \ldots k_n$$

$$\text{and } A_1 = \Sigma \pm b_2c_3 \ldots k_n = \begin{vmatrix} b_2 & c_2 & \cdot\cdot & \cdot\cdot & k_2 \\ b_3 & c_3 & \cdot\cdot & \cdot\cdot & k_3 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ b_n & c_n & \cdot\cdot & \cdot\cdot & k_n \end{vmatrix}$$

To find the value of A_2 interchange the first and second rows. This has the effect of changing the sign of the determinant. Arguing as before it follows that

$$A_2 = -\left| \begin{matrix} b_1 & c_1 & \cdot\cdot & \cdot\cdot & k_1 \\ & & & & k_3 \\ & & & & \\ & & & & \\ b_n & c_n & \cdot\cdot & \cdot\cdot & k_n \end{matrix} \right.$$

Similarly for $A_3, A_4, \ldots, A_n$. The determinants $A_1, A_2, A_3, \ldots, A_n$ are called the cofactors of $a_1, a_2, a_3, \ldots, a_n$ respectively in the expansion of the determinant.

When the development of the determinant is written in the form

$$\triangle = a_1A_1 + a_2A_2 + a_3A_3 + \ldots + a_nA_n$$

it is said to be expanded along the first column. By interchanging columns it is clear that we can expand in a similar way along any column.

Similarly we may expand the determinant along any row. Thus, *e.g.* if we expand along the first row,

$$\triangle = a_1A_1 + b_1B_1 + c_1C_1 + \ldots + k_1K_1,$$

where $A_1, B_1, C_1, \ldots, \mathrm{K}_1$ are determinants of order $(n-1)$. Thus, *e.g.*

$$B_1 = -\begin{vmatrix} a_2 & c_2 & d_2 & \cdot\cdot & \cdot\cdot & k_2 \\ a_3 & c_3 & d_3 & \cdot\cdot & \cdot\cdot & k_3 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_n & c_n & d_n & \cdot\cdot & \cdot\cdot & k_n \end{vmatrix}$$

$$C_1 = \begin{vmatrix} a_2 & b_2 & d_2 & \cdot\cdot & \cdot\cdot & k_2 \\ a_3 & b_3 & d_3 & \cdot\cdot & \cdot\cdot & k_2 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_n & b_n & d_n & \cdot\cdot & \cdot\cdot & k_n \end{vmatrix} \quad \text{and so on.}$$

12·31. Minor Determinants

Let Δ denote a given determinant and suppose that in it we remove a certain number of columns and the *same* number of rows. It is assumed that the elements in the other rows and columns remain unaltered. What is left is a new determinant. Such a determinant is called a **minor determinant** of the original determinant.

If only one column and one row are removed the resulting determinant is called a **first minor**, while if two of each are deleted we obtain a **second minor**, and so on.

The minor obtained by deleting the first row and the first column is called the **leading first minor**. It is convenient to adopt the following notation for first minors. If h_r denote any element of a determinant, then the first minor obtained by deleting the column and the row through h_r is denoted by Δ_{h_r}. Thus, *e.g.* the leading first minor is Δ_{a_1}.

The results of § 12·22 may now be expressed in terms of minors instead of cofactors. Consider first the form.

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + \ldots + a_nA_n.$$

$$\text{Thus } A_1 = \begin{vmatrix} b_2 & c_2 & d_2 & \cdot\cdot & \cdot\cdot & k_2 \\ b_3 & c_3 & d_3 & \cdot\cdot & \cdot\cdot & k_3 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ b_n & c_n & d_n & \cdot\cdot & \cdot\cdot & k_n \end{vmatrix} = \Delta_{a_1}$$

$$A_2 = - \begin{vmatrix} b_1 & c_1 & d_1 & \cdot\cdot & \cdot\cdot & k_1 \\ b_3 & c_3 & d_3 & \cdot\cdot & \cdot\cdot & k_3 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ b_n & c_n & d_n & \cdot\cdot & \cdot\cdot & k_n \end{vmatrix} = - \Delta_{a_2}.$$

$$\text{In general } A_r = (-1)^{r-1}\Delta_{a_r}.$$

Expressed in " minor " notation:

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - \ldots + (-1)^{n-1}a_n\Delta_{a_n}.$$

In a similar way we can write down the expansion along any row or column in terms of the corresponding minors.

Examples.—(1) *Expand the determinant* $\begin{matrix} a & h & g \\ h & b & f \\ g & f & c \end{matrix}$

Expanding along the first column

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & g \\ f & c \end{vmatrix} + g\begin{vmatrix} h & g \\ b & f \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - fg) + g(hf - bg)$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2.$$

(2) *Expand the determinant* $(a_1b_2c_3d_4)$ *in terms of the elements of the fourth column.*

The determinant is
$$\Delta = \begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & & d_2 \\ a_3 & & & d_3 \\ a_4 & b_4 & & d_4 \end{matrix}$$

$$\Delta = d_1D_1 + d_2D_2 + d_3D_3 + d_4D_4$$

$$= \pm(d_1\Delta_{d_1} - d_2\Delta_{d_2} + d_3\Delta_{d_3} - d_4\Delta_{d_4}).$$

We know that

$$D_1 = \pm \begin{matrix} a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{matrix}$$

and it is first necessary to determine the correct sign. Interchanging columns

$$\Delta = - \begin{matrix} & b_1 & d_1 \\ & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ & b_4 & d_4 \end{matrix} \qquad \begin{matrix} d_1 & b_1 \\ d_2 & b_2 \\ d_3 & b_3 \\ d_4 & b_4 \end{matrix} \qquad \begin{matrix} d_1 & a_1 & b_1 & \\ d_2 & a_2 & b_2 & \\ d_3 & & b_3 & \\ d_4 & & b_4 & d_4 \end{matrix}$$

Expanding along the first column we see that

$$\Delta = -d_1\Delta_{d_1} + d_2\Delta_{d_2} - d_3\Delta_{d_3} + d_4\Delta_{d_4}$$

$$= -d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & d_4 \end{vmatrix} + d_2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} - d_3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix} + d_4\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The third order determinants may now be expanded in the usual way.

It should be noted that we can determine the sign to be attached to Δ_{d_1} *without going through the process of interchanging columns.* Begin with the leading element a_1 and count along the row, taking $+$ and $-$ signs alternately beginning with $+$. Thus to a_1 we attach $+$, to b_1 $-$, to c_1 $+$, to d_1 $-$. The required sign is $-$ and

$$\Delta = -d_1\Delta_{d_1} + d_2\Delta_{d_2} - d_3\Delta_{d_3} + d_4\Delta_{d_4}.$$

The reader will observe that this process is a general one and is equivalent to the process of interchanging columns.

(3) *Evaluate*

$$\begin{matrix} 3 \\ 6 \\ 10 \end{matrix}$$

[*Lond. B.Sc.*]

Denoting the determinant by Δ and expanding along the first column,

$$\Delta = 3\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 6\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + 10\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix}$$

$$= 3(24 - 25) - 6(18 - 20) + 10(15 - 16) = -1.$$

(4) *Prove that if u_n denote the determinant of nth order*

$$\begin{matrix} a & 1 & 0 & 0 \\ 1 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1 & a \end{matrix}$$

then $u_{n+1} - au_n + u_{n-1}$. *Hence, prove that*

$$u_n = (p^{n+1} - q^{n+1})/(p - q),$$

where p and q are the roots of the equation $x^2 - ax + 1 = 0$. [*Camb. Sch.*]

Consider u_{n+1} and expand the determinant along the first column. Then

$$u_{n+1} = a\Delta'_n - \Delta''_n,$$

where Δ'_n, Δ''_n are two nth order determinants.

$$\Delta'_n = \begin{vmatrix} a & 1 & 0 & \cdots & \cdots & 0 \\ 1 & a & 1 & \cdots & \cdots & 0 \\ 0 & 1 & a & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & a \end{vmatrix} = u_n.$$

$$\Delta''_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & a & 1 & \cdots & \cdots & 0 \\ 0 & 1 & a & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & a \end{vmatrix}$$

Expanding along the first row it is clear that $\Delta''_n = u_{n-1}$.

Hence $u_{n+1} = au_n - u_{n-1}$.

This difference equation can be written in the form $(E^2 - aE + 1)\, a_n = 0$, the general solution of which is $u_n = c_1 p^n + c_2 q^n$, where p, q are the roots of the characteristic equation $x^2 - ax + 1 = 0$.

Now $u_1 = a = p + q$.

$$u_2 = \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} = a^2 - 1 = (p + q)^2 - pq = p^2 + pq + q^2.$$

Substituting $n = 1, 2$ in the general solution

$$p + q = c_1 p + c_2 q$$
$$p^2 + pq + q^2 = c_1 p^2 + c_2 q^2.$$

Solving these equations for c_1, c_2 we have

$$c_1 = \frac{p}{p - q}, \quad c_2 = -\frac{q}{p - q}.$$

Hence $$u_n = (p^{n+1} - q^{n+1})/(p - q).$$

The result can also be proved by induction. Clearly it is true for $n = 1$ and $n = 2$. Then assuming $u_{r-1} = (p^r - q^r)/(p - q)$, $u_r = (p^{r+1} - q^{r+1})/(p - q)$ it can be shown that

$$u_{r+1} = (p^{r+2} - q^{r+2})/(p - q)$$

which is the required form.

12·32. Further General Properties

THEOREM III.—**The value of a determinant is unaltered by changing rows into columns and columns into rows.**

$$\text{Write } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & \cdots & k_2 \\ a_3 & b_3 & c_3 & \cdots & \cdots & k_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & \cdots & k_n \end{vmatrix},$$

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_1 & k_2 & k_3 & \cdots & \cdots & k_n \end{vmatrix}$$

Then Δ' is obtained from Δ by interchanging the rows and columns.

The leading terms $a_1b_2c_3 \ldots k_n$ are the same in each case. The terms in Δ are obtained from $a_1b_2c_3 \ldots k_n$ by retaining the order of the letters and permuting the suffixes in every possible way.

The terms in Δ' are obtained from $a_1b_2c_3 \ldots k_n$ by retaining the order of the suffixes and permuting the n letters in every possible way.

Thus it is clear that as far as numerical values are concerned the two processes give rise to the same $n!$ terms.

Further, the signs of corresponding terms must be the same. For in the case of Δ the interchange of two suffixes produces a change in the sign, while in the case of Δ' an interchange of two letters produces a change in sign. Hence $\Delta \equiv \Delta'$.

THEOREM IV.—**If, in any determinant two rows or two columns are identical the determinant is zero.**

Let Δ denote the determinant and suppose that two rows are identical. Then the determinant is unaltered when the rows are interchanged. But the interchange of two rows changes the sign of the determinant. (Th. I., § 12·21.)

$$\text{Hence } \Delta = -\Delta.$$

This is only possible if $\Delta = 0$. Similarly for the case of two identical columns.

Example.—*Show that*

$$\begin{matrix} b^3 \\ b^2 \\ 1 \end{matrix} \qquad - (b - c)(c - a)(a - b)(bc + ca + ab).$$

[*Lond. B.Sc.*]

If $a = b$, the first and second columns become identical and hence the determinant vanishes. Hence $(a - b)$ is a factor of the determinant. Similarly $(b - c)$, $(c - a)$ are factors.

Clearly each term of the determinant is of the fifth degree in a, b, c. Further, the determinant changes sign if two letters are interchanged, for this is equivalent to interchanging two rows.

Again, the product $(a - b)(b - c)(c - a)$ changes sign if two of the letters are interchanged. Hence the other factor of the determinant must be symmetrical and of the second degree in a, b, c. Hence

$$= (a - b)(b - c)(c - a)\{\lambda(a^2 + b^2 + c^2) + \mu(ab + bc + ca)\}.$$

where λ and μ are numerical constants.

To find λ, μ we observe that in the determinant the highest power of a which can occur in any term is a^3. On the right hand side there is a term in a^4, viz. $-\lambda a^4(b - c)$.

Hence $\lambda = 0$. Again, the leading term in the determinant is a^3b^2. On the right hand side the corresponding term is $-\mu a^3b^2$. Hence $\mu = -1$.

THEOREM V.—**If every element in any column is multiplied by the same factor, the whole determinant is multiplied by that factor.**

This result follows immediately from the property that every term contains one, and only one, element from any row or any column.

Combining (IV) and (V) we have the following result. *If a determinant has two rows or two columns which differ only from one another by a constant factor, then the determinant is zero.* Thus:

$$\begin{vmatrix} ma_1 & nb_1 & c_1 \\ ma_2 & nb_2 & c_2 \\ ma_3 & nb_3 & c_3 \end{vmatrix} = mn \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$\begin{vmatrix} ma_1 & nb_1 & pb_1 \\ ma_2 & nb_2 & pb_2 \\ ma_3 & nb_3 & pb_3 \end{vmatrix} = mnp \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = 0.$$

Examples.—(1) *Prove that*

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} 1 & 1 & 1 \\ a_1bc & b_1ca & c_1ab \\ a_2bc & b_2ca & c_2ab \end{vmatrix}$$

Multiply the columns of the first determinant by bc, ca, ab respectively. Then

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \frac{1}{a^2b^2c^2}\begin{vmatrix} abc & bca & cab \\ a_1bc & b_1ca & c_1ab \\ a_2bc & b_2ca & c_2ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} 1 & 1 & 1 \\ a_1bc & b_1ca & c_1ab \\ a_2bc & b_2ca & c_2ab \end{vmatrix}$$

on dividing the first row by abc.

It is clear that the process employed in this example is a general one which may be used to reduce any determinant to one in which the elements of any specified row or column are each unity.

(2) *Prove that*

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}$$

Consider the problem of reducing the second determinant to one whose first column is unity. Multiply the second row by bc, the third by ca, and the fourth by ab. We obtain

$$\frac{1}{a^2b^2c^2}\begin{vmatrix} 0 & a & b & c \\ abc & 0 & bc^2 & b^2c \\ abc & ac^2 & 0 & a^2c \\ abc & ab^2 & a^2b & 0 \end{vmatrix} \equiv \frac{1}{abc}\begin{vmatrix} 0 & a & b & c \\ 1 & 0 & bc^2 & b^2c \\ 1 & ac^2 & 0 & a^2c \\ 1 & ab^2 & a^2b & 0 \end{vmatrix}$$

$$\equiv \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}$$

on taking the factor a from the second column, b from the third and c from the fourth.

12·41. Addition of Determinants

THEOREM VI.—If every element in any column or in any row can be expressed as the sum of two quantities, then the given determinant can be expressed as the sum of two determinants of the same order.

Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_2 & b_2 & c_2 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_3 & b_3 & c_3 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \end{vmatrix}$$

and suppose that each element of the first column can be expressed as follows:

$$a_1 = a_1' + a_1'',\ a_2 = a_2' + a_2'',\ a_3 = a_3' + a_3'', \ldots$$

$$\text{Then } \Delta = \begin{vmatrix} a_1' & b_1 & c_1 & \cdots & \cdots & \cdots \\ a_2' & b_2 & c_2 & \cdots & \cdots & \cdots \\ a_3' & b_3 & c_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 & \cdots & \cdots & \cdots \\ a_2'' & b_2 & c_2 & \cdots & \cdots & \cdots \\ a_3'' & b_3 & c_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

For expanding Δ along the column, we have, with the usual notation,

$$\begin{aligned} \Delta &= a_1A_1 + a_2A_2 + a_3A_3 + \ldots \\ &= (a_1' + a_1'')\,A_1 + (a_2' + a_2'')\,A_2 + (a_3' + a_3'')\,A_3 + \ldots \\ &= \{a_1'A_1 + a_2'A_2 + a_3'A_3 + \ldots\} \\ &\qquad + \{a_1''A_1 + a_2''A_2 + a_3''A_3 + \ldots\} \\ &= \Delta_1 + \Delta_2 \end{aligned}$$

where Δ_1 and Δ_2 are two determinants given immediately above. The method clearly applies to any row or column and thus the theorem is proved.

Further, the result extends directly to more than one row or column. Suppose, *e.g.* that we have a third order determinant of the form

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ a_2 + \alpha_2 & b_2 + \beta_2 & c_2 + \gamma_2 \\ a_3 + \alpha_3 & b_3 + \beta_3 & c_3 + \gamma_3 \end{vmatrix}$$

By repeated application of the theorem we obtain

$$\begin{vmatrix} a_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ a_2 & b_2 + \beta_2 & c_2 + \gamma_2 \\ a_3 & b_3 + \beta_3 & c_3 + \gamma_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ \alpha_2 & b_2 + \beta_2 & c_2 + \gamma_2 \\ \alpha_3 & b_3 + \beta_3 & c_3 + \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 + \gamma_1 \\ a_2 & b_2 & c_2 + \gamma_2 \\ a_3 & b_3 & c_3 + \gamma_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 + \gamma_1 \\ a_2 & \beta_2 & c_2 + \gamma_2 \\ a_3 & \beta_3 & c_3 + \gamma_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 + \gamma_1 \\ \alpha_2 & b_2 & c_2 + \gamma_2 \\ \alpha_3 & b_3 & c_3 + \gamma_3 \end{vmatrix}$$

$$+\begin{matrix} a_1 & \beta_1 & c_1+\gamma_1 \\ a_2 & \beta_2 & c_2+\gamma_2 \\ a_3 & \beta_3 & c_3+\gamma_3 \end{matrix} \qquad \begin{matrix} a_1 & b_1 & c_1 \\ & & \\ a_3 & b_3 & c_3 \end{matrix} \quad + \begin{Bmatrix}\text{seven other} \\ \text{determinants}\end{Bmatrix}$$

The eight determinants may be expressed concisely as follows:

$$(a_1b_2c_3)+(a_1b_2\gamma_3)+(a_1\beta_2c_3)+(a_1\beta_2\gamma_3)+(a_1b_2c_3)$$
$$+(a_1b_2\gamma_3)+(a_1\beta_2c_3)+(a_1\beta_2\gamma_3).$$

Again the theorem may be applied any number of times to the same column or row. Thus, *e.g.*

$$\begin{vmatrix} a_1 & b_1'-b_1''+b_1''' & c_1 \\ a_2 & b_2'-b_2''+b_2''' & c_2 \\ a_3 & b_3'-b_3''+b_3''' & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1' & c_1 \\ a_2 & b_2' & c_2 \\ a_3 & b_3' & c_3 \end{vmatrix}$$
$$- \begin{vmatrix} a_1 & b_1'' & c_1 \\ a_2 & b_2'' & c_2 \\ a_3 & b_3'' & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1''' & c_1 \\ a_2 & b_2''' & c_2 \\ a_3 & b_3''' & c_3 \end{vmatrix}$$

Examples.—(1) *Show that*

$$\begin{vmatrix} 1 & \cos\alpha-\sin\alpha & \cos\alpha+\sin\alpha \\ 1 & \cos\beta-\sin\beta & \cos\beta+\sin\beta \\ 1 & \cos\gamma-\sin\gamma & \cos\gamma+\sin\gamma \end{vmatrix} = 2\begin{vmatrix} 1 & \cos\alpha & \sin\alpha \\ 1 & \cos\beta & \sin\beta \\ 1 & \cos\gamma & \sin\gamma \end{vmatrix}$$

[*Lond. B.A.*]

The determinant on the left hand side may be written as the sum of four determinants:

$$\begin{vmatrix} 1 & \cos\alpha & \cos\alpha \\ 1 & \cos\beta & \cos\beta \\ 1 & \cos\gamma & \cos\gamma \end{vmatrix} + \begin{vmatrix} 1 & \cos\alpha & \sin\alpha \\ 1 & \cos\beta & \sin\beta \\ 1 & \cos\gamma & \sin\gamma \end{vmatrix}$$
$$+\begin{vmatrix} 1 & -\sin\alpha & \cos\alpha \\ 1 & -\sin\beta & \cos\beta \\ 1 & -\sin\gamma & \cos\gamma \end{vmatrix} + \begin{vmatrix} 1 & -\sin\alpha & \sin\alpha \\ 1 & -\sin\beta & \sin\beta \\ 1 & -\sin\gamma & \sin\gamma \end{vmatrix}$$

The first determinant is zero because two columns are identical. The fourth determinant is also zero, for when the factor -1 is taken out of the second column, two columns are identical. The third determinant is

$$-\begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \sin\beta & \cos\beta \\ 1 & \sin\gamma & \cos\gamma \end{vmatrix} = \begin{vmatrix} 1 & \cos\alpha & \sin\alpha \\ 1 & \cos\beta & \sin\beta \\ 1 & \cos\gamma & \sin\gamma \end{vmatrix}$$

on interchanging the second and third columns. Hence given determinant is

$$2\begin{vmatrix} 1 & \cos\alpha & \sin\alpha \\ 1 & \cos\beta & \sin\beta \\ 1 & \cos\gamma & \sin\gamma \end{vmatrix}$$

(2) *The roots of* $x^4 = 3px + q$ *are* x_1, x_2, x_3, x_4 *and*

$$A_n = \begin{vmatrix} 1 & x_1^n & x_1^{n+1} & x_1^{n+2} \\ 1 & x_2^n & x_2^{n+1} & x_2^{n+2} \\ 1 & x_3^n & x_3^{n+1} & x_3^{n+2} \\ 1 & x_4^n & x_4^{n+1} & x_4^{n+2} \end{vmatrix}$$

Find the values of A_4/A_1 *and* A_5/A_1 *in terms of* p *and* q. [*M.T.*]

Substituting $x_i^4 = 3px_i + q,\ i = 1, 2, 3, 4,$ it follows that

$$A_4 = \begin{vmatrix} 1 & 3px_1 + q & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 \\ 1 & 3px_2 + q & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 \\ 1 & 3px_3 + q & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 \\ 1 & 3px_4 + q & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 \end{vmatrix}$$

A_4 may be expressed in the sum of the determinants

$$\begin{vmatrix} 1 & 3px_1 & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 \\ 1 & 3px_2 & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 \\ 1 & 3px_3 & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 \\ 1 & 3px_4 & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 \end{vmatrix} + \begin{vmatrix} 1 & q & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 \\ 1 & q & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 \\ 1 & q & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 \\ 1 & q & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 \end{vmatrix}$$

The second determinant is zero because when the common factor q is removed, the first and second columns are identical. Similarly we may express the first determinant as the sum of four determinants, three of which vanish in a similar way. The one which does not vanish is

$$\begin{vmatrix} 1 & 3px_1 & 3px_1^2 & 3px_1^3 \\ 1 & 3px_2 & 3px_2^2 & 3px_2^3 \\ 1 & 3px_3 & 3px_3^2 & 3px_3^3 \\ 1 & 3px_4 & 3px_4^2 & 3px_4^3 \end{vmatrix} = 27p^3 \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix} = 27p^3 A_1$$

Hence $A_4/A_1 = 27p^3$.

Next consider

$$A_5 = \begin{vmatrix} 1 & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 & 3px_1^4 + qx_1^3 \\ 1 & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 & 3px_2^4 + qx_2^3 \\ 1 & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 & 3px_3^4 + qx_3^3 \\ 1 & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 & 3px_4^4 + qx_4^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 & qx_1^3 + 9p^2x_1 + 3pq \\ 1 & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 & qx_2^3 + 9p^2x_2 + 3pq \\ 1 & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 & qx_3^3 + 9p^2x_3 + 3pq \\ 1 & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 & qx_4^3 + 9p^2x_4 + 3pq \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 & qx_1^3 + 9p^2x_1 \\ 1 & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 & qx_2^3 + 9p^2x_2 \\ 1 & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 & qx_3^3 + 9p^2x_3 \\ 1 & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 & qx_4^3 + 9p^2x_4 \end{vmatrix}$$

since the determinant

$$\begin{vmatrix} 1 & 3px_1^2 + qx_1 & 3px_1^3 + qx_1^2 & 3pq \\ 1 & 3px_2^2 + qx_2 & 3px_2^3 + qx_2^2 & 3pq \\ 1 & 3px_3^2 + qx_3 & 3px_3^3 + qx_3^2 & 3pq \\ 1 & 3px_4^2 + qx_4 & 3px_4^3 + qx_4^2 & 3pq \end{vmatrix} = 0.$$

for on removing the common factors $3pq$, two columns become identical. Again:

$$A_5 = \begin{vmatrix} 1 & 3px_1^2 & 3px_1^3 + qx_1^2 & qx_1^3 + 9p^2x_1 \\ 1 & 3px_2^2 & 3px_2^3 + qx_2^2 & qx_2^3 + 9p^2x_2 \\ 1 & 3px_3^2 & 3px_3^3 + qx_3^2 & qx_3^3 + 9p^2x_3 \\ 1 & 3px_4^2 & 3px_4^3 + qx_4^2 & qx_4^3 + 9p^2x_4 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & qx_1 & 3px_1^3 + qx_1^2 & qx_1^3 + 9p^2x_1 \\ 1 & qx_2 & 3px_2^3 + qx_2^2 & qx_2^3 + 9p^2x_2 \\ 1 & qx_3 & 3px_3^3 + qx_3^2 & qx_3^3 + 9p^2x_3 \\ 1 & qx_4 & 3px_4^3 + qx_4^2 & qx_4^3 + 9p^2x_4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3px_1^2 & 3px_1^3 & qx_1^3 + 9p^2x_1 \\ 1 & 3px_2^2 & 3px_2^3 & qx_2^3 + 9p^2x_2 \\ 1 & 3px_3^2 & 3px_3^3 & qx_3^3 + 9p^2x_3 \\ 1 & 3px_4^2 & 3px_4^3 & qx_4^3 + 9p^2x_4 \end{vmatrix} + \begin{vmatrix} 1 & qx_1 & 3px_1^3 + qx_1^2 & qx_1^3 \\ 1 & qx_2 & 3px_2^3 + qx_2^2 & qx_2^3 \\ 1 & qx_3 & 3px_3^3 + qx_3^2 & qx_3^3 \\ 1 & qx_4 & 3px_4^3 + qx_4^2 & qx_4^3 \end{vmatrix}$$

since the two determinants which have been split off are zero as before,

$$= \begin{vmatrix} 1 & 3px_1^2 & 3px_1^3 & 9p^2x_1 \\ 1 & 3px_2^2 & 3px_2^3 & 9p^2x_2 \\ 1 & 3px_3^2 & 3px_3^3 & 9p^2x_3 \\ 1 & 3px_4^2 & 3px_4^3 & 9p^2x_4 \end{vmatrix} + \begin{vmatrix} 1 & qx_1 & qx_1^2 & qx_1^3 \\ 1 & qx_2 & qx_2^2 & qx_2^3 \\ 1 & qx_3 & qx_3^2 & qx_3^3 \\ 1 & qx_4 & qx_4^2 & qx_4^3 \end{vmatrix}$$

In this last step two more zero determinants have been removed. Hence

$$A_5 = 81p^4 \begin{vmatrix} 1 & x_1^2 & x_1^3 & x_1 \\ 1 & x_2^2 & x_2^3 & x_2 \\ 1 & x_3^2 & x_3^3 & x_3 \\ 1 & x_4^2 & x_4^3 & x_4 \end{vmatrix} + q^3 \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix}$$

Now $$\begin{vmatrix} 1 & x_1^2 & x_1^3 & x_1 \\ 1 & x_2^2 & x_2^3 & x_2 \\ 1 & x_3^2 & x_3^3 & x_3 \\ 1 & x_4^2 & x_4^3 & x_4 \end{vmatrix} = - \begin{vmatrix} 1 & r_1^2 & x_1 & x_1^3 \\ 1 & x_2^2 & x_2 & x_2^3 \\ 1 & x_3^2 & x_3 & x_3^3 \\ 1 & x_4^2 & x_4 & x_4^3 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix}$$

It follows that $A_5 = (81p^4 + q^3)\, A_1$.

We now prove some properties which are of frequent application in the evaluation of determinants.

THEOREM VII.—A determinant is unaltered if to each element of any row (or column) is added the corresponding element of any other row (or column) multiplied by a given factor.

Thus, *e.g.* $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_3 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 + mb_1 & b_1 & c_1 \\ a_2 + mb_2 & b_2 & c_2 \\ a_3 + mb_3 & b_3 & c_3 \end{vmatrix}$$

where m is a given number.

The result becomes obvious when we write the second determinant as the sum of two others, one being the original determinant and the other one, which is zero, for it will have two rows (or columns) identical after the common factor is removed. By repeated application of the theorem it follows that *a determinant is unaltered when to each element of any row (or column) are added those of any number of other rows (or columns) multiplied respectively by constant factors.*

Thus, *e.g.* $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 + pc_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 + pc_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 + pc_3 & c_3 \end{vmatrix}$$

It is important to observe that when more than one row (or column) is altered by a series of changes of this kind, *one row (or*

column), at least, remains unaltered. Thus in the example just given the third column is unchanged.

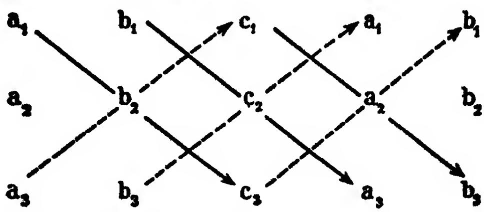
Important particular cases occur when the factors involved are ± 1. The theorem then asserts that we may *add or subtract the elements of any row from the corresponding elements of any other row without altering the value of the determinant.* In the process elements of the row which is added or subtracted *remain unchanged.* Similarly for columns.

THEOREM VIII.—If the elements of a determinant Δ are polynomials in a variable x and the determinant vanishes when $x = \lambda$ then $x - \lambda$ is a factor of Δ. *In particular the result will be true if the substitution $x = \lambda$ makes two columns or two rows identical.*

The theorem follows immediately from the Remainder theorem, for Δ when expanded will be a polynomial in x.

12·42. Rule of Sarrus

This is a rule for the expansion of a determinant of the third order. Consider $\Delta = (a_1 b_2 c_3)$.



Imagine the first two columns to be repeated after the third and draw a system of diagonals as indicated, the direction of the diagonals being indicated by the arrows. Count the three diagonals sloping down as positive, and the three sloping up (shown in dotted lines) as negative. Then form the corresponding products, attaching the sign of the corresponding diagonal. Thus

$$a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1.$$

Comparison with the expansion in § 12·11 shows that the expression is equal to Δ.

Examples.—(1) *Prove that*

$$\begin{vmatrix} Bb + Cc & Bc & Cb \\ Ac & Cc + Aa & Ca \\ Ab & Ba & Aa + Bb \end{vmatrix}$$

$$4ABC\,abc + BC\,(bc - a^2)\,(Bb + Cc)$$
$$+ CA\,(ca - b^2)\,(Cc + Aa) + AB\,(ab - c^2)\,(Aa + Bb).$$

[*Lond. B.Sc.*]

The elements of the determinant are set out as above.

$$\begin{array}{ccccc} Bb+Cc & Bc & Cb & Bb+Cc & Bc \\ Ac & Cc+Aa & Ca & Ac & Cc+Aa \\ Ab & Ba & Aa+Bb & Ab & Ba\ . \end{array}$$

If Δ denote the determinant, then

$$\Delta = (Bb+Cc)(Cc+Aa)(Aa+Bb) + ABC\,abc + ABC\,abc - ACb^2(Cc+Aa) - BCa^2(Bb+Cc) - ABc^2(Aa+Bb).$$

Now $(Bb+Cc)(Cc+Aa)(Aa+Bb)$

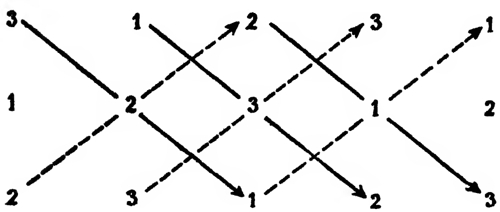
$$= (Abab + BCbc + ACac + B^2b^2)(Cc+Aa)$$

$$= ACac(Cc+Aa) + ABCabc + B^2Cb^2c + A^2Ba^2b + BC^2bc^2 + ABCabc + AB^2ab^2$$

$$= ACac(Cc+Aa) + ABab(Aa+Bb) + BCbc(Bb+Cc) + 2ABCabc.$$

Substitution of this result gives the required form for Δ.

(2) *Evaluate the determinant* $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$.



$$= 6+6+6-8-27-1 = -18.$$

12·43. Illustrative Examples

In each question the given determinant is denoted by Δ.

(1) *Prove that* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

[*Madras, B.A.*]

Subtracting the second column from the third and the first from the second

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & 0 \\ b^2 & (c+a)^2-b^2 & b^2-(c+a)^2 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a-b-c)(a+b+c) & 0 \\ b^2 & (c+a-b)(a+b+c) & (b-c-a)(a+b+c) \\ c^2 & 0 & (a+b-c)(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & a-b+c & -a+b-c \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Subtract from the first row the sum of the second and third rows. Then

$$\Delta = (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b+2c \\ b^2 & a-b+c & -a+b-c \\ c^2 & 0 & a+b-c \end{vmatrix}$$

$$2\,(a+b+c)^2 \begin{vmatrix} bc & -c & -b+c \\ b^2 & a-b+c & -a+b-c \\ & 0 & a+b-c \end{vmatrix}$$

Now add the second column on to the third and we have

$$2\,(a+b+c)^2 \begin{vmatrix} bc & -c & -b \\ b^2 & a-b+c & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Multiply the third column by c and add to the first.

$$\Delta = 2\,(a+b+c)^2 \begin{vmatrix} 0 & -c & -b \\ b^2 & a-b+c & 0 \\ c\,(a+b) & 0 & a+b-c \end{vmatrix}$$

Expanding along the first column,

$$\begin{aligned} \Delta &= 2\,(a+b+c)^2\,\{+\,b^2c\,(a+b-c) + c\,(a+b)\,b\,(a-b+c)\} \\ &= 2bc\,(a+b+c)^2\,\{b\,(a+b-c) + (a+b)\,(a-b+c)\} \\ &= 2abc\,(a+b+c)^3. \end{aligned}$$

(2) Show that $\begin{vmatrix} \cos(x+y) & \sin(x+y) & -\cos(x+y) \\ \sin(x-y) & \cos(x-y) & \sin(x-y) \\ & & \sin 2y \end{vmatrix} = \sin 2(x+y)$

[*Lond. B.Sc.*]

Subtracting the third column from the first,

$$\Delta = \begin{vmatrix} 2\cos(x+y) & \sin(x+y) & -\cos(x+y) \\ 0 & \cos(x-y) & \sin(x-y) \\ \sin 2x - \sin 2y & 0 & \sin 2y \end{vmatrix}$$

$$\begin{vmatrix} 2\cos(x+y) & \sin(x+y) & -\cos(x+y) \\ 0 & \cos(x-y & \sin(x-y) \\ 2\cos(x+y)\sin(x-y) & 0 & \sin 2y \end{vmatrix}$$

$$2\cos(x+y) \begin{vmatrix} 1 & \sin(x+y) & -\cos(x+y) \\ 0 & \cos(x-y) & \sin(x-y) \\ \sin(x-y) & 0 & \sin 2y \end{vmatrix}$$

Multiply the first row by $\sin(x-y)$ and subtract from the third, then

$$\Delta = 2\cos(x+y)$$

$$\begin{vmatrix} 1 & \sin(x+y) & -\cos(x+y) \\ 0 & \cos(x-y) & \sin(x-y) \\ 0 & -\sin(x+y)\sin(x-y) & \sin 2y + \sin(x-y)\cos(x+y) \end{vmatrix}$$

$$2\cos(x+y)$$

$$\times \begin{vmatrix} 1 & \sin(x+y) & -\cos(x+y) \\ 0 & \cos(x-y) & \sin(x-y) \\ 0 & -\sin(x+y)\sin(x-y) & \sin(x+y)\cos(x-y) \end{vmatrix}$$

$$= 2\cos(x+y)\sin(x+y)\begin{vmatrix} \cos(x-y) & \sin(x-y) \\ -\sin(x-y) & \cos(x-y) \end{vmatrix}$$

$$= \sin 2(x+y)\{\cos^2(x-y) + \sin^2(x-y)\} = \sin 2(x+y).$$

(3) *Evaluate the determinant* $\begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$ [*N.Sc. Prelim.*]

We give two methods of evaluating this determinant.

(*a*) Subtract the first row from the second, third and fourth rows and then expand along the fourth column. Thus

$$\Delta = \begin{matrix} a^3 & a^2 & a \\ b^3 - a^3 & b^2 - a^2 & b - a \\ c^3 - a^3 & c^2 - a^2 & c - a \\ d^3 - a^3 & d^2 - a^2 & d - a \end{matrix} \qquad \begin{matrix} b^3 - a^3 & b^2 - a^2 & b - a \\ c^3 - a^3 & c^2 - a^2 & c - a \\ d^3 - a^3 & d^2 - a^2 & d - a \end{matrix}$$

$$-(b-a)(c-a)(d-a)\begin{matrix} a^2 + ab + b^2 & a + b & 1 \\ a^2 + ac + c^2 & a + c & 1 \\ a^2 + ad + d^2 & a + d & 1 \end{matrix}$$

Now subtract the first row from the second and third and expand along the third column.

$$\Delta = -(b-a)(c-a)(d-a)\begin{vmatrix} a^2 + ab + a^2 & a + b & 1 \\ c^2 - b^2 + ac - ab & c - b & 0 \\ d^2 - b^2 + ad - ab & d - b & 0 \end{vmatrix}$$

$$= -(b-a)(c-a)(d-a)\begin{vmatrix} (c-b)(a+b+c) & c - b \\ (d-b)(a+b+d) & d - b \end{vmatrix}$$

$$= -(b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} a+b+c & 1 \\ a+b+d & 1 \end{vmatrix}$$

$$= -(b-a)(c-a)(d-a)(c-b)(d-b)(c-d)$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

(*b*) *Alternative Method.*—If in Δ we write $a = b$ the first two rows become identical. Hence $(a-b)$ is a factor. Similarly $(a-c)$, $(a-d)$, $(b-c)$, $(b-d)$, $(c-d)$ are factors. Further it is clear that each of the terms in the expanded determinant are of the sixth degree in a, b, c, d. Also if we interchange any two of the letters, the determinant changes sign, for the process is equivalent to the interchange of two rows. Hence

$$\Delta = \lambda(a-b)(a-c)(a-d)(b-c)(b-d)(c-d),$$

where λ is a numerical factor.

The leading term in the determinant is a^3b^2c. The corresponding term in the product form is λa^3b^2c, Hence $\lambda = 1$ and

$$\Delta = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

(4) *If* $c_{ij} = a_ib_j - a_jb_i$, $i, j = 1, 2, \ldots 5$, *show by consideration of the determinant*

$$\begin{vmatrix} a_1 & a_2 & & a_4 \\ b_1 & b_2 & & b_4 \\ 0 & a_2 & & a_4 \\ 0 & b_2 & & b_4 \end{vmatrix}$$

that $c_{12}c_{34} + c_{13}c_{42} + c_{14}c_{23} = 0$.

Show further that if $c_{34} \neq 0$, *and*

$$\left.\begin{aligned} c_{14}c_{15} + c_{24}c_{25} + c_{34}c_{35} &= 0 \\ c_{13}c_{15} + c_{23}c_{25} + c_{43}c_{45} &= 0 \end{aligned}\right\}$$

then $c_{12}c_{15} + c_{32}c_{35} + c_{42}c_{45} = 0$. [*M.T.*]

Subtract third row from first, fourth row from second. Then

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & a_4 \\ 0 & b_2 & b_3 & b_4 \end{vmatrix} = a_1 \quad b_2 \quad b_3 \quad b_4$$

Again expanding Δ along the first column,

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ a_2 & & \\ b_2 & b_3 & b_4 \end{vmatrix} - b_1 \begin{vmatrix} & a_3 & \\ & a_3 & \\ b_2 & b_3 & b_4 \end{vmatrix}$$

$$= a_1\,[b_2c_{34} - b_3c_{24} + b_4c_{23}] - b_1\,[a_2c_{34} - a_3c_{24} + a_4c_{23}]$$

$$c_{34}c_{12} - c_{24}c_{13} + c_{23}c_{14}.$$

Since $c_{34} = -c_{43}$ it follows that

$$c_{12}c_{34} + c_{13}c_{42} + c_{14}c_{23} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

From $c_{14}c_{15} + c_{24}c_{25} + c_{34}c_{35} = 0$, $c_{13}c_{15} + c_{23}c_{25} + c_{43}c_{45} = 0$ and $c_{34} = -c_{43}$,

$$\frac{}{-c_{24}c_{45} - c_{23}c_{35}} = \frac{c_{25}}{c_{35}c_{13} + c_{14}c_{45}} = \frac{c_{34}}{c_{14}c_{23} - c_{13}c_{24}} = \frac{c_{34}}{-c_{12}c_{34}}, \text{ from (1).}$$

$\therefore\ c_{15}c_{12} - c_{24}c_{45} - c_{23}c_{35} = 0$, i.e. $c_{12}c_{15} + c_{32}c_{35} + c_{42}c_{45} = 0$.

(5) *Show that* $\begin{vmatrix} t_1+x & a+x & a+x & a+x & a+x \\ b+x & t_2+x & a+x & a+x & a+x \\ b+x & b+x & t_3+x & a+x & a+x \\ b+x & b+x & b+x & t_4+x & a+x \\ b+x & b+x & b+x & b+x & t_5+x \end{vmatrix}$

is of the form $A + Bx$, *where* A *and* B *are independent of* x. *Hence, by giving particular values to* x, *prove that*

$$A = \{af(b) - bf(a)\}/(a-b), \quad B = \{f(b) - f(a)\}/(a-b),$$

where $f(t) \equiv (t_1 - t)(t_2 - t)(t_3 - t)(t_4 - t)(t_5 - t)$. *Also find the values of* A *and* B *when* $a = b$. [*Lond. B.Sc.*]

Let $\Delta(x)$ denote the determinant. Subtracting the first column from the second, third, fourth, and fifth columns,

$$\Delta(x) \quad \begin{vmatrix} t_1+x & a-t_1 & a-t_1 & a-t_1 & a-t_1 \\ b+x & t_2-b & a-b & a-b & a-b \\ b+x & 0 & t_3-b & a-b & a-b \\ b+x & 0 & 0 & t_4-b & a-b \\ b+x & 0 & 0 & 0 & t_5-b \end{vmatrix} \qquad \text{(i)}$$

Expanding along the first column it is clear that $\triangle$ is of the form $A + Bx$. If A and B have the given values it follows that $f(a)$ and $f(b)$ must satisfy the equations

$$A - aB = f(a), \quad A - bB = f(b), \quad a \neq b.$$

Thus it is sufficient to prove that when $x = -a$, $\triangle(x) = f(a)$, and when $x = -b$, $\triangle(x) = f(b)$. Substituting $x = -a$,

$$\triangle(-a) = \begin{vmatrix} t_1 - a & 0 & 0 & 0 & 0 \\ b - a & t_2 - a & 0 & 0 & 0 \\ b - a & b - a & t_3 - a & 0 & 0 \\ b - a & b - a & b - a & t_4 - a & 0 \\ b - a & b - a & b - a & b - a & t_5 - a \end{vmatrix}$$

Expanding along the first row we obtain:

$$\triangle(-a) = (t_1 - a)\begin{vmatrix} t_2 - a & 0 & 0 & 0 \\ b - a & t_3 - a & 0 & 0 \\ b - a & b - a & t_4 - a & 0 \\ b - a & b - a & b - a & t_5 - a \end{vmatrix}$$

Continuing this method of expansion it follows that

$$\triangle(-a) = (t_1 - a)(t_2 - a)(t_3 - a)(t_4 - a)(t_5 - a) = f(a).$$

Similarly by substituting $x = -b$ it is easily seen that $\triangle(-b) = f(b.)$

If $a = b$ the determinant becomes

$$F(x) = \begin{vmatrix} t_1 + x & a + x & a + x & a + x & a + x \\ a + x & t_2 + x & a + x & a + x & a + x \\ a + x & a + x & t_3 + x & a + x & a + x \\ a + x & a + x & a + x & t_4 + x & a + x \\ a + x & a + x & a + x & a + x & t_5 + x \end{vmatrix}$$

$$\therefore F(-a) = \begin{vmatrix} t_1 - a & 0 & 0 & 0 & 0 \\ 0 & t_2 - a & 0 & 0 & 0 \\ 0 & 0 & t_3 - a & 0 & 0 \\ 0 & 0 & 0 & t_4 - a & 0 \\ 0 & 0 & 0 & 0 & t_5 - a \end{vmatrix}$$

$$= (t_1 - a)(t_2 - a)(t_3 - a)(t_4 - a)(t_5 - a) = f(a).$$

Thus $A - aB = f(a)$. Again from (i), if $a = b$,

$$F(x) = x\begin{vmatrix} \frac{t_1}{x} + 1 & a - t_1 & a - t_1 & a - t_1 & a - t_1 \\ \frac{a}{x} + 1 & t_2 - a & 0 & 0 & 0 \\ \frac{a}{x} + 1 & 0 & t_3 - a & 0 & 0 \\ \frac{a}{x} + 1 & 0 & 0 & t_4 - a & 0 \\ \frac{a}{x} + 1 & 0 & 0 & 0 & t_5 - a \end{vmatrix}$$

$$\therefore \lim_{x \to \infty} \frac{F(x)}{x} = \begin{vmatrix} 1 & a - t_1 & a - t_1 & a - t_1 & a - t_1 \\ 1 & t_2 - a & 0 & 0 & 0 \\ 1 & 0 & t_3 - a & 0 & 0 \\ 1 & 0 & 0 & t_4 - a & 0 \\ 1 & 0 & 0 & 0 & t_5 - a \end{vmatrix}$$

Since $F(x) = A + Bx$ it follows that this determinant gives the value of B. Expanding along the first column,

$$B = \begin{vmatrix} t_2 - a & 0 & 0 & 0 \\ 0 & t_3 - a & 0 & 0 \\ 0 & 0 & t_4 - a & 0 \\ 0 & 0 & 0 & t_5 - a \end{vmatrix} - (a - t_1) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & t_3 - a & 0 & 0 \\ 0 & 0 & t_4 - a & 0 \\ 0 & 0 & 0 & t_5 - a \end{vmatrix}$$

$$+ (a - t_1) \begin{vmatrix} 1 & 1 & 1 & 1 \\ t_2 - a & 0 & 0 & 0 \\ 0 & 0 & t_4 - a & 0 \\ 0 & 0 & 0 & t_5 - a \end{vmatrix} - (a - t_1) \begin{vmatrix} 1 & 1 & 1 & 1 \\ t_2 - a & 0 & 0 & 0 \\ 0 & t_3 - a & 0 & 0 \\ 0 & 0 & 0 & t_5 - a \end{vmatrix}$$

$$+ (a - t_1) \begin{vmatrix} 1 & 1 & 1 & 1 \\ t_2 - a & 0 & 0 & 0 \\ 0 & t_3 - a & 0 & 0 \\ 0 & 0 & t_4 - a & 0 \end{vmatrix}$$

$$= (t_2 - a)\ (t_3 - a)\ (t_4 - a)\ (t_5 - a) + (t_1 - a)\ (t_3 - a)\ (t_4 - a)\ (t_5 - a)$$
$$+ (t_1 - a)\ (t_2 - a)\ (t_4 - a)\ (t_5 - a) + (t_1 - a)\ (t_2 - a)(t_3 - a)\ (t_5 - a)$$
$$+ (t_1 - a)\ t_2 - a)(t_3 - a)\ (t_4 - a).$$

Then A is given by $f(a) + aB$.

(6) *Evaluate the determinant*

$$\begin{vmatrix} 1 + a_1 & a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ a_1 & 1 + a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ a_1 & a_2 & 1 + a_3 & a_4 & \cdot\cdot & a_n \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_1 & a_2 & a_3 & a_4 & \cdot\cdot & 1 + a_n \end{vmatrix}$$

[*N.Sc. Prelim.*]

Adding all the columns to the first,

$$\Delta = \begin{vmatrix} 1 + \Sigma a_r & a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ 1 + \Sigma a_r & 1 + a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ 1 + \Sigma a_r & a_2 & 1 + a_3 & a_4 & \cdot\cdot & a_n \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 + \Sigma a_r & a_2 & a_3 & a_4 & \cdot\cdot & 1 + a_n \end{vmatrix}$$

$$= (1 + \Sigma a_r) \begin{vmatrix} 1 & a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ 1 & 1 + a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ 1 & a_2 & 1 + a_3 & a_4 & \cdot\cdot & a_n \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 & a_2 & a_3 & a_4 & \cdot\cdot & 1 + a_n \end{vmatrix}$$

Now subtract the first row from each of the other rows. Then

$$\Delta = (1 + \Sigma a_r) \begin{vmatrix} 1 & a_2 & a_3 & a_4 & \cdot\cdot & a_n \\ 0 & 1 & 0 & 0 & \cdot\cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & 0 & \cdot\cdot & 1 \end{vmatrix}$$

$$= (1 + \Sigma a_r) \begin{vmatrix} 1 & 0 & 0 & \cdot\cdot & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 \\ 0 & 0 & 1 & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & \cdot\cdot & 1 \end{vmatrix}$$

on expanding along the first column. Since the new determinant has each of the elements of the principal diagonal equal to unity, and all the other elements zero, it follows that this determinant is unity. Hence

$$\Delta = 1 + \Sigma a_r = 1 + a_1 + a_2 + a_3 + \ldots + a_n.$$

12·44. Equations involving Determinants

We now consider some examples of equations expressed in determinant form. As in § 12·43 the given determinants are denoted by Δ, so that each equation takes the form $\Delta = 0$.

(1) *Solve*
$$\begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix} = 0$$

for x, where a, b, c are unequal. [*Camb. Sch.*]

Adding the second row to the third,

$$\Delta = \begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 2x(x^2+3a^2) & 2x(x^2+3b^2) & 2x(x^2+3c^2) \end{vmatrix}$$

$$= 2x \begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ x^2+3a^2 & x^2+3b^2 & x^2+3c^2 \end{vmatrix}$$

Adding the first row to the third,

$$\Delta = 4x \begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ x^2+a^2 & x^2+b^2 & x^2+c^2 \end{vmatrix}$$

Subtracting the first column from the second and third columns,

$$\Delta = 4x \begin{vmatrix} x^2 - a^2 & a^2 - b^2 & a^2 - c^2 \\ (x-a)^3 & (a-b)(3x^2 - 3ax - 3bx + b^2 + ab + a^2) & (a-c)(3x^2 - 3ax - 3cx + c^2 + ac + a^2) \\ x^2 + a^2 & b^2 - a^2 & c^2 - a^2 \end{vmatrix}$$

$$= 4x(a-b)(a-c) \begin{vmatrix} x^2 - a^2 & a+b & a+c \\ (x-a)^3 & 3x^2 - 3ax - 3bx + b^2 + ab + a^2 & 3x^2 - 3ax - 3cx + c^2 + ac + a^2 \\ x^2 + a^2 & -(a+b) & -(c+a) \end{vmatrix}$$

Adding the first row to the third,

$$\Delta = 4x(a-b)(a-c) \begin{vmatrix} x^2 - a^2 & a+b & a+c \\ (x-a)^3 & 3x^2 - 3ax - 3bx + b^2 + ab + a^2 & 3x^2 - 3ax - 3cx + c^2 + ac + a^2 \\ 2x^2 & 0 & 0 \end{vmatrix}$$

$$= 8x^3(a-b)(a-c) \begin{vmatrix} a+b & a+c \\ 3x^2 - 3ax - 3bx + b^2 + ab + a^2 & 3x^2 - 3ax - 3cx + c^2 + ac + a^2 \end{vmatrix}$$

$$= 8x^2 (a-b)(a-c) \begin{vmatrix} a+b & 3x^2 - 3ax - 3bx + b^2 + ab + a^2 \\ c-b & 3bx - 3cx + c^2 - b^2 + ac - ab \end{vmatrix}$$

$$= 8x^2 (a-b)(a-c)(c-b) \begin{vmatrix} a+b & 3x^2 - 3ax - 3bx + b^2 + ab + a^2 \\ 1 & -3x + a + b + c \end{vmatrix}$$

$$= 8x^3 (a-b)(a-c)(c-b)(-3x^2 + ab + bc + ca).$$

Hence the required solutions are $0, 0, 0, \pm \{\frac{1}{3}(ab + bc + ca)\}^{\frac{1}{2}}$.

(2) *Show that* $-(a+b+c)$ *is one root of the equation*

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$$

and solve the equation completely. [*Lond. B.Sc.*]

Adding the second and third rows to the first,

$$\Delta = \begin{vmatrix} x+a+b+c & x+a+b+c & x+a+b+c \\ b & x+c & a \\ c & a & x+b \end{vmatrix}$$

$$= (x+a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & x+c & a \\ c & a & x+b \end{vmatrix}$$

Hence $x = -(a+b+c)$ is one solution of the equation. Now subtract the first column from the second and from the third. We have

$$\Delta = (x+a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & x+c-b & a-b \\ c & a-c & x+b-c \end{vmatrix}$$

$$= (x+a+b+c) \begin{vmatrix} x+c-b & a-b \\ a-c & x+b-c \end{vmatrix}$$

$$= (x+a+b+c)(x^2 - a^2 - b^2 - c^2 + ab + bc + ca).$$

Thus the roots of the equation are

$$x = -(a+b+c), \pm \sqrt{\{a^2 + b^2 + c^2 - ab - bc - ca\}}.$$

(3) *Solve the equation* $\begin{vmatrix} x^3 - a^3 & x^2 & x \\ b^3 - a^3 & b^2 & b \\ c^3 - a^3 & c^2 & c \end{vmatrix} = 0$, *where* b, c *are unequal.*

[*M.T.*]

Subtract the first row from the second and third. Then

$$\Delta = \begin{vmatrix} x^3 - a^3 & x^2 & x \\ b^3 - x^3 & b^2 - x^2 & b - x \\ c^3 - x^3 & c^2 - x^2 & c - x \end{vmatrix}$$

$$= (b-x)(c-x) \begin{vmatrix} x^3 - a^3 & x^2 & x \\ b^2 + bx + x^2 & b + x & 1 \\ c^2 + cx + x^2 & c + x & 1 \end{vmatrix}$$

Subtract the second row from the third,

$$\Delta = (b - x)(c - x)\begin{vmatrix} x^3 - a^3 & x^2 & x \\ b^2 + bx + x^2 & b + x & 1 \\ c^2 - b^2 + cx - bx & c - b & 0 \end{vmatrix}$$

$$= (b - x)(c - x)(c - b)\begin{vmatrix} x^3 - a^3 & x^2 & x \\ b^2 + bx + x^2 & b + x & 1 \\ b + c + x & 1 & 0 \end{vmatrix}$$

Expanding along the third row,

$$\Delta = (b - x)(c - x)(c - b)\{(b + c + x)(-bx) - (x^3 - a^3 - b^2x - bx^2 - x^3)\}$$

$$= (b - x)(c - x)(c - b)(a^3 - bcx).$$

Hence the solutions of $\Delta = 0$ are $x = b, c$ or a^3/bc.

(4) *Solve the equation* $\begin{vmatrix} 1 + x & 2 & 3 & 4 \\ 1 & 2 + x & 3 & 4 \\ 1 & 2 & 3 + x & 4 \\ 1 & 2 & 3 & 4 + x \end{vmatrix} = 0.$

[*Madras, B.Sc.*]

Subtract the second row from the first and from the third and fourth. Then

$$\Delta = \begin{vmatrix} x & -x & 0 & 0 \\ 1 & 2 + x & 3 & 4 \\ 0 & -x & x & 0 \\ 0 & -x & 0 & x \end{vmatrix} = x^3\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 + x & 3 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

Now subtract the first row from the second. Then

$$\Delta = x^3\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 + x & 3 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix} = x^3\begin{vmatrix} 3 + x & 3 & 4 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Add the first column to the third and expand along the third row. Thus

$$\Delta = x^3\begin{vmatrix} 3 + x & 3 & 7 + x \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{vmatrix} = -x^3(-3 - 7 - x) = x^3(x + 10).$$

Hence $x = 0, 0, 0$ or -10.

EXERCISES XII

1. Show that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (a^3 - 1)^2.$ [*Lond. B.A.*]

2. Evaluate the determinants

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$, (ii) $\begin{vmatrix} 1 & 1 & 1 \\ b + c & c + a & a + b \\ b^2 + c^2 & c^2 + a^2 & a^2 + b^2 \end{vmatrix}$

[*Lond. B.Sc.*]

3. Express $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ as a product of linear factors in a, b, c. Hence

prove that, if $\alpha + \beta + \gamma = \pi$, then

$$\begin{vmatrix} 1 & \sin\alpha & \sin 3\alpha \\ 1 & \sin\beta & \sin 3\beta \\ 1 & \sin\gamma & \sin 3\gamma \end{vmatrix} = -16 \sin\alpha \sin\beta \sin\gamma \sin\tfrac{1}{2}(\beta-\gamma) \sin\tfrac{1}{2}(\gamma-\alpha) \sin\tfrac{1}{2}(\alpha-\beta).$$

4. Evaluate $\begin{vmatrix} 10 & 17 & 4 \\ 5 & 11 & 7 \\ 3 & 19 & 6 \end{vmatrix}$ and prove that

$$\begin{vmatrix} a & b & c \\ b & c & d \\ 1 & -x & x^2 \end{vmatrix} = \begin{vmatrix} ax+b & bx+c \\ bx+c & cx+d \end{vmatrix}$$

[*Lond. B.Sc.*]

5. If x, y and z are unequal and

$$\begin{vmatrix} x^3 & (x+a)^3 & (x-a)^3 \\ y^3 & (y+a)^3 & (y-a)^3 \\ z^3 & (z+a)^3 & (z-a)^3 \end{vmatrix} = 0$$

prove that $a^2(x+y+z) - 3xyz = 0$. [*N.Sc. Prelim.*]

6. Express $\begin{vmatrix} x+2a & a+1 & a+1 \\ a+1 & x+2a & a^2+1 \\ a+1 & a^2+1 & x+2a \end{vmatrix}$ as the product of the three factors each linear in x. [*Lond. B.Sc.*]

7. Show that $x+1$ is a factor of the determinant

$$\begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+1 & 3 \\ 3 & -6 & x+1 \end{vmatrix}$$

and factorise it completely. [*Lond. B.Sc.*]

8. Prove that $\begin{vmatrix} x & a_1 & a_2 \\ a_1 & x & a_2 \\ a_1 & a_2 & x \end{vmatrix} = (x-a_1)(x-a_2)(x+a_1+a_2).$

Hence find *all* the values of θ satisfying the equation

$$\begin{vmatrix} \sin\theta & \sin 2\theta & \sin 3\theta \\ \sin 2\theta & \sin\theta & \sin 3\theta \\ \sin 2\theta & \sin 3\theta & \sin\theta \end{vmatrix} = 0.$$

[*Lond. B.Sc.*]

9. Prove that, if to the terms of any column of a determinant linear multiples of the corresponding terms of other columns of the determinant are added, the value of the determinate is unaltered. Show that

$$\begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix} = \pm \frac{a^2 \Pi(a_i - a_j)}{\Pi a_i \; \Pi(a-a_i)^2}.$$

Write out the terms of the product in the numerator, and give the resulting expression its correct sign. [*M.T.*]

10. Expand and evaluate the determinant

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

[*Lond. B.Sc*]

11. Show that $\begin{vmatrix} a_1 + A & a_2 \\ b_1 + B & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} A & a_2 \\ B & b_2 \end{vmatrix}$

Find the coefficient of the first power of X in the expansion of the determinant

$$\begin{vmatrix} 1 + X & m & sk \\ m & m^2 + \frac{1}{2}(1 - s) + X & \frac{1}{2}(1 + s)\, mk \\ sk & \frac{1}{2}(1 + s)\, mk & k^2 + \frac{1}{2}(1 - s)\, m^2 \end{vmatrix}$$

[*Lond. B.Sc., Eng.*]

12. Prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3 (s-a)(s-b)(s-c)$$

where $2s = a + b + c$. [*Camb. Sch.*]

13. Prove that $\begin{vmatrix} (x+1)(x+2) & x+2 & 1 \\ (x+2)(x+3) & x+3 & 1 \\ (x+3)(x+4) & x+4 & 1 \end{vmatrix} = -2.$ [*Lond. B.Sc.*]

14. Prove that $(\lambda + \mu)(bc + ca + ab) + \lambda\mu(a + b + c) + 3abc$ is a factor of the determinant

$$\begin{vmatrix} a^3 & b^3 & c^3 \\ (a+\lambda)^3 & (b+\lambda)^3 & (c+\lambda)^3 \\ (a+\mu)^3 & (b+\mu)^3 & (c+\mu)^3 \end{vmatrix}$$

and find the other factors. [*Camb. Sch.*]

15. Prove that $\begin{vmatrix} x+y & y+z & z+x \\ y+z & z+x & x+y \\ z+x & x+y & y+z \end{vmatrix} = 2 \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$

[*Lond. B.A.*]

16. Calculate the determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$ [*Madras, B.A.*]

17. In n denote a positive integer, show that $(b - c)(c - a)(a - b)$ is a factor of

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^{n+2} & b^{n+2} & c^{n+2} \end{vmatrix}$$

and that the remaining factor is the sum of the homogeneous products, of n dimensions, of a, b, c. Deduce an analogous theorem for

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^{n+3} & b^{n+3} & c^{n+3} & d^{n+3} \end{vmatrix}$$

and proceed to a generalisation. [*Camb. Sch.*]

18. Evaluate $\begin{vmatrix} 1 + x_1 & x_2 & x_3 & x_4 \\ x_1 & 1 + x_2 & x_3 & x_4 \\ x_1 & x_2 & 1 + x_3 & x_4 \\ x_1 & x_2 & x_3 & 1 + x_4 \end{vmatrix}$

19. (i) Evaluate the determinant $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$

(ii) Prove that $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = -(x+y+z)(y+z-x)(z+x-y)(x+y-z).$

[*Madras, B.A.*]

20. Prove that $\begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ 3\alpha^2 & 2\alpha & 1 & 0 \\ \beta^3 & \beta^2 & \beta & 1 \\ 3\beta^2 & 2\beta & 1 & 0 \end{vmatrix} = (\alpha - \beta)^4.$

[*Camb. Sch.*]

21. Show that the determinant

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = (a+b+c+d)(a-b+c-d)\{(a-c)^2+(b-d)^2\}.$$

[*Lond, B.Sc.*]

22. Show that $x = 0$ satisfies the equation

$$\begin{vmatrix} 1 & 3 & 5+x \\ 7 & 8+x & 9 \\ 5+x & 2+x & -1+x \end{vmatrix} = 0,$$

and solve it. [*Lond. B.Sc.*]

23. Prove that $x = 1$ is a root of the equation

$$\begin{vmatrix} x+2 & 3 & 3 \\ 3 & x+4 & 5 \\ 3 & 5 & x+4 \end{vmatrix} = 0$$

and find the other two roots. [*Lond. B.A.*]

24. (i) Solve the equation $\begin{vmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0.$

(ii) Show that the determinant

$$\begin{vmatrix} -x & b & c \\ b & c & -x \\ c & -x & b \end{vmatrix} = (x-b-c)(x-\omega b-\omega^2 c)(x-\omega^2 b-\omega c),$$

where $1, \omega, \omega^2$ are the cube roots of unity. [*Lond. B.A.*]

25. Solve the equation $\begin{vmatrix} 3-2x & 2 & 6 \\ 4-x & 4 & 12 \\ 1-x & 1 & 4 \end{vmatrix} = 0.$ [*N.Sc. Prelim.*]

26. Prove that one root of the equation

$$\begin{vmatrix} 11-x & -6 & 2 \\ -6 & 10-x & -4 \\ 2 & -4 & 6-x \end{vmatrix} = 0$$

is 6, and find the other roots. [*Lond. B.Sc.*]

27. Solve the equation $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$ [*Lond. B.Sc.*]

28. Expand the determinant $\begin{vmatrix} x & c+x & b+x \\ c+x & x & a+x \\ b+x & a+x & x \end{vmatrix}$
and find for what values or value of x it vanishes. [*Lond. B.A.*]

29. If a, b, c, have all different values and

$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0,$$

prove that $abc = 1$. [*Lond. B.Sc., Eng.*]

30. Find the value of $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$

If a, b, c are all different, and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0,$
prove that $abc + 1 = 0$. [*Lond. B.A.*]

31. Solve the equation $\begin{vmatrix} 1 & 2x+1 & 3x+3 & 3 \\ 2 & 4x+5 & 6x+5 & 7 \\ 1 & 3x+6 & 5x+6 & 5 \\ 3 & 7x+2 & 3x+4 & 2 \end{vmatrix} = 0.$ [*Madras, B.A.*]

32. If $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = 0$, show that two of the numbers a, b, c, d must be equal. [*Madras, B.Sc.*]

12·5. Expansion in Terms of Leading Elements

We consider a determinant of the fourth order, but the method is quite general. Let Δ denote the determinant

$$\begin{vmatrix} \alpha & b_1 & c_1 & d_1 \\ a_2 & \beta & c_2 & d_2 \\ a_3 & b_3 & \gamma & d_3 \\ a_4 & b_4 & c_4 & \delta \end{vmatrix} -$$

where the leading elements a_1, b_2, c_3, d_4 are replaced by α, β, γ, δ in order to give special emphasis to them. The expanded determinant will consist of terms of the following forms:

(*a*) those which contain no leading element;
(*b*) those containing one leading element;
(*c*) those containing two leading elements;
(*d*) the term $\alpha\beta\gamma\delta$.

Clearly there will be no terms involving the product $\alpha\beta\gamma$, except the leading term $\alpha\beta\gamma\delta$. Hence the expanded determinant may be expressed in the form

$$\Delta \equiv \lambda + (\mu_1\alpha + \mu_2\beta + \mu_3\gamma + \mu_4\delta) + (\nu_1\alpha\beta + \nu_2\alpha\gamma + \nu_3\alpha\delta + \nu_4\beta\gamma + \nu_5\beta\delta + \nu_6\gamma\delta) + \alpha\beta\gamma\delta,$$

where the quantities λ, μ, ν do not involve α, β, γ, δ. In this identity write $\alpha = \beta = \gamma = \delta = 0$. Then

$$\begin{array}{cccc} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{array} \Big| = \lambda.$$

To determine the value of μ_1, write $\beta = \gamma = \delta = 0$. Then

$$\begin{array}{cccc} & b_1 & c_1 & d_1 \\ a_2 & 0 & & d_2 \\ & b_3 & & d_3 \\ & b_4 & & 0 \end{array} = \lambda + \mu_1\alpha.$$

The coefficient of α on the left hand side is $\begin{array}{cc} & c_2 \\ & 0 \\ b_4 & c_4 \end{array}$

In a similar way μ_2, μ_3, μ_4 may be found. Thus

$$\mu_2 = \begin{array}{ccc} 0 & c_1 & \\ a_3 & 0 & d_2 \\ a_4 & c_4 & \end{array} \qquad \mu_3 \quad \begin{array}{cc} b_1 & d_1 \\ 0 & d_2 \\ b_4 & 0 \end{array} \qquad \mu_4 = \begin{array}{cc} 0 & b_1 \\ a_2 & 0 \\ a_3 & b_3 \end{array}$$

To find ν_1, write $\gamma = \delta = 0$. Then ν_1 is the coefficient of $\alpha\beta$ in the determinant

$$\begin{array}{cccc} \alpha & b_1 & & d_1 \\ a_2 & \beta & & d_2 \\ a_3 & b_3 & & d_3 \\ & b_4 & & 0 \end{array}$$

This is clearly the second order determinant $\begin{array}{cc} c_4 & 0 \end{array}$

Similarly we find

$$\nu_2 = \begin{vmatrix} 0 & d_2 \\ b_4 & 0 \end{vmatrix}, \quad \nu_3 = \begin{vmatrix} 0 & c_2 \\ b_3 & 0 \end{vmatrix}, \quad \nu_4 = \begin{vmatrix} 0 & d_1 \\ a_4 & 0 \end{vmatrix},$$

$$\nu_5 = \begin{vmatrix} 0 & c_1 \\ a_3 & 0 \end{vmatrix}, \quad \nu_6 = \begin{vmatrix} 0 & b_1 \\ a_2 & 0 \end{vmatrix}$$

Thus the expansion is obtained.

It will be observed that each of the determinants in the expansion has the leading elements zero. Such a determinant has been called by some writers *zero-axial*.

12·61. Multiplication of Determinants

THEOREM IX.—**The product of two determinants of any order is a determinant of the same order.**

We consider the case of two determinants of the third order, but it will be observed that the method is quite general. We prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_3 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

Observe carefully *the method* of writing down the elements of the last determinant.

The *first column* involves the elements of the *first column* of $(\alpha_1\beta_2\gamma_3)$, and no other elements of $(\alpha_1\beta_2\gamma_3)$, the *second column* involves the elements of the *second column* of $(\alpha_1\beta_2\gamma_3)$, the *third column* involves the elements of the *third column* $(\alpha_1\beta_2\gamma_3)$. Then the *first* element of the first column is obtained by multiplying the elements of the *first* row of $(a_1b_2c_3)$ by α_1, α_2, α_3. The *second* element of the first column is obtained by multiplying the elements of the *second* row of $(a_1b_2c_3)$ by α_1, α_2, α_3, the *third* element by multiplying the elements of the *third* row of $(a_1b_2c_3)$ by α_1, α_2, α_3. The second and third columns are obtained by using β_1, β_2, β_3 and γ_1, γ_2, γ_3 respectively, as multipliers. The method can obviously be extended so as to apply to the multiplication of two determinants of *any* order.

To prove the result we use the theorem on the addition of determinants. Since each column consists of the sum of three terms, the determinant can be expanded as the sum of 27 determinants.

When this has been done it is clear that each column in each determinant will have a common factor. Further, after the common factor has been removed, some of the determinants will vanish because of two columns being identical. There will be two types, those which vanish identically and those which do not. The following two are typical:—

$$\begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix}, \qquad \begin{vmatrix} a_1\alpha_1 & a_1\beta_1 & b_1\gamma_2 \\ a_2\alpha_1 & a_2\beta_1 & b_2\gamma_2 \\ a_3\alpha_1 & a_3\beta_1 & b_3\gamma_2 \end{vmatrix}$$

The first determinant is $\alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

while the second is $\alpha_1\beta_1\gamma_2 \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0.$

There will be six determinants of the first type, the other twenty-one being zero. The determinants of the first type are

$$\alpha_1\beta_2\gamma_3 \times (a_1b_2c_3) + \alpha_1\beta_3\gamma_2 \times (a_1c_2b_3) + \alpha_2\beta_1\gamma_3\,(b_1a_2c_3)$$
$$+ \alpha_2\beta_3\gamma_1\,(b_1c_2a_3) + \alpha_3\beta_1\gamma_2\,(c_1a_2b_3) + \alpha_3\beta_2\gamma_1 \times (c_1b_2a_3).$$

Interchanging columns we see that

$(a_1c_2b_3) = -(a_1b_2c_3)$, $(b_1a_2c_3) = -(a_1b_2c_3)$, $(b_1c_2a_3) = (a_1b_2c_3)$, $(c_1a_2b_3) = (a_1b_2c_3)$, $(c_1b_2a_3) = -(a_1b_2c_3)$.

Hence the value of the determinant is

$$\{\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 + \alpha_2\beta_3\gamma_1 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1\} \times (a_1b_2c_3)$$
$$= (\alpha_1\beta_2\gamma_3) \times (a_1b_2c_3).$$

Since a determinant is unaltered when the rows and columns are interchanged it is clear that the product may be represented in more than one form.

We give a brief outline of an alternative method. Consider the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & -1 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & -1 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

which is easily seen on expansion to be the product

$$(a_1b_2c_3) \times (\alpha_1\beta_2\gamma_3).$$

To the fourth column add the first multiplied by α_1, the second multiplied by α_2, the third multiplied by α_3. To the fifth add the first multiplied by β_1, the second multiplied by β_2, and the third multiplied by β_3. To the sixth column add the first column multiplied by γ_1, the second column multiplied by γ_2, and the third column multiplied by γ_3. We obtain

$$\begin{vmatrix} a_1 & b_1 & c_1 & A_1 & B_1 & C_1 \\ a_2 & b_2 & c_2 & A_2 & B_2 & C_2 \\ a_3 & b_3 & c_3 & A_3 & B_3 & C_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix},$$

where
$$\begin{aligned} A_1 &= a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3, & A_2 &= a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3, \\ A_3 &= a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3, & B_1 &= a_1\beta_1 + b_1\beta_2 + c_1\beta_3, \\ B_2 &= a_2\beta_1 + b_2\beta_2 + c_2\beta_3, & B_3 &= a_3\beta_1 + b_3\beta_2 + c_3\beta_3, \\ C_1 &= a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3, & C_2 &= a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3. \\ C_3 &= a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3. \end{aligned}$$

This determinant is easily seen on expansion to be

$$\begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_3 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}.$$

Examples.—(1) *By multiplying together the two determinants*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \begin{vmatrix} bc - f^2 & fg - ch & hf - bg \\ fg - ch & ca - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{vmatrix}$$

or otherwise, prove that the second is the square of the first. [*Camb. Sch.*]

Denote the first determinant by Δ_1 and the second by Δ_2. Then

$$\Delta_1\Delta_2 = \begin{vmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{vmatrix}$$

where $A = abc + 2fgh - af^2 - bg^2 - ch^2$,

i.e. $\Delta_1\,\Delta_2 = (abc + 2fgh - af^2 - bg^2 - ch^2)^3 = \Delta_1^3$.

Hence $\Delta_2 = \Delta_1^2$.

(2) *Show that if the quadratic form*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

is transformed by the substitutions

$$\begin{aligned} x &= l_1X + m_1Y + n_1Z, \\ y &= l_2X + m_2Y + n_2Z, \\ z &= l_3X + m_3Y + n_3Z \end{aligned}$$

into the form

$$a'X^2 + b'Y^2 + c'Z^2 + 2f'YZ + 2g'ZX + 2h'XY,$$

$$\text{then} \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}^2$$

[*M.T.*]

Consider the product:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \times \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} al_1 + hl_2 + gl_3 & hl_1 + bl_2 + fl_3 & gl_1 + fl_2 + cl_3 \\ am_1 + hm_2 + gm_3 & hm_1 + bm_2 + fm_3 & gm_1 + fm_2 + cm_3 \\ an_1 + hn_2 + gn_3 & hn_1 + bn_2 + fn_3 & gn_1 + fn_2 + cn_3 \end{vmatrix}$$

Now multiplying this determinant by $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$ we obtain

$$\begin{vmatrix} al_1^2 + hl_1l_2 + gl_1l_3 + hl_1l_2 + bl_2^2 + fl_2l_3 + gl_1l_3 + fl_2l_3 + cl_3^2 & \ldots\ldots & \ldots\ldots \\ al_1m_1 + hl_1m_2 + gl_1m_3 + hl_2m_1 + bl_2m_2 + fl_2m_3 + gl_3m_1 & & \\ \qquad + fl_3m_2 + cl_3m_3 & \ldots\ldots & \ldots\ldots \\ al_1n_1 + hl_1n_2 + gl_1n_3 + hl_2n_1 + bl_2n_2 + fl_2n_3 + gl_3n_1 & & \\ \qquad + fl_3n_2 + cl_3n_3 & \ldots\ldots & \ldots\ldots \end{vmatrix}$$

Now $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$
$= a\,(l_1X + m_1Y + n_1Z)^2 + b\,(l_2X + m_2Y + n_2Z)^2 + c\,(l_3X + m_3Y + n_3Z)^2$
$+ 2f\,(l_2X + m_2Y + n_2Z)\,(l_3X + m_3Y + n_3Z) + 2g\,(l_3X + m_3Y + n_3Z)$
$(l_1X + m_1Y + n_1Z) + 2h\,(l_1X + m_1Y + n_1Z)\,(l_2X + m_2Y + n_2Z)$
$= a'X^2 + b'Y^2 + c'Z^2 + 2f'YZ + 2g'ZX + 2h'XY.$

Equating corresponding coefficients,

$a' = al_1^2 + bl_2^2 + cl_3^2 + 2fl_2l_3 + 2gl_3l_1 + 2hl_1l_2,$

$h' = al_1m_1 + bl_2m_2 + cl_3m_3 + fl_2m_3 + fl_3m_2 + gl_3m_1 + gl_1m_3 + hl_1m_2 + hl_2m_1,$

$g' = al_1n_1 + bl_2n_2 + cl_3n_3 + fl_2n_3 + fl_3n_2 + gl_3n_1 + gl_1n_3 + hl_1n_2 + hl_2n_1,$

. . . .

Hence $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}^2 = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$

(3) *If* $s_r = \alpha^r + \beta^r + \gamma^r$, *prove by considering the square of the determinant*

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

that $\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = (\alpha - \beta)^2\,(\beta - \gamma)^2\,(\gamma - \alpha)^2.$

Now $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \times \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 \\ \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 & \alpha^4+\beta^4+\gamma^4 \end{vmatrix} = \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

If in the original determinant we write $\alpha = \beta$ two columns become identical and the determinant vanishes. Hence $(\alpha - \beta)$ is a factor. Similarly $(\beta - \gamma)$, $(\beta - \alpha)$ are factors. Since the determinant is of the third degree in α, β, γ it follows that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = \lambda(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha), \text{ where } \lambda \text{ is independent of } \alpha, \beta, \gamma.$$

Considering the leading term $\beta\gamma^2$ and comparing with the corresponding term on the right-hand side it is seen that $\lambda = 1$. Hence

$$(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 = \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

It is clear from the method of argument that this example will generalise. Thus for the case of four quantities it may be seen by the same argument that

$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix} = (\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2,$$

where $s_r = \alpha^r + \beta^r + \gamma^r + \delta^r$.

(4) *Using the rule for the product of two determinants show that the product*

$$(x^3 + y^3 + z^3 - 3xyz)(a^3 + b^3 + c^3 - 3abc)$$

may be expressed in the form $A^3 + B^3 + C^3 - 3ABC$. [*Camb. Sch.*]

If Δ_1, Δ_2 denote the determinants

$$\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}, \quad \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

then $\Delta_1 = a^3 + b^3 + c^3 - 3abc$, $\Delta_2 = x^3 + y^3 + z^3 - 3xyz$. The product $\Delta_1\Delta_2$ is the determinant

$$\begin{vmatrix} ax+cz+by & ay+cx+bz & az+cy+bx \\ bx+az+cy & by+ax+cz & bz+ay+cx \\ cx+bz+ay & cy+bx+az & cz+by+ax \end{vmatrix}$$

Writing $A = ax + by + cz$, $B = bx + az + cy$, $C = cx + bz + ay$ this determinant takes the form

$$\begin{vmatrix} A & C & B \\ B & A & C \\ C & B & A \end{vmatrix} = A^3 + B^3 + C^3 - 3ABC.$$

12·62. Adjoint Determinants

Consider the determinant Δ' formed by the cofactors $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$, which occur in the expansion of the determinant $(a_1b_2c_3)$. Thus

$$\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

Δ' is called the adjoint determinant of $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and in a similar way the adjoint of a determinant of any order is defined.

THEOREM X.—If Δ is a determinant of the nth order, Δ' the adjoint determinant, then $\Delta' = \Delta^{n-1}$.

We give the proof for the case $n = 3$, but the method is quite general. Consider the product $\Delta\Delta'$. Then

$$\Delta\Delta' = \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

Now Δ
$$= a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3.$$

Also $a_2A_1 + b_2B_1 + c_2C_1$

$$= a_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Similarly for the other five elements of $\Delta\Delta'$. Hence

$$\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3.$$

In general if Δ is of the nth order, $\Delta\Delta' = \Delta^n$. Thus $\Delta' = \Delta^{n-1}$.

Any minor of the adjoint determinant can be expressed in terms of the elements of the original determinant.

Consider, *e.g.*, a determinant of the fourth order so that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{vmatrix}$$

To determine the leading first minor in Δ' multiply Δ by the determinant

$$\Delta'' = \begin{vmatrix} 1 & A_2 & A_3 & A_4 \\ 0 & B_2 & B_3 & B_4 \\ 0 & C_2 & C_3 & C_4 \\ 0 & D_2 & D_3 & D_4 \end{vmatrix} = \begin{vmatrix} B_2 & B_3 & B_4 \\ C_2 & C_3 & C_4 \\ D_2 & D_3 & D_4 \end{vmatrix}.$$

Then $\Delta\Delta''$

$$= \begin{vmatrix} a_1 & a_1A_2 + b_1B_2 + c_1C_2 + d_1D_2 & a_1A_3 + b_1B_3 + c_1C_3 + d_1D_3 & a_1A_4 + b_1B_4 + c_1C_4 + d_1D_4 \\ a_2 & a_2A_2 + b_2B_2 + c_2C_2 + d_2D_2 & a_2A_3 + b_2B_3 + c_2C_3 + d_2D_3 & a_2A_4 + b_2B_4 + c_2C_4 + d_2D_4 \\ a_3 & a_3A_2 + b_3B_2 + c_3C_2 + d_3D_2 & a_3A_3 + b_3B_3 + c_3C_3 + d_3D_3 & a_3A_4 + b_3B_4 + c_3C_4 + d_3D_4 \\ a_4 & a_4A_2 + b_4B_2 + c_4C_2 + d_4D_2 & a_4A_3 + b_4B_3 + c_4C_3 + d_4D_3 & a_4A_4 + b_4B_4 + c_4C_4 + d_4D_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & \Delta & 0 & 0 \\ a_3 & 0 & \Delta & 0 \\ a_4 & 0 & 0 & \Delta \end{vmatrix} = a_1 \Delta^3.$$

$$\text{Hence} \quad \begin{vmatrix} B_2 & B_3 & B_4 \\ C_2 & C_3 & C_4 \\ D_2 & D_3 & D_4 \end{vmatrix} = a_1 \Delta^2.$$

Note that the argument assumes that $\Delta \neq 0$.

Similarly for the case of other minors. Examples of the method will occur later.

12·7. Symmetrical Determinants

For a discussion of determinants of this type it is convenient to take a standard form which exhibits more clearly the symmetry

relation between rows and columns. Thus a determinant of the nth order may be represented in the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{vmatrix} = (a_{11}\, a_{22}\, a_{33} \ldots a_{nn}).$$

Then a determinant is said to be *symmetric* or symmetrical if $a_{rs} = a_{sr}$, and to be *skew-symmetric* or skew-symmetrical if $a_{rs} = -a_{sr}$ where $r, s = 1, 2, 3, \ldots n$. In the latter case the determinant is zero-axial, *i.e.* the elements in the leading diagonal are zero. For $a_{rr} = -a_{rr}$, i.e. $a_{rr} = 0$.

$$\text{Thus} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}$$

are symmetrical determinants, while

$$\begin{vmatrix} 0 & h & g & l \\ -h & 0 & f & m \\ -g & -f & 0 & n \\ -l & -m & -n & 0 \end{vmatrix} \text{ is skew-symmetric.}$$

THEOREM XI.—The adjoint of a symmetrical determinant is itself symmetrical. For let A_{rs} and A_{sr} be the cofactors a_{rs}, a_{sr} of a symmetric determinant. Then since A_{rs} is transformed into A_{sr} by interchanging rows and columns it follows that $A_{rs} = A_{sr}$.

Thus, *e.g.*, consider the third order determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{Then } A_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = A_{13}.$$

12·71. An Important Case

We now prove some properties of the symmetrical determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

If A, B, C, F, G, H are the cofactors of a, b, c, f, g, h, then

(i) $BC - F^2 = a\Delta$, $CA - G^2 = b\Delta$, $AB - H^2 = c\Delta$;

(ii) $GH - AF = f\Delta$, $HF - BG = g\Delta$, $FG - CH = h\Delta$.

It will be sufficient to prove one of each set. Consider first $CA - G^2 = b\Delta$. Now

$$CA - G^2 = \begin{vmatrix} A & G \\ G & C \end{vmatrix} = \begin{vmatrix} A & 0 & G \\ H & 1 & F \\ G & 0 & C \end{vmatrix}.$$

$$\text{Then } \Delta(CA - G^2) = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} A & 0 & G \\ H & 1 & F \\ G & 0 & C \end{vmatrix}$$

$$= \begin{vmatrix} aA + hH + gG & h & aG + hF + gC \\ hA + bH + fG & b & hG + bF + fC \\ gA + fH + cG & f & gG + fF + cC \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & h & 0 \\ 0 & b & 0 \\ 0 & f & \Delta \end{vmatrix} = b\Delta^2.$$

Hence $CA - G^2 = b\Delta$.

Next consider $HF - BG = g\Delta$. Now

$$HF - BG = \begin{vmatrix} H & G \\ B & F \end{vmatrix} = \begin{vmatrix} 0 & H & G \\ 0 & B & F \\ 1 & F & C \end{vmatrix}.$$

$$\text{Hence } \Delta(HF - BG) = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} 0 & H & G \\ 0 & B & F \\ 1 & F & C \end{vmatrix}$$

$$= \begin{vmatrix} g & aH + hB + gF & aG + hF + gC \\ f & hH + bB + fF & hG + bF + fC \\ c & gH + fB + cF & gG + fF + cC \end{vmatrix}$$

$$= \begin{vmatrix} g & 0 & 0 \\ f & \Delta & 0 \\ c & 0 & \Delta \end{vmatrix} = g\Delta^2.$$

Hence $HF - BC = g\Delta$.

We now consider *the condition that the general homogeneous expression of the second degree in three variables may be written as the product of two factors linear in the three variables.*

The general homogeneous expression of the second degree in the three variables x, y, z may be written in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

Consider the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

where $a \neq 0$. Rearranging as a quadratic in x^2,

$$ax^2 + 2x\,(gz + hy) + by^2 + 2fyz + cz^2 = 0.$$

The roots of this quadratic in x are

$$x = [-(gz + hy) \pm \{(gz + hy)^2 - a\,(by^2 + 2fyz + cz^2)\}^{\frac{1}{2}}]/a.$$

In order that x may be expressed linearly in terms of y and z the expression

$$(gz + hy)^2 - a\,(by^2 + 2fyz + cz^2)$$

must be a perfect square. This expression may be written in the form

$$(h^2 - ab)\,y^2 + 2yz\,(hg - af) + (g^2 - ac)\,z^2.$$

In order that this may be a perfect square

$$(hg - af)^2 = (h^2 - ab)\,(g^2 - ac)$$

which simplifies on division by a, to

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Now the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

Hence *the condition that the expression*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

may be written as the product of two factors linear in x, y, z is that the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ *vanish.*

Example.—*Prove that in a triangle ABC*

$$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$$

[*Camb. Sch.*]

Consider the expression:

$E = x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C + 2yz \sin A + 2zx \sin B + 2xy \sin C.$

We prove the result by showing that E can be expressed as the product of two factors linear in x, y, z. Now in any triangle ABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}.$$

Using this property, and the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ we obtain

$$E/2k = ax^2 \cos A + by^2 \cos B + cz^2 \cos C + ayz + bzx + cxy.$$

$$\text{Again, } a = b \cos C + c \cos B,$$
$$b = a \cos C + c \cos A,$$
$$c = a \cos B + b \cos A.$$

$$\therefore\ E/2k = ax^2 \cos A + by^2 \cos B + cz^2 \cos C + (b \cos C + c \cos B)\,yz + (a \cos C + c \cos A)\,zx + (a \cos B + b \cos A)\,xy$$
$$= (ax + by + cz)(x \cos A + y \cos B + z \cos C).$$

Hence E can be written as the product of two factors linear in x, y, z. Thus

$$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$$

12·81. Skew-Symmetric Determinants

THEOREM XII.—In a skew-symmetric determinant

$$(a_{11}a_{22}a_{33}\ldots a_{nn}),\ A_{rs} = (-1)^{n-1}A_{sr},$$

where A_{rs} and A_{sr} are the cofactors of a_{rs} and a_{sr} respectively.

Now A_{sr} is changed into A_{rs} by changing the sign of every element and then interchanging rows and columns. Further A_{sr} is a determinant of order $n - 1$. Taking out a factor -1 from each column, it follows that $A_{rs} = (-1)^{n-1} A_{sr}$. Thus, *e.g.* in the skew-determinant

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix}$$

$$A_{14} = -\begin{vmatrix} a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \\ a_{41} & a_{42} & a_{43} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{31} & a_{41} \\ 0 & a_{32} & a_{42} \\ a_{23} & 0 & a_{43} \end{vmatrix}$$

$$= -\begin{vmatrix} -a_{12} & -a_{13} & -a_{14} \\ 0 & -a_{23} & -a_{24} \\ -a_{32} & 0 & -a_{34} \end{vmatrix} = (-1)^4 \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ 0 & a_{23} & a_{24} \\ a_{32} & 0 & a_{34} \end{vmatrix} = (-1)^3 A_{41}.$$

It follows that *if n be even the adjoint of a skew-symmetric determinant is skew-symmetric, while if n be odd the adjoint determinant is symmetric.*

THEOREM XIII.—**A skew-symmetric determinant of odd order is zero.**

The determinant is unaltered by changing rows into columns and changing the sign of all the elements. But when the determinant is of odd order this is equivalent to introducing a factor $(-1)^n = -1$. Thus the determinant is equal to itself with changed sign and so must be zero.

THEOREM XIV.—**A skew-symmetric determinant of even order is a perfect square.** Consider the skew-symmetric determinant Δ_4 of the fourth order,

$$\Delta_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\text{Then } A_{11}A_{22} - A_{12}A_{21} = \begin{vmatrix} A_{11} & A_{21} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 \\ A_{13} & A_{23} & 1 & 0 \\ A_{14} & A_{24} & 0 & 1 \end{vmatrix}.$$

Applying the ordinary rule for the multiplication of two determinants of the fourth order, and simplifying we have

$$(A_{11}A_{22} - A_{12}A_{21})\,\Delta_4 =$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \times \begin{vmatrix} A_{11} & A_{21} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 \\ A_{13} & A_{23} & 1 & 0 \\ A_{14} & A_{24} & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta_4 & 0 & a_{13} & a_{14} \\ 0 & \Delta_4 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix}$$

$= \Delta_4^2\,(a_{33}a_{44} - a_{34}a_{43}) = a^2_{34}\,\Delta_4^2$, since $a_{33} = a_{44} = 0$ and $a_{34} = -a_{43}$.

Now A_{11}, A_{22} are skew-symmetric determinants of odd order and hence are zero. Also $A_{12} = -A_{21}$, since the adjoint determinant of Δ_4 is skew-symmetric.

Hence $\Delta_4 = (A_{12}/a_{34})^2$ and Δ_4 is a perfect square.

The argument may be extended to $\Delta_6, \Delta_8, \ldots, \Delta_{2p}$.

12·82. Skew Determinants

A skew determinant $(a_{11}\,a_{22}\,a_{33}\,\ldots\,a_{nn})$ is one in which

$$a_{rs} = -a_{sr},\ r, s = 1, 2, 3, \ldots n \text{ and } r \neq s.$$

Thus a skew determinant differs from a skew-symmetric determinant only in that it is not zero-axial, *i.e.* the elements of the leading diagonal are not zero. Such a determinant may be reduced to the consideration of skew-symmetric determinants.

Example.—*Prove that the skew determinant* $\begin{vmatrix} x & a & b & c \\ -a & x & d & e \\ -b & -d & x & f \\ -c & -e & -f & x \end{vmatrix}$

is equal to $x^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)\,x^2 + (af - be + cd)^2$.

What is required here is the expansion in powers of x. This may be obtained directly by expanding along the principal diagonal, *i.e.* in terms of the leading elements. Let Δ denote the determinant. Then proceeding as in § 12·5,

$$\Delta = \lambda + (\mu_1 + \mu_2 + \mu_3 + \mu_4)\,x + (\nu_1 + \nu_2 + \ldots + \nu_6)\,x^2 + x^4, \text{ where}$$

$$\lambda = \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix},\ \mu_1 = \begin{vmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{vmatrix}, \ldots,$$

$$\nu_1 = \begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix}, \ldots$$

Each of the determinants is skew-symmetric. Thus

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0,$$

since the determinants are of odd order. The other determinants must be perfect squares.

From § 12·81 it follows that

$$\lambda = (A_{12}/a_{34})^2 = \frac{1}{f^2}\begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix}^2 = \frac{1}{f^2}\{cdf + af^2 - bef\}^2$$

$$= (cd + af - be)^2.$$

Further, $\nu_1 = f^2,\ \nu_2 = e^2,\ \nu_3 = d^2,\ \nu_4 = c^2,\ \nu_5 = b^2,\ \nu_6 = a^2$.
Substituting the values the required result is obtained.

12·9. Solution of Linear Equations

Consider the linear equations

$$a_1x + b_1y + c_1z = d_1 \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

$$a_3x + b_3y + c_3z = d_3 \quad \ldots\ldots\ldots\ldots \quad \text{(iii)}$$

in the three variables x, y, z.

Let Δ denote the determinant $(a_1b_2c_3)$, Δ_1 the determinant $(d_1b_2c_3)$, Δ_2 the determinant $(a_1d_2c_3)$, Δ_3 the determinant $(a_1b_2d_3)$, $A_1, A_2, A_3, B_1, \ldots$ the cofactors of $a_1, a_2, a_3, b_1, \ldots$ in Δ.

We suppose that $\triangle \neq 0$. Multiply (i) by A_1, (ii) by A_2, (iii) by A_3 and add. Then

$$x(a_1A_1 + a_2A_2 + a_3A_3) + y(b_1A_1 + b_2A_2 + b_3A_3) + z(c_1A_1 + c_2A_2 + c_3A_3) = d_1A_1 + d_2A_2 + d_3A_3 \quad \ldots \text{(iv)}$$

Now $a_1A_1 + a_2A_2 + a_3A_3 = \triangle$. Also

$$b_1A_1 + b_2A_2 + b_3A_3 = \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0,$$

$$c_1A_1 + c_2A_2 + c_3A_3 = \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = 0,$$

since two columns are identical in each case, and

$$d_1A_1 + d_2A_2 + d_3A_3 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \triangle_1.$$

Hence from (iv), $x\triangle = \triangle_1$, i.e. $x = \triangle_1/\triangle$, $\triangle \neq 0$.

In a similar way it may be proved that

$$y = \triangle_2/\triangle, \quad z = \triangle_3/\triangle.$$

The various cases that can arise when $\triangle = 0$ are discussed fully in the next chapter.

Examples.—(1) *Prove that* $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - c)(c - a)(a - b).$

Hence solve the equations

$$\begin{aligned} x + y + z &= 1 \\ ax + by + cz &= k \\ a^2x + b^2y + c^2z &= k^2, \end{aligned}$$

where a, b, c, k are given, and no two of a, b, c are equal. [*N.Sc.*]

The determinant has been evaluated in § 12·61, Ex. 3. Solving the three linear equations by determinants,

$$x = \triangle_1/\triangle, \; y = \triangle_2/\triangle, \; z = \triangle_3/\triangle, \text{ where}$$

$$\triangle = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - c)(c - a)(a - b),$$

$$\triangle_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (b - c)(c - k)(k - b),$$

$$\triangle_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (k - c)(c - a)(a - k),$$

$$\triangle_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (b - k)(k - a)(a - b).$$

Hence
$$\begin{aligned} x &= (c - k)(k - b)/(c - a)(a - b), \\ y &= (k - c)(a - k)/(b - c)(a - b), \\ z &= (b - k)(k - a)/(b - c)(c - a). \end{aligned}$$

(2) *If* $\dfrac{x}{a+\lambda}+\dfrac{y}{b+\lambda}+\dfrac{z}{c+\lambda}=1,\ \dfrac{x}{a+\mu}+\dfrac{y}{b+\mu}+\dfrac{z}{c+\mu}=1,$

$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}=1, \text{ prove that}$$

$$x=(a+\lambda)(a+\mu)(a+\nu)/(a-b)(a-c). \qquad \text{[Camb. Sch.]}$$

The value of x which satisfies the equations is Δ_1/Δ, where

$$\Delta_1=\begin{vmatrix} 1 & 1/(b+\lambda) & 1/(c+\lambda)\\ 1 & 1/(b+\mu) & 1/(c+\mu)\\ 1 & 1/(b+\nu) & 1/(c+\nu)\end{vmatrix},\ \Delta=\begin{vmatrix} 1/(a+\lambda) & 1/(b+\lambda) & 1/(c+\lambda)\\ 1/(a+\mu) & 1/(b+\mu) & 1/(c+\mu)\\ 1/(a+\nu) & 1/(b+\nu) & 1/(c+\nu)\end{vmatrix}$$

Hence $x=(a+\lambda)(a+\mu)(a+\nu)\,\Delta'_1/\Delta'$, where

$$\Delta_1'=\begin{vmatrix} 1 & (b+\mu)(b+\nu) & (c+\mu)(c+\nu)\\ 1 & (b+\nu)(b+\lambda) & (c+\nu)(c+\lambda)\\ 1 & (b+\lambda)(b+\mu) & (c+\lambda)(c+\mu)\end{vmatrix},$$

$$\Delta'=\begin{vmatrix} (a+\mu)(a+\nu) & (b+\mu)(b+\nu) & (c+\mu)(c+\nu)\\ (a+\nu)(a+\lambda) & (b+\nu)(b+\lambda) & (c+\nu)(c+\lambda)\\ (a+\lambda)(a+\mu) & (b+\lambda)(b+\mu) & (c+\lambda)(c+\mu)\end{vmatrix}$$

In Δ' subtract the first row from the second and third rows. Then

$$\Delta'=\begin{vmatrix} (a+\mu)(a+\nu) & (b+\mu)(b+\nu) & (c+\mu)(c+\nu)\\ (\lambda-\mu)(a+\nu) & (\lambda-\mu)(b+\nu) & (\lambda-\mu)(c+\nu)\\ (\lambda-\nu)(a+\mu) & (\lambda-\nu)(b+\mu) & (\lambda-\nu)(c+\mu)\end{vmatrix}$$

$$=(\lambda-\mu)(\lambda-\nu)\begin{vmatrix} (a+\mu)(a+\nu) & (b+\mu)(b+\nu) & (c+\mu)(c+\nu)\\ a+\nu & b+\nu & c+\nu\\ a+\mu & b+\mu & c+\mu\end{vmatrix}$$

Subtracting the second row from the third and taking out a common factor,

$$\Delta'=(\lambda-\mu)(\lambda-\nu)(\mu-\nu)$$

$$\times\begin{vmatrix} (a+\mu)(a+\nu) & (b+\mu)(b+\nu) & (c+\mu)(c+\nu)\\ a+\nu & b+\nu & c+\nu\\ 1 & 1 & 1\end{vmatrix}$$

Now subtract the third column from the first and second columns and expand along the third row. Then

$$\Delta'=(\lambda-\mu)(\lambda-\nu)(\mu-\nu)$$

$$\times\begin{vmatrix} a^2-c^2+(\mu+\nu)(a-c) & b^2-c^2+(\mu+\nu)(b-c)\\ a-c & b-c\end{vmatrix}$$

$$=(\lambda-\mu)(\lambda-\nu)(\mu-\nu)(a-c)(b-c)$$

$$\times\begin{vmatrix} a+c+\mu+\nu & b+c+\mu+\nu\\ 1 & 1\end{vmatrix}$$

$$=(\lambda-\mu)(\lambda-\nu)(\mu-\nu)(a-c)(b-c)(a-b).$$

In a similar way we find that $\Delta'_1=(\lambda-\mu)(\lambda-\nu)(\mu-\nu)(b-c)$.

$$\therefore\ x=(a+\lambda)(a+\mu)(a+\nu)/(a-b)(a-c).$$

(3) *Determine all the values of x, y, z which satisfy the equations*

$$x + y + z = 3,$$
$$a^2x + b^2y + c^2z = a^2 + b^2 + c^2,$$
$$a^3x + b^3y + c^3z = a^3 + b^3 + c^3,$$

where no two of the numbers a, b, c are equal, and $bc + ca + ab \neq 0$. Show also that if $bc + ca + ab = 0$, the complete solution of the equations is given by

$$x = 1 + \lambda (b^2 - c^2), \quad y = 1 + \lambda (c^2 - a^2), \quad z = 1 + \lambda (a^2 - b^2),$$

where λ is arbitrary. [*M.T.*]

The values of x, y, z are given by

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta} \text{ where}$$

$$\Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ a^2 + b^2 + c^2 & b^2 & c^2 \\ a^3 + b^3 + c^3 & b^3 & c^3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ a^2 & a^2 + b^2 + c^2 & c^2 \\ a^3 & a^3 + b^3 + c^3 & c^3 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ a^2 & b^2 & a^2 + b^2 + c^2 \\ a^3 & b^3 & a^3 + b^3 + c^3 \end{vmatrix}, \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$\text{Now } \Delta = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2 - a^2 & c^2 - a^2 \\ a^3 & b^3 - a^3 & c^3 - a^3 \end{vmatrix}$$

$$= (b - a)(c - a) \begin{vmatrix} b + a & c + a \\ b^2 + ab + a^2 & c^2 + ac + a^2 \end{vmatrix}$$

$$= (b - a)(c - a) \begin{vmatrix} b + a & c - b \\ b^2 + ab + a^2 & (c - b)(a + b + c) \end{vmatrix}$$

$$= (b - a)(c - a)(c - b) \begin{vmatrix} b + a & 1 \\ b^2 + ab + a^2 & a + b + c \end{vmatrix}$$

$$= (a - b)(b - c)(c - a)(ab + bc + ca).$$

Now consider Δ_1.

Then on subtracting the sum of the second and third columns from the first we find

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \Delta.$$

Since $a - b \neq 0$, $b - c \neq 0$, $c - a \neq 0$, $ab + bc + ca \neq 0$ it follows that the only values of x, y, z are $1, 1, 1$.

If $bc + ca + ab = 0$ then each of the determinants is zero, so that the determinants give indeterminate forms for x, y, z.

The given equations may be written in the form

$$u + v + w = 0,$$
$$a^2u + b^2v + c^2w = 0,$$
$$a^3u + b^3v + c^3w = 0,$$

where $u = x - 1$, $v = y - 1$, $w = z - 1$. From the first pair of equations

$$\frac{u}{c^2 - b^2} = \frac{v}{a^2 - c^2} = \frac{w}{b^2 - a^2} = -\lambda \text{ (say)} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Substituting in the third equation

$$\lambda \{a^3 (b^2 - c^2) + b^3 (c^2 - a^2) + c^3 (a^2 - b^2)\} = 0.$$

Now $a^3 (b^2 - c^2) + b^3 (c^2 - a^2) + c^3 (a^2 - b^2)$

$$= -(a - b)(b - c)(c - a)(ab + bc + ca).$$

Since $ab + bc + ca = 0$ it follows that

$$\lambda \{a^3 (b^2 - c^2) + b^3 (c^2 - a^2) + c^3 (a^2 - b^2)\} = 0$$

for all values of λ. Thus the solutions of the equations are given by (1) where λ is arbitrary.

12·91. Generalisation to n Variables

The method of § 12·9 can clearly be extended to the case of n linear equations involving n variables. Thus if we have the n equations involving the n unknowns $x_1, x_2, x_3, \ldots, x_n$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= k_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= k_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= k_3 \\
\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= k_n
\end{aligned}$$

then $x_1 = \Delta_1/\Delta$, $x_2 = \Delta_2/\Delta$, $x_3 = \Delta_3/\Delta$, $\ldots$, $x_n = \Delta_n/\Delta$,
where $\Delta = (a_{11}a_{22}a_{33}\ldots a_{nn})$, $\Delta_1 = (k_1a_{22}a_{33}\ldots a_{nn})$,

$$\Delta_2 = (a_{11}k_2a_{33}\ldots a_{nn}), \quad \Delta_3 = (a_{11}a_{22}k_3\ldots a_{nn}), \ \ldots,$$

$$\Delta_n = (a_{11}a_{22}a_{33}\ldots k_n).$$

As in § 12·9 special cases occur when $\Delta = 0$. These will be discussed in § 13·71.

EXERCISES XII

33. If $ax_r^2 + by_r^2 + cz_r^2 = 0$, $(r = 1, 2, 3)$ prove that

$$(ax_2x_3 + by_2y_3 + cz_2z_3)(ax_3x_1 + by_3y_1 + cz_3z_1)(ax_1x_2 + by_1y_2 + cz_1z_2)$$
$$= \tfrac{1}{2}abc \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2$$

34. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$
$$= \tfrac{1}{2} \begin{vmatrix} 2(a_1c_1 - b_1^2) & a_1c_2 + a_2c_1 - 2b_1b_2 & a_1c_3 + a_3c_1 - 2b_1b_3 \\ a_1c_2 + a_2c_1 - 2b_1b_2 & 2(a_2c_2 - b_2^2) & a_2c_3 + a_3c_2 - 2b_2b_3 \\ a_1c_3 + a_3c_1 - 2b_1b_3 & a_2c_3 + a_3c_2 - 2b_2b_3 & 2(a_3c_3 - b_3^2) \end{vmatrix}$$

35. Prove that the square of a determinant of the third order may be represented in the form

$$\begin{array}{ccc} A & H & G \\ H & B & \\ G & F & \end{array}$$

36. Prove that

$$\begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix} \times \begin{vmatrix} \alpha - i\beta & -\gamma - i\delta \\ \gamma - i\delta & \alpha + i\beta \end{vmatrix} = \begin{vmatrix} A - iB & C - iD \\ -C - iD & A + iB \end{vmatrix}$$

where $A \equiv a\alpha + b\beta + c\gamma + d\delta$, $\quad B \equiv a\beta - \alpha b + c\delta - \gamma d$,
$C \equiv c\alpha - \gamma a + b\delta - \beta d$, $\quad D \equiv b\gamma - \beta c + a\delta - \alpha d$.

Deduce Euler's theorem that *the product of two sums each of four squares can be expressed as the sum of four squares.*

37. By considering the determinants

$$\begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{array} \qquad \begin{array}{ccc} -b_1 & a_1 & -d_1 \\ -b_2 & a_2 & -d_2 \\ -b_3 & a_3 & -d_3 \\ -b_4 & a_4 & -d_4 \end{array}$$

prove that the square of any determinant of the fourth order may be expressed as a skew-symmetric determinant.

38. Form the adjoint determinant of $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and show that it is symmetrical.

39. Prove for the case of third order determinants that the product of two adjoint determinants is the adjoint determinant of the product of the original determinants.

40. Solve the equations

$$2x - y - z = 4,$$
$$-x + 2y - z = -5$$
$$x - y + 2z = 1.$$

[*Lond. B.A.*]

41. Show how to solve the equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

by means of determinants, assuming that

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}$$

Complete the solution for the following numerical case:

$$3x + 5y - 7z = 13,$$
$$4x + y - 12z = 6,$$
$$2x + 9y - 3z = 20.$$

[*M.T.*]

42. Solve by the aid of determinants, or otherwise,

$$53x - 37y + 29z = 32,$$
$$22x + 31y - 99z = 23,$$
$$99x - 44y - 37z = 3.$$

[*Lond. B.Sc. Eng.*]

43. Prove that if Δ denote the determinant

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}$$

then $\Delta = -(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm)$, where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

44. Prove that $\begin{vmatrix} -1 & x & y & z \\ w & -1 & y & z \\ w & x & -1 & z \\ w & x & y & -1 \end{vmatrix}$ is equal to

$$(w+1)(x+1)(y+1)(z+1)\left\{1 - \frac{w}{w+1} - \frac{x}{x+1} - \frac{y}{y+1} - \frac{z}{z+1}\right\}.$$

45. Prove that the two quadratic expressions

$$Q \equiv ax^2 + 2bx + c, \quad Q' \equiv a'x^2 + b'x + c',$$

can in general be expressed in the forms

$$Q \equiv \lambda(x-\alpha)^2 + \mu(x-\beta)^2,$$
$$Q' \equiv \lambda'(x-\alpha)^2 + \mu'(x-\beta)^2,$$

where $(x - \alpha)$, $(x - \beta)$ are the factors of the quadratic expression

$$\begin{vmatrix} ax+b & bx+c \\ a'x+b' & b'x+c' \end{vmatrix}$$

Hence find the turning points of the expression $(x^2+1)/(x^2+6x+1)$.

[M.T.]

46. Prove that the determinant of the nth order

$$\begin{vmatrix} 1+x & 1-x & 1-x & \cdots & \cdots & 1-x \\ 1-x & 1+x & 1-x & \cdots & \cdots & 1-x \\ 1-x & 1-x & 1+x & \cdots & \cdots & 1-x \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1-x & 1-x & 1-x & \cdots & \cdots & 1+x \end{vmatrix}$$

is equal to $(2x)^{n-1}\{n-(n-2)x\}$.

47. If $u_n = \begin{vmatrix} x_1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & x_2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & x_3 & 1 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & -1 & x_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & -1 & x_n \end{vmatrix}$ prove that

$$u_n = x_n u_{n-1} + u_{n-2}.$$

48. Show that if $l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3$ are real quantities satisfying relations

$$l_r^2 + m_r^2 + n_r^2 = 1, \quad (r = 1, 2, 3),$$
$$l_p l_q + m_p m_q + n_p n_q = 0, \quad \left(\begin{Bmatrix} p \\ q \end{Bmatrix} = 1, 2, 3;\ p \neq q\right),$$

then $\Sigma l_r^2 = \Sigma m_r^2 = \Sigma n_r^2 = 1$; $\Sigma l_r m_r = \Sigma m_r n_r = \Sigma n_r l_r = 0$

$$\text{and} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$$

[*Camb. Sch.*]

49. Use the determinant $\begin{vmatrix} w & x & y & z \\ 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \end{vmatrix}$ to prove the identity

$$w(y-z)(z-x)(x-y) - x(z-w)(w-y)(y-z)$$
$$+ y(w-x)(x-z)(z-w) - z(x-y)(y-w)(w-x) \equiv 0.$$

Use the substitutions $w = e^{2i\alpha}$, $x = e^{2i\beta}$, $y = e^{2i\gamma}$, $z = e^{2i\delta}$ to prove that if α, β, γ, δ are any four angles then

$$\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha) - \sin(\beta-\gamma)\sin(\gamma-\delta)\sin(\delta-\beta)$$
$$+ \sin(\gamma-\delta)\sin(\delta-\alpha)\sin(\alpha-\gamma) - \sin(\delta-\alpha)\sin(\alpha-\beta)\sin(\beta-\delta) \equiv 0.$$

50. If in the determinant $\begin{vmatrix} a_1 & -a_2 & -a_3 \\ -b_1 & b_2 & -b_3 \\ -c_1 & -c_2 & c_3 \end{vmatrix}$ all the numbers $a_1, a_2, a_3, b_1, \ldots, c_1 \ldots$ are positive, and if the sum of the elements in any row is positive, show that the determinant is positive. [*Madras, B.Sc.*]

51. Prove that the determinant of the nth order:

$$\begin{vmatrix} \cos\theta & 1 & 0 & 0 & \cdot\cdot & \cdot\cdot & 0 \\ 1 & 2\cos\theta & 1 & 0 & \cdot\cdot & \cdot\cdot & 0 \\ 0 & 1 & 2\cos\theta & 1 & \cdot\cdot & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & \cdot\cdot & \cdot\cdot & 1 & 2\cos\theta & 1 \\ 0 & 0 & \cdot\cdot & \cdot\cdot & 0 & 1 & 2\cos\theta \end{vmatrix} = \cos n\theta.$$

CHAPTER XIII

MATRICES

IN this chapter we discuss the properties of matrices, a set of mathematical objects quite different from those we have so far encountered. Unlike a determinant, we cannot attach a *value* to a matrix. Matrices are not functions, but we shall find that, with certain important exceptions, they obey the ordinary laws of algebra, if the operations of addition, multiplication, etc., of matrices are suitably defined. Matrices are of fundamental importance in both pure and applied mathematics, and we can most easily understand their nature by considering some examples.

13·11. Linear Equations and Transformations

Consider the pair of simultaneous linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2.$$

Provided that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, we can solve these equations for x and y, writing each in terms of the coefficients a_1, a_2, b_1, b_2, c_1, c_2. This method of writing the equations is cumbersome if there are many equations and many unknowns, and a natural abbreviation is to write them in the form

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The array of elements $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is called a matrix. It contains two rows and two columns and is therefore called a 2×2 matrix, or a square matrix of order 2×2 (read 2 by 2).

$\begin{bmatrix} x \\ y \end{bmatrix}$ is a matrix of order 2×1, since it contains two rows and only one column. Such a matrix, containing only a single column, is more commonly known as a *column vector*. Similarly, a matrix containing a single row is called a *row vector*.

$[a_1 \ b_1]$ is a row vector of order 2; it is also a matrix of order 1×2.

The single equation $a_1x + b_1y = c_1$ may be written in the form

$$[a_1 \ b_1]\begin{bmatrix} x \\ y \end{bmatrix} = c_1;$$

$[a_1 \ b_1]\begin{bmatrix} x \\ y \end{bmatrix} = a_1x + b_1y$ is called the *inner product* of the two vectors $[a_1 \ b_1]$ and $\begin{bmatrix} x \\ y \end{bmatrix}$. The row vector is always written on the *left* of the product and the column vector on the *right.*

We return now to the pair of equations represented by

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The two equations can be written separately in the form

$$[a_1 \ b_1]\begin{bmatrix} x \\ y \end{bmatrix} = c_1$$

$$[a_2 \ b_2]\begin{bmatrix} x \\ y \end{bmatrix} = c_2.$$

Thus the first equation is obtained by equating to c_1 the inner product of the first row vector of the matrix and the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. The second equation is formed by equating to c_2 the inner product of the second row vector of the matrix and the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

We can abbreviate the form still further if we adopt a different notation. We write the original equations in the form

$$a_{11}x_1 + a_{12}x_2 = c_1$$
$$a_{21}x_1 + a_{22}x_2 = c_2.$$

The variables x, y are replaced by x_1, x_2, and the equations can now be written

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

This notation has the great advantage that the subscripts tell us at once the position of the element to which they are attached. For example, x_1, c_1 are the first elements of the column vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ respectively, and similarly for the second elements. a_{11} is the element in the first row and first column of the matrix,

a_{21} is in the second row and first column, and so on. The first subscript gives the number of the row and the second the number of the column. Since a column vector has only one column, we usually omit the column subscript and write only the row subscript. Similarly, for a row vector we write only the column subscript.

We are now in a position to write the equations in a very compact form. The column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is denoted by a single letter x, and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is denoted by c.

The matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is denoted by A, or sometimes by $[a_{ij}]$. We must now specify the order of the matrix, and in this example we say that A is a matrix of order 2×2. The equations are now written in the form

$$Ax = c.$$

A matrix, unlike a determinant, is not necessarily square. If the matrix A has m rows and n columns, we say that it is of order $m \times n$. Note that the row number always comes *first*. Suppose, for example, that a three-dimensional object is to be represented by a drawing on a plane. Three coordinates, say x, y, and z, are needed to fix a point in the object, but only two coordinates, say X and Y, specify the corresponding point in the plane. In general, X and Y will each depend on x, y, and z, and we may, in particular, have a relationship between them of the following type:—

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The matrix $A = [a_{ij}]$ is of order 2×3.

13·12. Matrix Algebra. Addition

Consider the following sets of equations:

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or $y = Ax$ and $z = Bx$, where A and B are matrices of the same order 2×3, y and z are column vectors of order 2, and x is a column vector of order 3. It is important to note that the symbol Ax has no meaning unless the number of columns in A is equal to the order of the vector x.

Let w be a column vector of order 2 such that

$$w_1 = y_1 + z_1$$

and

$$w_2 = y_2 + z_2.$$

It is easily seen that

$$w_1 = (a_{11} + b_{11})\,x_1 + (a_{12} + b_{12})\,x_2 + (a_{13} + b_{13})\,x_3$$

$$w_2 = (a_{21} + b_{21})\,x_1 + (a_{22} + b_{22})\,x_2 + (a_{23} + b_{23})\,x_3,$$

or

$$w = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} x = [a_{ij} + b_{ij}]\,x.$$

Now we regard the process of obtaining this equation from the two equations $y = Ax$, $z = Bx$ as one of addition, and we therefore define w to be the sum of y and z.

$[a_{ij} + b_{ij}]\,x$ is thus the sum of Ax and Bx, which we write $(A + B)\,x$.

Thus $$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

We may extend this rule to define the sum of any finite number of matrices all of the same order. The sum is a matrix of the same order, each of whose elements is the sum of the corresponding elements of the given matrices. The sum does not exist if the matrices are not of the same order.

13·13. Scalar Multiplication

Suppose that $y = Ax$, as in the last section, and that

$$z_1 = ky_1, \quad z_2 = ky_2.$$

We write

$$z = ky = kAx.$$

It is easily seen that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Hence the elements of the matrix kA are obtained by multiplying the corresponding elements of the matrix A by the scalar k. This is known as *scalar multiplication.*

If we take $k = -1$, we have $-[b_{ij}] = [-b_{ij}]$.

Hence $A - B = A + (-B) = [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}]$.

In particular, if $B = A$, we have $A - A$ which we call the *null* (or zero) matrix and denote by O.

$$A - A = [a_{ij} - a_{ij}],$$

so that every element of the null matrix is 0.

There are null matrices of all possible orders $m \times n$.

The null matrix $A - A$ is of the same order as A.

Two matrices are said to be *equal* if and only if they are of the same order and their difference is the null matrix. This implies that every element of one of them is equal to the corresponding element of the other.

13·14. Multiplication

Let us apply two successive transformations to a set of variables x_1, x_2, x_3. We first transform to two new variables y_1, y_2, by means of the equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

As before, we may write these equations

$$y = Ax,$$

where x, y are column vectors of order 3, 2 respectively, and A is a 2×3 matrix.

We now transform to the variables z_1, z_2, by means of the equations

$$\begin{aligned} z_1 &= b_{11}y_1 + b_{12}y_2 \\ z_2 &= b_{21}y_1 + b_{22}y_2. \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

This may be written

$$z = By,$$

where z and y are column vectors of order 2, and B is a 2×2 matrix.

We could transform directly from the variables x_1, x_2, x_3 to the variables z_1, z_2, and, by substituting for y_1, y_2 from Equations (i) into Equations (ii), we obtain the corresponding equations

$$\begin{aligned} z_1 &= b_{11}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3) + b_{12}(a_{21}x_1 + a_{22}x_2 + a_{23}x_3) \\ z_2 &= b_{21}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3) + b_{22}(a_{21}x_1 + a_{22}x_2 + a_{23}x_3). \end{aligned}$$

Simplifying these equations we obtain

$$\begin{aligned} z_1 &= (b_{11}a_{11} + b_{12}a_{21})x_1 + (b_{11}a_{12} + b_{12}a_{22})x_2 + (b_{11}a_{13} + b_{12}a_{23})x_3. \\ z_2 &= (b_{21}a_{11} + b_{22}a_{21})x_1 + (b_{21}a_{12} + b_{22}a_{22})x_2 + (b_{21}a_{13} + b_{22}a_{23})x_3. \end{aligned}$$

If C is the matrix of the transformation, we have

$$C = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \end{bmatrix},$$

and $z = Cx$.

Now $z = By$, and $y = Ax$, so that it is natural to write

$$z = B\,(Ax) = BAx = Cx,$$

and we regard the matrix C as the product of the two matrices B and A.

Thus $C = BA$, in that order.

We transformed first from 3 variables x to 2 variables y, so that A is of order 2×3, and finally from 2 variables y to 2 variables z, so that B is of order 2×2. Thus the number of columns in B (the left-hand factor of the product) is the same as the number of rows in A (the right-hand factor of the product). In exactly the same way we see that, whatever the order of the column vectors x, y, z, where $y = Ax$, $z = By$, the number of columns in B must be the same as the number of rows in A.

Now $z = Cx$, so that the number of columns in C must be equal to the order of x, *i.e.* to the number of columns in A, and the number of rows in C is equal to the order of z, *i.e.* to the number of rows in B.

Thus, in general, if B is of order $m \times n$ and A is of order $n \times p$, the product BA exists and is of order $m \times p$.

We cannot define the product of two matrices if the number of columns in the left-hand factor is not equal to the number of rows in the right-hand factor, and we say that the product does not exist. We now examine the way in which the product is formed. In the above example

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$
$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \end{bmatrix}.$$

We notice that

$$c_{11} = [b_{11} \;\; b_{12}] \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix},$$

so that c_{11} is the inner product of row 1 of B and column 1 of A.

Similarly, $c_{12} = [b_{11} \;\; b_{12}] \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, i.e. c_{12} is the inner product of row 1 of B and column 2 of A.

In general $\qquad c_{ij} = b_{i1}a_{1j} + b_{i2}a_{2j}$

$$= [b_{i1} \;\; b_{i2}] \begin{bmatrix} a_{1j} \\ a_{2j} \end{bmatrix}.$$

This can be extended to matrices of any order. Thus, if $BA = C$, where B is of order $m \times n$, A of order $n \times p$, we define c_{ij} to be the inner product of the i-th row of the left-hand factor B and the j-th column of the right-hand factor A.

Hence $$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Examples.—(1) $$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 1 + -1 \times 2 & 2 \times -2 + -1 \times 3 & 2 \times 4 + -1 \times -1 \\ 3 \times 1 + \quad 4 \times 2 & 3 \times -2 + \quad 4 \times 3 & 3 \times 4 + \quad 4 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -7 & 9 \\ 11 & 6 & 8 \end{bmatrix}.$$

(2) $$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 26 & 7 \\ 8 & 4 \end{bmatrix}.$$

It is important to note that, if $AB = O$, it does not necessarily follow that either $A = O$, or $B = O$.

e.g. $$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

13·15. Non-commutativity of Matrix Multiplication

It is clear that neither of the above products would exist if the order of the factors were reversed. If B is of order $m \times n$ and A is of order $p \times q$, BA exists only if $n = p$, and AB exists only if $m = q$. Thus if both products AB and BA exist, and B is of order $m \times n$, A must be of order $n \times m$. AB is then a square matrix of order $n \times n$ and BA is square of order $m \times m$. Hence AB and BA are not even of the same order unless $m = n$, in which case A and B are both square and of the same order. Then AB and BA both exist and are square and of the same order as A and B. However, the ij-th element of AB is the inner product of the i-th *row* of A and the j-th *column* of B, whereas the ij-th element of BA is the inner product of the i-th *row* of B and the j-th *column* of A.

Hence, $AB \neq BA$ in general, and multiplication of matrices is not commutative.

Example.— $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -4 \\ 3 & 5 \end{bmatrix}$.

$AB = \begin{bmatrix} 5 & 1 \\ 13 & 7 \end{bmatrix}$, $BA = \begin{bmatrix} -6 & -10 \\ 13 & 18 \end{bmatrix}$.

If we denote the ij-th element of AB by $(AB)_{ij}$, then

$$(AB)_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

$$(BA)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

AA, which we write A^2, exists only if $m = n$ and A is square. A^2 is also square and of the same order.

13·16. The Associative Law

If A is of order $m \times n$, B of order $n \times p$, C of order $p \times q$, AB exists and is of order $m \times p$, and

$$(AB)_{ik} = \sum_{h=1}^{n} a_{ih}b_{hk}.$$

It follows that $(AB)C$ exists and is of order $m \times q$, and

$$[(AB)C]_{ij} = \sum_{k=1}^{p} (AB)_{ik}c_{kj}$$

$$= \sum_{k=1}^{p} \sum_{h=1}^{n} a_{ih}b_{hk}c_{kj} \quad \dots\dots\dots \quad \text{(iii)}$$

Similarly, (BC) exists and is of order $n \times q$, and

$$(BC)_{hj} = \sum_{k=1}^{p} b_{hk}c_{kj}.$$

Hence, $A(BC)$ exists and is of order $m \times q$, and

$$[A(BC)]_{ij} = \sum_{h=1}^{n} a_{ih}(BC)_{hj}$$

$$= \sum_{h=1}^{n} \sum_{k=1}^{p} a_{ih}b_{hk}c_{kj} \quad \dots\dots\dots\dots \text{(iv)}$$

But the double sum in (iv) is obtained from (iii) by interchanging the order of summation. This we may do, since we may rearrange the order of terms in any finite series without altering the sum.

Thus $$(AB)C = A(BC).$$

It follows that it is immaterial whether we first find AB and then multiply on the right by C, or first find BC and then multiply on the left by A. The order of the factors must, of course, remain unchanged. We may, without ambiguity, write the product ABC.

If A and B are of the same order $n \times p$, then

$$(A+B)_{ij} = a_{ij} + b_{ij}.$$

If C is of order $p \times q$,

$$[(A+B)C]_{ij} = \sum_{k=1}^{p} (a_{ik} + b_{ik}) c_{kj}$$

$$= \sum_{k=1}^{p} a_{ik}c_{kj} + \sum_{k=1}^{p} b_{ik}c_{kj}$$

$$= (AC)_{ij} + (BC)_{ij}.$$

Hence $$(A+B)C = AC + BC.$$

Similarly, if E is of order $m \times n$,

$$E(A+B) = EA + EB.$$

13·17. The Unit Matrix

In order to build up an algebra of matrices we need an element which will take the place of the number unity. We seek a matrix I which is such that

$$Ix = x \text{ for every } x.$$

If x is of order 3, then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This matrix is square and has unit elements in the *principal diagonal*, *i.e.* the diagonal running from the top left-hand to the bottom right-hand corner, and zero elements elsewhere. A square matrix whose only non-zero elements are in the principal diagonal is called a *diagonal matrix*. The diagonal matrix of order n, all

of whose diagonal elements are equal to unity, is called the ***unit matrix I*** of order n. If it is necessary to specify the order we write I_n. We usually denote the elements of the unit matrix by δ_{ij} (known as the Kronecker delta). Thus I is given by

$$\delta_{ij} = 0\ (i \neq j),\ \delta_{ii} = 1.$$

Let A be a matrix of order $m \times n$.

Then $$I_m A = A.$$

For $$(I_m A)_{ij} = \sum_{k=1}^{m} \delta_{ik} a_{kj} = \delta_{ii} a_{ij} = a_{ij}.$$

Similarly, $$AI_n = A.$$

It should be noted that for multiplication on the left we need a unit matrix of order $m \times m$, whereas on the right we need a unit matrix of order $n \times n$.

If I is of any order, the reader will easily verify that

$$I^2 = I,\ I^3 = I,\ \ldots,\ I^p = I,$$

where p is any positive integer.

13·18. Scalar Matrices

A diagonal matrix all of whose diagonal elements are equal to a number λ is called a *scalar matrix.*

The scalar matrix of order 3 is

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda I_3.$$

Let A be of order $m \times n$.

Then $$\lambda I_m . A = \lambda A, \text{ and } A.\lambda I_n = \lambda A.$$

Thus multiplication of a matrix by a scalar matrix is equivalent to multiplication by a scalar, as already defined.

Examples.—(1) *Prove that the product of two diagonal matrices of the same order is a diagonal matrix.*

(2) *Prove that the product of two scalar matrices of the same order is a scalar matrix.*

(3) *If* $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$, *verify that*

$$(I + A)^2 = I + 2A + A^2,$$
$$(A + B)^2 = A^2 + AB + BA + B^2,$$
$$(A + B)(A - B) = A^2 + BA - AB - B^2.$$

(4) *If A is of order 3×3 and B is the diagonal matrix* $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, *show that AB is obtained by multiplying the columns of A by $\lambda_1, \lambda_2, \lambda_3$ respectively, and BA is obtained by multiplying the rows of A by $\lambda_1, \lambda_2, \lambda_3$ respectively.*

(5) *If* $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$, *prove that*

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ but } BA = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}.$$

13·21. The Determinant of a Square Matrix

If A is a square matrix, the determinant whose elements are exactly the same as those of A, and in the same positions, is called the determinant of A, and is written $|A|$ (occasionally det. A is used). Clearly a matrix which is not square has no determinant. Now the rule for the multiplication of two determinants of the same order is exactly the same as the rule for the multiplication of matrices. Hence, if A and B are both square and of the same order $n \times n$, then

$$|AB| = |A| \times |B|, \text{ and } |BA| = |B| \times |A|,$$

so that $|AB| = |BA|$, although $AB \neq BA$ in general.

13·22. Transposition

The matrix obtained from A by interchanging its rows and its columns is called the *transpose* of A and is denoted by A'. If A is of order $m \times n$, A' is therefore of order $n \times m$, and $a'_{ij} = a_{ji}$.

In particular, the transpose of a column vector is a row

x_n

vector $[x_1, x_2, \ldots, x_n]$, and vice versa. For convenience in printing we write a column vector horizontally as follows:

$$\{x_1, x_2, \ldots, x_n\}.$$

The single letter x is used to denote this column vector, and x' is then used to denote the corresponding row vector. If A is square, A' is also square, and since the value of a determinant is unaltered by interchanging its rows and its columns, it follows that $|A| = |A'|$.

13·23. The Transpose of a Product

Let $AB = C$, where A and B are of orders $m \times n$, $n \times p$ respectively. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Now $$c'_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} a'_{kj}b'_{ik}$$

$$= \sum_{k=1}^{n} b'_{ik}a'_{kj} = (B'A')_{ij},$$

i.e. $$C' = B'A', \text{ or } (AB)' = B'A'.$$

Thus the transpose of the product of two matrices is equal to the product of their transposes, taken in the reverse order. B', A' are of orders $p \times n$, $n \times m$ respectively, so that $B'A'$ is of order $p \times m$.

13·24. Singular Matrices

If A is square and $|A| = 0$, A is said to be a *singular* matrix. If $|A| \neq 0$, A is said to be *non-singular*.

If $y = Ax$ represents a linear transformation from n variables x to n variables y, $|A|$ is called the *modulus* of the transformation. The transformation is said to be singular if its modulus is zero, and non-singular otherwise.

13·25. The Reciprocal of a Matrix

We have defined the sum, difference, and product of two matrices. It now remains to consider the quotient, which may be regarded as the product of one matrix and the reciprocal of the other. Given a matrix A, we require to find a matrix A^{-1} such that $AA^{-1} = I$, or $A^{-1}A = I$.

Suppose that A is square of order $n \times n$, so that it has a determinant $|A|$. Let A_{ij} be the cofactor of the element a_{ij} in this determinant. Then we know that, if $i \leqslant n$, $j \leqslant n$,

$$\sum_{k=1}^{n} a_{ik}A_{ik} = |A|,$$

and $$\sum_{k=1}^{n} a_{ik}A_{jk} = 0, \quad (i \neq j).$$

Let us now form a new matrix by replacing each a_{ij} in A by A_{ij}, and then transposing. This matrix is called the *adjoint* of A, written adj. A, and is given by

$$(\text{adj. } A)_{ij} = A_{ji}.$$

This is also square and of the same order as A, so that $A.\text{adj. } A$ and adj. $A.A$ both exist.

Now if
$$i \neq j,$$

$$(A.\text{adj. } A)_{ij} = \sum_{k=1}^{n} a_{ik}\,(\text{adj. } A)_{kj} = \sum_{k=1}^{n} a_{ik}A_{jk} = 0,$$

and
$$(A.\text{adj. } A)_{ii} = \sum_{k=1}^{n} a_{ik}\,(\text{adj. } A)_{ki} = \sum_{k=1}^{n} a_{ik}A_{ik} = |\,A\,|.$$

$A.\text{adj. } A$ is therefore a scalar matrix, all of whose diagonal elements are equal to $|\,A\,|$.

Hence
$$A.\text{adj. } A = |\,A\,|\,I.$$

In exactly the same way, using the fact that, if $i \leqslant n$, $j \leqslant n$,

$$\sum_{k=1}^{n} a_{ki}A_{ki} = |\,A\,|,$$

$$\sum_{k=1}^{n} a_{ki}A_{kj} = 0\ (i \neq j),$$

we can show that
$$\text{adj. } A.A = |\,A\,|\,I.$$

If A is non-singular, we may multiply by the scalar $1/|\,A\,|$, and obtain

$$A.\frac{\text{adj. } A}{|\,A\,|} = \frac{\text{adj. } A}{|\,A\,|}.A = I.$$

Hence
$$A^{-1} = \frac{\text{adj. } A}{|\,A\,|}, \text{ and } \overset{-1}{a_{ij}} = \frac{A_{ji}}{|\,A\,|}.$$

We have therefore established a rule for finding the reciprocal A^{-1} of a non-singular square matrix A.

We first replace each element of A by its cofactor in $|\,A\,|$, we next transpose this matrix, and we finally divide each element by $|\,A\,|$.

Example.—*Let*
$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 5 & 1 & -1 \end{bmatrix}; \quad |A| = 2.$$

Replacing each element in A by A_{ij} we obtain the matrix
$$\begin{bmatrix} -2 & 6 & -4 \\ 3 & -8 & 7 \\ 1 & -2 & 1 \end{bmatrix}$$
and transposing this we have
$$\text{adj. } A = \begin{bmatrix} -2 & 3 & 1 \\ 6 & -8 & -2 \\ -4 & 7 & 1 \end{bmatrix}.$$

Dividing by $|A| = 2$, we have
$$A^{-1} = \begin{bmatrix} -1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -4 & -1 \\ -2 & \frac{7}{2} & \frac{1}{2} \end{bmatrix}.$$

The reader will easily verify that $AA^{-1} = A^{-1}A = I$.

13·26. Solution of Linear Equations

Using the reciprocal of a matrix, we can now solve a set of simultaneous linear equations. We shall illustrate the method by an example.

Consider the three equations
$$\begin{aligned} x + y + 2z &= 4 \\ 2x - y + 3z &= 9 \\ 3x - y - z &= 2 \end{aligned}$$
in the three unknowns x, y, z.

We may write these equations $Ax = h$,

where $$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}, \; x = \{x, y, z\}, \;\; h = \{4, 9, 2\}.$$

Now $$|A| = 17 \text{ and adj. } A = \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix},$$

so that $$A^{-1} = \frac{1}{17} \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

If we multiply each side of the equation on the left by A^{-1}, we obtain
$$A^{-1}.Ax = A^{-1}h.$$

But $A^{-1}.Ax = A^{-1}A.x = Ix = x$, so that

$$x = A^{-1}h = \tfrac{1}{17}\begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix} = \tfrac{1}{17}\begin{bmatrix} 17 \\ -17 \\ 34 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Hence $x = 1$, $y = -1$, $z = 2$ is the solution of the equations.

Examples.—

(1) *If* $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -3 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 2 \\ -3 & -1 & 1 \\ 2 & 5 & 0 \end{bmatrix}$,
evaluate $A^{-1}B$ *and* $B^{-1}A$, *and show that each of these products is the reciprocal of the other.*

(2) *Prove that the reciprocal of a non-singular diagonal matrix is a non-singular diagonal matrix.*

13·27. Uniqueness of the Reciprocal

We have found a reciprocal $A^{-1} = \dfrac{\text{adj.}\,A}{|A|}$ of the square matrix A, and we now show that this is the only reciprocal. Suppose B is another reciprocal, so that $AB = I$.

Then $B - A^{-1} = I(B - A^{-1}) = A^{-1}A(B - A^{-1})$

$$= A^{-1}(AB - AA^{-1}) = A^{-1}(I - I) = O.$$

Hence $\qquad B = A^{-1}.$

Thus A^{-1} is the only right-hand reciprocal of A, and it can be shown similarly that it is the only left-hand reciprocal. We showed earlier that, in matrix algebra, $AB = O$ does not necessarily imply that either $A = O$ or $B = O$.

We can, however, assert that if A is square and non-singular, and $AB = O$, then $B = O$.

For, if A is non-singular, A^{-1} exists, and we have

$$A^{-1}(AB) = (A^{-1}A)B = IB = B,$$

and $\qquad A^{-1}(AB) = A^{-1}O = O,$

so that $\qquad B = O.$

Similarly, if B is square and non-singular and $AB = O$, then $A = O$.

13·28. The Reciprocal of the Product of Two Matrices

If A and B are non-singular square matrices of the same order, A^{-1} and B^{-1} exist, and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Hence $B^{-1}A^{-1}$ is the unique reciprocal of AB; i.e.

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Similarly $\quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, etc.

Thus the reciprocal of the product of any number of non-singular matrices is the product of their reciprocals in the reverse order.

13·29. The Reciprocal of a Transpose

Let A be square and non-singular.

Then $\quad a^{-1}_{ij} = \dfrac{A_{ji}}{|A|}$, and $(a^{-1})'_{ij} = \dfrac{A_{ij}}{|A|}$.

But $\quad (a')^{-1}_{ij} = \dfrac{A'_{ji}}{|A'|} = \dfrac{A_{ij}}{|A|}$.

Hence $\quad (A^{-1})' = (A')^{-1}$.

The reciprocal of the transpose of a matrix is therefore equal to the transpose of its reciprocal.

13·31. Some Special Types of Matrices

If $A' = A$, we say that A is *symmetric*. This implies that A is square, and $a_{ij} = a_{ji}$. Thus the matrix is symmetrical about its principal diagonal.

If $A' = -A$, we say that A is *skew symmetric*. A is again square, and $a_{ij} = -a_{ji}$. In particular, $a_{ii} = -a_{ii}$, so that $a_{ii} = 0$ for every i. Thus the principal diagonal contains only zero elements.

The elements of a matrix are not necessarily real numbers. The matrix obtained from A by replacing each element by its complex conjugate $\bar{a}_{ij}$ is called the *conjugate* of A, and denoted by $\bar{A}$. If $\bar{A}' = A$, the matrix A is said to be *Hermitian*. Hermitian matrices are of importance in some branches of physics.

If A is real and Hermitian, then $A = \bar{A}' = A'$, so that a real Hermitian matrix is symmetric.

If $\bar{A}' = -A$, then A is said to be *skew Hermitian*. A real skew Hermitian matrix is skew symmetric.

It is easily seen that $\bar{\bar{A}} = A$, and $(\bar{A})' = (\bar{A'})$.

Examples.—(1) *If A is any matrix (not necessarily square), AA' is symmetric, and $A\bar{A}'$ is Hermitian.*

(2) *If x is any column vector of order n, $x'x$ is a scalar, but xx' is a symmetric $n \times n$ matrix.*

(3) *If A is Hermitian, iA is skew Hermitian ($i^2 = -1$).*

(4) *If A is any square matrix, $A' + A$ is symmetric, and $A' - A$ is skew symmetric.*

13·32. Orthogonal and Unitary Matrices

Suppose that x_1, x_2, x_3 are the rectangular Cartesian coordinates of a point P in space. We may regard these as the elements of a vector x of order 3. If O is the origin,

$$OP^2 = x_1^2 + x_2^2 + x_3^2 = x'x.$$

Now suppose that the origin remains fixed, but the axes are rotated about it in such a way that they remain rectangular. If y_1, y_2, y_3 are the coordinates of P referred to the new system, it is well known that

$$\begin{aligned} y_1 &= l_1x_1 + m_1x_2 + n_1x_3 \\ y_2 &= l_2x_1 + m_2x_2 + n_2x_3 \\ y_3 &= l_3x_1 + m_3x_2 + n_3x_3, \end{aligned}$$

where (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are the direction cosines of the new axes referred to the old.

Thus the vectors x and y are connected by a relationship of the form $y = Ax$, where A is a 3×3 matrix.

Now $\quad OP^2 = y_1^2 + y_2^2 + y_3^2 = y'y = (Ax)'(Ax)$
$\quad\quad = (x'A')(Ax) = x'(A'A)x.$

But $\quad OP^2 = x'x$, so that $A'A = I$.

Writing this in the form

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we see that it is equivalent to the six familiar relations

$$l_1^2 + l_2^2 + l_3^2 = 1, \quad l_1m_1 + l_2m_2 + l_3m_3 = 0, \text{ etc.}$$

In two dimensions, a rotation of axes through an angle θ is equivalent to the transformation

$$\begin{aligned} y_1 &= x_1 \cos\theta + x_2 \sin\theta \\ y_2 &= -x_1 \sin\theta + x_2 \cos\theta, \end{aligned}$$

so that in this case

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

The reader will easily verify that $A'A = I$.

We use the same geometrical language to describe the corresponding algebraic process when the order of x exceeds 3. We regard the elements of the real column vector $x = \{x_1, x_2, \ldots, x_n\}$ as the rectangular Cartesian coordinates of a point in a space of n dimensions, and $x'x = x_1^2 + x_2^2 + \ldots + x_n^2$ as the square of the distance of this point from the origin. If we now refer the point to a new set of rectangular axes with the same origin, and if $y = Ax$ is the equation of the transformation, then $y'y = x'x$, and, as in the 3-dimensional case, we have $A'A = I$. A square matrix A for which $A'A = I$ is called an *orthogonal* matrix. Hence $A' = A^{-1}$, and therefore $AA' = I$ also.

We may generalise this process when the elements of the column vector $x = \{x_1, x_2, \ldots, x_n\}$ are not necessarily real. We then define the length of the vector x to be

$$\bar{x}'x = |x_1^2| + |x_2^2| + \ldots + |x_n^2|.$$

If we now apply the transformation $y = Ax$, we have

$$\bar{y}'y = (\overline{Ax})'.(Ax) = \bar{x}'\bar{A}'Ax.$$

Thus $\bar{y}'y = \bar{x}'x$ if, and only if, $\bar{A}'A = I$. A square matrix A for which $\bar{A}'A = I$ is called a *unitary* matrix. $\bar{A}' = A^{-1}$, so that $A\bar{A}' = I$ also.

13·33. Elementary Transformations

Consider the product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{bmatrix}.$$

The second matrix on the left is obtained by interchanging the second and third columns of the unit matrix, and the effect of multiplying a matrix A by this matrix on the right is to interchange the second and third columns of A. The modified unit matrix is also obtained by interchanging the second and third rows of the unit matrix, and it is easily seen that the effect of multiplying A by this matrix on the left is to interchange the second and third rows of A. We can generalise this result.

Let $A_{(pq)}$ be the matrix obtained by interchanging the p-th and q-th columns of A. Then it is easily seen that

$$A_{(pq)} = A.I_{(pq)}.$$

We note that $I'_{(pq)} = I_{(pq)}$, and $I_{(pq)}\,I_{(pq)} = I^2_{(pq)} = I$.

Similarly, $A'_{(pq)} = A'I_{(pq)}$ and transposing this equation gives

$$(A'_{(pq)})' = I'_{(pq)}A = I_{(pq)}A.$$

The matrix on the left is obtained by interchanging the p-th and q-th rows of A.

Now suppose that we rearrange the order of the columns of A in any way whatever. We can suppose this done in a number of steps, each step consisting of the interchange of two columns. Each interchange is achieved by post-multiplying A by a matrix of type $I_{(pq)}$, so that the whole rearrangement can be effected by postmultiplying A by a matrix Q, which is the product of matrices of type $I_{(pq)}$.

Now $|I_{(pq)}| \neq 0$ (it is, in fact, either $+1$ or -1).

Hence $|Q| \neq 0$; i.e. Q is non-singular.

In the same way we can rearrange the order of the rows of A in any way whatever by pre-multiplying by a suitable non-singular matrix P. Combining these two results, if B is a matrix obtained from A by rearranging its rows and also its columns in any way,

then $\qquad B = PAQ$, where $|P| \neq 0$, $|Q| \neq 0$.

We note that if A is of order $m \times n$, P is of order $m \times m$, Q of order $n \times n$, and B of order $m \times n$.

Now consider the product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & \lambda a_{11} + a_{12} & a_{13} \\ a_{21} & \lambda a_{21} + a_{22} & a_{23} \\ a_{31} & \lambda a_{31} + a_{32} & a_{33} \end{bmatrix}$$

The second matrix on the left is obtained by adding to the unit matrix a single element λ in its 1st row and 2nd column. Post-multiplication of A by this matrix has the effect of adding λ times its first column to its second column. Pre-multiplication of A by this matrix would have the effect of adding λ times its second row to its first. The modified unit matrix has determinant 1.

This result can be generalised as in the last example, and we see that adding to any column of A a linear combination of any of

the other columns of A is equivalent to post-multiplying by a non-singular matrix Q.

$|Q| = 1$, since Q is the product of square matrices each of determinant 1.

A similar result holds for rows, the transformation being equivalent to pre-multiplication by a non-singular matrix P whose determinant is 1.

Finally, consider the product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \mu a_{12} & a_{13} \\ a_{21} & \mu a_{22} & a_{23} \\ a_{31} & \mu a_{32} & a_{33} \end{bmatrix}.$$

The second matrix on the left is obtained by replacing the second unit in the principal diagonal of the unit matrix by $\mu \neq 0$. The determinant of this matrix is μ, so that the matrix is non-singular. The effect of post-multiplying A by this matrix is to multiply the second column by μ. The effect of pre-multiplying is to multiply the second row by μ. The result can be generalised as in the last two examples, and we see that, if B is obtained from A by multiplying its j_1-th, j_2-th, ... columns by the scalars $\mu_1, \mu_2, \ldots$ and its i_1-th, i_2-th, ... rows by the scalars $\lambda_1, \lambda_2, \ldots$ respectively, then

$$B = PAQ, \text{ where } |P| \neq 0, \ |Q| \neq 0.$$

The following three types of transformation of a matrix are known as *elementary transformations*:—

(1) *Interchange of any two columns (or rows),*

(2) *Addition to any column (or row) of a multiple of any other column (or row),*

(3) *Multiplication of any column (or row) by a scalar.*

Any transformation of a matrix which could be carried out by a succession of elementary transformations is known as a *chain of elementary transformations*. Combining the above results, it is clear that a chain of elementary transformations of a matrix A is equivalent to a transformation of the type

$$PAQ, \text{ where } |P| \neq 0 \text{ and } |Q| \neq 0.$$

13·41. The Minors of a Matrix

Let A be any matrix of order $m \times n$, and let r be any integer less than or equal to the lesser of the two integers m and n. If we now delete from A all elements except those belonging to a

particular set of r columns, and also to a particular set of r rows, we are left with a square $r \times r$ matrix. The determinant of this matrix is called a *minor of order* r of A. For example, the 3×3 matrix A has 1 minor of order 3 (viz. the determinant of A itself), 9 minors of order 2, and 9 minors of order 1.

Suppose that $r < m$ and $r < n$, and that every minor of A of order r vanishes. Now any minor of order $r + 1$ can be expanded by, say, its first row, and the cofactors of this row are all minors of order r, and hence all zero. Thus every minor of order $r + 1$ also vanishes. By continuing this argument we see that, if all minors of A of order r vanish, then all minors of higher order also vanish.

Any matrix formed from a given matrix A by deleting some of its rows or columns, or both, is called a *submatrix* of A. Thus every minor of A is the determinant of a submatrix of A. The submatrix consisting of elements of the first r rows and the first r columns of A is called the *leading submatrix* of order r. Its determinant is called the *leading minor* of order r.

We now define a concept which is of importance in the solution of sets of linear equations.

13·42. The Rank of a Matrix

If A is a matrix of order $m \times n$, all of whose minors of order $r + 1$ vanish, whilst at least one minor of order r does not vanish, A is said to be of *rank* r.

Clearly $$r \leqslant m \text{ and } r \leqslant n.$$

The null matrix has all its elements (*i.e.* minors of order 1) zero, and we regard it as having rank 0. All other matrices have at least one non-zero element, and are therefore of rank $\geqslant 1$. If A is square of order $n \times n$, and if $|A| \neq 0$ (.i.e A is non-singular), A is of rank n. If A is singular, $|A| = 0$, and $r < n$.

Examples.—(1) *A null vector is of rank* 0; *a vector which is not null is of rank* 1.

(2) *The rank of a diagonal matrix is equal to the number of its non-zero elements.*

(3) *P is the matrix* $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. *Prove that P is of rank* 3, *and that P^2, P^3, P^4 are of rank* 2, 1, 0 *respectively.*

13·43. The Rank of a Matrix is Unaltered by Elementary Transformations

A determinant is unaltered, except possibly in sign, if we interchange two rows (or columns). Hence, if a matrix is subjected to an elementary transformation of type (1) (see § 13·33), its minors of any order are the same as those of the same order of the original matrix, except that some may have different signs. In particular, if all minors of a given order vanish, all minors of this order of the transformed matrix also vanish. If at least one minor of a given order does not vanish, then at least one minor of the same order of the transformed matrix does not vanish. Thus the rank remains the same.

In the same way we see that an elementary transformation of type (2) leaves all minors unchanged, and that one of type (3) leaves some minors unchanged, and multiplies the remaining minors by a non-zero number.

Hence the rank of a matrix is unaltered by elementary transformations.

13·44. Solution of a Set of Homogeneous Linear Equations

If A is an $m \times n$ matrix and x a column vector of order n, the matrix equation $Ax = O$ is equivalent to a set of m homogeneous linear equations in the n unknowns $x_1, x_2, \ldots, x_n$. $x = O$, i.e. $x_1 = x_2 = \ldots = x_n = 0$, clearly satisfies the equations, whatever the matrix A. This solution is called *trivial*. We seek *non-trivial* (*i.e.* non-zero) solutions of these equations, when they exist. The solution will not be unique. For, suppose that $x_1, x_2, \ldots, x_n$ satisfy the equations. Then, since the equations are homogeneous, $\lambda x_1, \lambda x_2, \ldots, \lambda x_n$ also satisfy them, for all values of λ. In matrix notation, if x is a solution, so is λx for all values of the scalar λ. We now prove the following important theorem.

THEOREM I.—If A is a matrix of order $m \times n$ and rank r, and x a column vector of order n, the equations $Ax = O$ have a non-trivial solution if, and only if, $r < n$.

(1) **Necessity.**—Suppose that a non-trivial solution x exists. Let us suppose that $r = n$. Since $r \leq m$, this implies that $m \geq n$.

There is at least one non-zero minor of A of order n. We may therefore suppose that the order of the equations is arranged in such a way that the submatrix A_n of A, consisting of its first n

rows, is non-singular. The first n equations may then be written

$$A_n x = O, \text{ where } |A_n| \neq 0.$$

A_n^{-1} exists, and we have

$$O = A_n^{-1}(A_n x) = (A_n^{-1}A_n)\,x = Ix = x.$$

But $x \neq 0$, by hypothesis.

This contradiction proves that $r \neq n$.

Hence, since $r \leqslant n$, it follows that $r < n$.

(2) **Sufficiency.**—Suppose that $r < n$.

A has at least one non-zero minor of order r, and we may suppose the order of the equations and the order of the unknowns to be such that the leading submatrix A_r of order r is non-singular. A_{r+1}, the leading submatrix of order $(r + 1) \times (r + 1)$, is, however, singular.

Let $x_1, x_2, \ldots, x_{r+1}$ be the cofactors of the elements in the last row of the determinant of A_{r+1}, and let the remaining elements of x be zeros. Now $x_{r+1} = \pm |A_r| \neq 0$, so that x is not null.

The inner product of any of the first r row vectors of A_{r+1} and the column vector $\{x_1, x_2, \ldots, x_{r+1}\}$ is zero, since it is the sum of the products of elements of a row of a determinant and the cofactors of another row.

By the definition of $x_1, x_2, \ldots, x_{r+1}$, the inner product of the last row vector of A_{r+1} with $\{x_1, x_2, \ldots, x_{r+1}\}$ is $|A_{r+1}| = 0$.

Thus $$A_{r+1}\{x_1, x_2, \ldots, x_{r+1}\} = O.$$

Since every minor of order $r + 1$ is zero, the minor formed by elements from the first $r + 1$ columns of $\underline{A}$, and the first r rows and any one other row, vanishes.

But $x_1, x_2, \ldots, x_{r+1}$ are also the cofactors of the last row of this minor, so that the inner product of this last row and the vector $\{x_1, x_2, \ldots, x_{r+1}\}$ is zero.

We have thus shown that the inner product of the vector formed by the first $r + 1$ elements of any row of A and the vector $\{x_1, x_2, \ldots, x_{r+1}\}$ is zero.

Since the remaining elements of x are zeros, it follows that the inner product of any row vector of A and x is zero. Thus $Ax = O$, and we have found a non-zero solution of the equations.

The theorem is now proved.

If the number of equations m is equal to the number of unknowns n, the condition $r < n$ reduces to the familiar form $|A| = 0$.

If there are fewer equations than unknowns, *i.e.* if $m < n$, then r is certainly less than n, and the equations always have a non-trivial solution.

If $m > n$, the condition is satisfied only if every matrix formed from n rows of the matrix A is singular.

13·51. Linear Dependence

Let $x_1, x_2, \ldots, x_n$ be a set of vectors all of the same order. If they satisfy an equation of the form

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = O \quad \ldots\ldots\ldots\ldots \quad \text{(v)}$$

where the scalars $c_1, c_2, \ldots, c_n$ are not all zero, then the vectors are said to be *linearly dependent*. Note that the zero on the right of the equation is the null vector of the same order. If, on the other hand, no such relation exists, *i.e.* if the Equation (v) is satisfied only if $c_1 = c_2 = \ldots = c_n = 0$, the vectors are said to be *linearly independent*.

Let $A_1, A_2, \ldots, A_n$ be the column vectors of the $m \times n$ matrix A. These columns are linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1A_1 + c_2A_2 + \ldots + c_nA_n = O.$$

Since each vector is of order m, this is equivalent to m equations in the n unknowns $c_1, c_2, \ldots, c_n$. These equations may be written $Ac = O$, where $c = \{c_1, c_2, \ldots, c_n\}$. Thus the columns of A are linearly independent if, and only if, the equations $Ac = O$ have a non-trivial solution.

In the particular case when A is square, its columns are linearly dependent if, and only if, $|A| = 0$.

Now if $|A| = 0$, $|A'| = |A| = 0$, so that the columns of A', *i.e.* the rows of A, are also linearly dependent.

If $|A| \neq 0$, then the columns of A, and also the rows of A, are linearly independent.

We now show that if A is any $m \times n$ matrix of rank r, it contains r, and not more than r, linearly independent columns. The same is true of its rows.

A contains at least one non-vanishing minor, $|A_r|$, of order r, and the columns of A_r are linearly independent. The r columns of A which contain A_r are therefore also linearly independent. On

the other hand, every minor of order $r + 1$ vanishes, so that no set of $r + 1$ columns of A can be linearly independent. Similarly, the r rows of A which contain A_r are linearly independent, but no set of $r + 1$ rows is linearly independent.

We could therefore define the rank of a matrix A to be the maximum number of linearly independent rows (or columns) of A. This definition has been shown to be equivalent to our previous one.

13·52. Equivalent Matrices

Two matrices are said to be *equivalent* if it is possible to pass from one to the other by a chain of elementary transformations. If A and B are equivalent we write $A \simeq B$. By § 13·43, A and B have the same rank.

To determine the rank of a square matrix A we replace it by an equivalent matrix B, all of whose elements above the leading diagonal are zeros. $|B|$ is then the product of the leading diagonal elements, and we can determine the rank by inspection.

Example.—*Find the rank of* $A = \begin{bmatrix} 3 & 4 & -6 \\ 2 & -1 & 7 \\ 1 & -2 & 8 \end{bmatrix}$.

$$A \simeq \begin{bmatrix} 1 & -2 & 8 \\ 2 & -1 & 7 \\ 3 & 4 & -6 \end{bmatrix} \quad \text{(by interchanging 1st and 3rd rows)}$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -9 \\ 3 & 10 & -30 \end{bmatrix} \quad \begin{array}{l}\text{(column 2 + 2} \times \text{column 1;} \\ \text{column 3} - 8 \times \text{column 1)}\end{array}$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 10 & 0 \end{bmatrix} = B \quad \text{(column 3 + 3} \times \text{column 2).}$$

$|B| = 1 \times 3 \times 0 = 0$, so that B is of rank less than 3.

The leading minor of order 2 is $\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3 \neq 0$.

Hence the rank of B, and so also of A, is 2.

The object of interchanging the first and third rows of A was to bring the element 1 into the leading position. If there is no unit element, we begin by bringing the simplest element into this position. This simplifies the subsequent arithmetic.

The method can be modified to find the rank of a rectangular matrix.

Example.—*Find the rank of* $A = \begin{bmatrix} 3 & 11 & 1 & 5 \\ 5 & 13 & -1 & 11 \\ -2 & 2 & 4 & -8 \end{bmatrix}$.

$$A \cong \begin{bmatrix} 1 & 11 & 3 & 5 \\ -1 & 13 & 5 & 11 \\ 4 & 2 & -2 & -8 \end{bmatrix} \quad \text{(interchange 1st and 3rd columns)}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 24 & 8 & 16 \\ 4 & -42 & -14 & -28 \end{bmatrix} \quad \text{(from columns 2, 3, 4 subtract 11, 3, 5 times column 1 respectively)}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 4 & 4 \\ 4 & -7 & -7 & -7 \end{bmatrix} \quad \text{(divide columns 2, 3, 4 by 6, 2, 4 respectively)}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 4 & -7 & 0 & 0 \end{bmatrix} \quad \text{(subtract column 2 from columns 3, 4).}$$

Every minor of order 3 has a zero column and so vanishes.

The leading minor of order 2 is $\begin{matrix} 1 \\ 1 \end{matrix} \quad 4 \neq 0$.

Hence A is of rank 2.

13·53. Vector Spaces

We next show that, if $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are n linearly independent vectors of order n, we can express any other vector of order n as a linear combination of these n-vectors. Since the column vectors of any non-singular $n \times n$ matrix are linearly independent, we can certainly find n linearly independent n-vectors, and there is an unlimited number of such sets. Consider, in particular, the columns of the unit matrix I_n. Let these be $e^{(1)}$, $e^{(2)}$ $e^{(n)}$.

Thus
$$e^{(1)} = \{1, 0, 0, \ldots, 0\}$$
$$e^{(2)} = \{0, 1, 0, \ldots, 0\}$$
$$e^{(n)} = \{0, 0, 0, \ldots, 1\}.$$

These form one set of n linearly independent n-vectors, since $|I_n| \neq 0$, and we can express any n-vector $x = \{x_1, x_2, \ldots, x_n\}$ as a linear combination of $e^{(1)}, e^{(2)}, \ldots, e^{(n)}$.

Clearly $\quad x = x_1 e^{(1)} + x_2 e^{(2)} + \ldots + x_n e^{(n)}$.

We now prove this result in the general case.

Let A be the matrix whose column vectors are the n linearly independent n-vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$.

Then A is square and non-singular, so that A^{-1} exists.

Let $y = \{y_1, y_2, \ldots, y_n\}$ be any n-vector, and consider the matrix equation $Ac = y$, where $c = \{c_1, c_2, \ldots, c_n\}$.

Multiplying each side by A^{-1} on the left, we have

$$A^{-1}y = A^{-1}(Ac) = (A^{-1}A)\,c = c.$$

If c is null, this would imply that $y = Ac$ is null.

Hence, if y is not null, there exists a non-null vector $c = A^{-1}y$, such that $Ac = y$.

But this is equivalent to the relation

$$c_1x^{(1)} + c_2x^{(2)} + \ldots + c_nx^{(n)} = y,$$

where $c_1, c_2, \ldots, c_n$ are not all zero.

Consequently, if y is not null it can be expressed as a linear combination of $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$.

Let us consider what this implies if the vectors are of order 2. If $x = \{x_1, x_2\}$, we may regard x_1, x_2 as the Cartesian coordinates (not necessarily rectangular) of a point P in a plane, and x as representing the vector $\overrightarrow{OP}$ (O being the origin) with components of magnitude x_1, x_2 respectively in the directions of the axes. Let $y = \{y_1, y_2\}$ represent another point Q. If x and y are linearly dependent, this implies that $c_1x + c_2y = O$, where c_1 and c_2 are not both zero. If $c_2 = 0$, then $c_1x = 0$, so that $c_1 = 0$ also if $x \neq O$, contrary to hypothesis. Hence $c_2 \neq 0$, and we may write

$$y = -c_1/c_2 . x, \text{ or } y = \lambda x, \text{ where } \lambda \neq 0.$$

This implies that $y_1 = \lambda x_1$, $y_2 = \lambda x_2$, so that $y_1/y_2 = x_1/x_2$. Hence the lines OP and OQ coincide.

Thus two linearly dependent 2-vectors have the same direction in a plane. Conversely, if x and y represent the same direction, $y = \lambda x$ and x and y are linearly dependent. Linearly independent 2-vectors, on the other hand, represent different directions in a plane.

We have shown that, if $x^{(1)}$ and $x^{(2)}$ are linearly independent 2-vectors, then any 2-vector y can be expressed in the form

$$y = c_1x^{(1)} + c_2x^{(2)}.$$

This is equivalent to the well-known result that any vector in a plane can be decomposed into two components, one in each of two given directions.

The totality of all vectors $x = \{x_1, x_2\}$ obtained by giving x_1 and x_2 all possible values may be regarded as corresponding to the whole plane, or two-dimensional space.

The same argument may be used for three dimensions. Thus the totality of all vectors $x = \{x_1, x_2, x_3\}$ corresponds to the whole three-dimensional space.

If $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are three linearly independent 3-vectors, (*i.e.* in three different directions), any 3-vector y can be expressed in the form $y = c_1x^{(1)} + c_2x^{(2)} + c_3x^{(3)}$.

We may therefore also say that the totality of vectors of the form $c_1x^{(1)} + c_2x^{(2)} + c_3x^{(3)}$, where $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are linearly independent, is the whole three-dimensional space. We use the same geometrical language for vectors of higher order. We say that the totality of all n-vectors form the *vector space* R_n of n dimensions. Alternatively, we may regard R_n as consisting of all vectors $c_1x^{(1)} + c_2x^{(2)} + \ldots + c_nx^{(n)}$, where $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are n linearly independent n-vectors. We say that the space R_n is *spanned* by the vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$, or that these vectors form a *basis* for R_n. The basis is, of course, not unique.

13·54. Vector Subspaces in R_n

Suppose that L is a set of n-vectors containing at least one member, and suppose that:—

(1) If x belongs to L, then λx also belongs to L, where λ is any real scalar,

(2) If x and y belong to L, then $x + y$ also belongs to L.

Then L is called a *vector subspace* of R_n.

As a special case it may, of course, be the whole space R_n. For example, R_n is spanned by the unit vectors $e^{(1)}, e^{(2)}, \ldots, e^{(n)}$ Let L be the set of all vectors of the form

$$c_1e^{(1)} + c_2e^{(2)} + \ldots + c_ke^{(k)}, \ (k < n),$$

where all the c's are real. It is immediately obvious that conditions (1) and (2) are both satisfied in this case, so that L is a vector subspace of R_n. We say that L is spanned by the vectors $e^{(1)}, e^{(2)}, \ldots, e^{(k)}$.

If $n = 3$, $k = 2$, all the vectors of L lie in one of the planes of reference. If x and y are any two linearly independent 3-vectors, the space L of vectors of the form $c_1x + c_2y$ is a vector subspace of the whole three-dimensional space, viz. the plane of the two vectors x and y.

Example.—*Express the vector* $\{-3, 16, -7\}$ *in terms of the three linearly independent vectors* $\{2, 3, 1\}$, $\{-1, 3, 0\}$, $\{4, -2, 5\}$.

Let A be the matrix $\begin{bmatrix} 2 & -1 & 4 \\ 3 & 3 & -2 \\ 1 & 0 & 5 \end{bmatrix}$, whose columns are the three given linearly independent vectors.

$$\begin{matrix} 2 & -1 & 4 \\ & 3 & -2 \end{matrix} = -10 + 5 \times 9 = 35.$$

$$A^{-1} = \tfrac{1}{35}\begin{bmatrix} 15 & 5 & -10 \\ -17 & 6 & 16 \\ -3 & -1 & 9 \end{bmatrix}.$$

If $\quad A\{c_1, c_2, c_3\} = \{-3, 16, -7\}$, then

$$\{c_1, c_2, c_3\} = A^{-1}\{-3, 16, -7\}$$

$$= \tfrac{1}{35}\begin{bmatrix} 15 & 5 & -10 \\ -17 & 6 & 16 \\ -3 & -1 & 9 \end{bmatrix}\begin{bmatrix} -3 \\ 16 \\ -7 \end{bmatrix} = \tfrac{1}{35}\begin{bmatrix} 105 \\ 35 \\ -70 \end{bmatrix}$$

$$= \{3, 1, -2\}.$$

Hence $\{-3, 16, -7\} = 3\{2, 3, 1\} + \{-1, 3, 0\} - 2\{4, -2, 5\}$.

13·61. Any Non-singular Square Matrix is Equivalent to the Unit Matrix

Let $\qquad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad |A| \neq 0.$

By subtracting suitable multiples of the first column from each of the other columns in turn, we can reduce this to the form

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{bmatrix}$$

in which all elements of the first row are zero except the first. Next, by subtracting suitable multiples of the second column from each of the other columns in turn, we obtain

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & b_{22} & 0 & \dots & 0 \\ c_{31} & b_{32} & c_{33} & \dots & c_{3n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ c_{n1} & b_{n2} & c_{n3} & \dots & c_{nn} \end{bmatrix}$$

in which all elements in the second row are zero except the second. Continuing in this way, after n stages we reach an equivalent diagonal matrix of the form

$$\begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & b_{22} & 0 & \ldots & 0 \\ 0 & 0 & c_{33} & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & \kappa_{nn} \end{bmatrix}.$$

A is of rank n, and since rank is unaltered by elementary transformations, the diagonal matrix is also of rank n. Thus

$$a_{11} \; b_{22} \; c_{33} \; \ldots \; \kappa_{nn} \neq 0,$$

and we may now divide the n columns by $a_{11}, b_{22}, c_{33}, \ldots, \kappa_{nn}$ respectively, obtaining the unit matrix I_n.

All the above operations are elementary operations on columns, and are therefore equivalent to multiplying A on the right by a suitable non-singular matrix P.

Thus $\quad AP = I$, where $|A| \neq 0$, $|P| \neq 0$.

Hence $\quad A = P^{-1}$.

But the inverse of an elementary transformation is clearly an elementary transformation of the same type, so that, since P represents a chain of elementary transformations, its inverse P^{-1} also represents a chain of elementary transformations. We have thus shown that *any* non-singular matrix A is equivalent to a chain of elementary transformations.

13·62. Partitioned Matrices

It is sometimes convenient to "partition" matrices into submatrices. For example, suppose A is an $m \times n$ matrix, and p and q are integers such that $p < m$, $q < n$.

$$A = \left[\begin{array}{cccc:cccc} a_{11} & a_{12} & \ldots & a_{1q} & a_{1,q+1} & a_{1,q+2} & \ldots & a_{1,n} \\ a_{p1} & a_{p2} & \ldots & a_{pq} & a_{p,q+1} & a_{p,q+2} & \ldots & a_{p,n} \\ a_{p+1,1} & a_{p+1,2} & \ldots & a_{p+1,q} & a_{p+1,q+1} & a_{p+1,q+2} & \ldots & a_{p+1,n} \\ a_{m1} & a_{m2} & \ldots & a_{mq} & a_{m,q+1} & a_{m,q+2} & \ldots & a_{m,n} \end{array}\right]$$

We draw a horizontal line between rows p and $p+1$, and a vertical line between columns q and $q+1$, thereby dividing the matrix into four submatrices.

We write $$= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$
where A_1, A_2, A_3, A_4 are of orders $p \times q$, $p \times (n-q)$, $(m-p) \times q$, $(m-p) \times (n-q)$ respectively.

We may partition the columns only, and write
$$A = [A_1 \;\; A_2],$$
where A_1, A_2 are of orders $m \times p$, $m \times (n-p)$ respectively. We may partition the rows only.

The n-vector x may be written $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}$, where $x^{(1)}$ contains p elements, $x^{(2)}$ contains $n-p$ elements.

Consider the set of m homogeneous equations in n unknowns given by the matrix equation $Ax = O$.

A is of order $m \times n$, x of order $n \times 1$. The zero on the right-hand side is a null vector of order n.

Let us partition A and x as follows:—
$$A = [A_1 \;\; A_2], \qquad x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix},$$
where A_1 consists of the first q columns of A, A_2 the remaining $n-q$ columns, and $x^{(1)}$ consists of the first q elements of x, $x^{(2)}$ of the remaining $n-q$ elements. Note that the column partitioning of A is exactly the same as the row partitioning of x. We may therefore write
$$Ax = [A_1 \;\; A_2] \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = A_1x^{(1)} + A_2x^{(2)} = O.$$
$A_1x^{(1)}$ exists *only* if the number of columns in A_1 is equal to the number of elements in $x^{(1)}$, and this is automatically true if the column partitioning of A is the same as the row partitioning of x.

We could also partition the rows of A as in the example above, and write $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ and the equations then become
$$Ax = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix},$$
where the zero n-vector on the right is partitioned in rows in exactly the same way as A. The reader should satisfy himself that the orders of the matrices and vectors involved are such that the equations may be written
$$A_1x^{(1)} + A_2x^{(2)} = O \text{ (the null vector of order } p\text{)},$$

and $\quad A_3x^{(1)} + A_4x^{(2)} = O$ (the null vector of order $m - p$).

We see that these equations are obtained by using the rule for matrix multiplication and treating the submatrices as elements. The first is equivalent to the first p equations of the set $Ax = O$, and the second is equivalent to the remaining $m - p$ equations.

13·63. Fundamental Sets of Solutions

Consider the homogeneous equations $Ax = O$, where A is of order $m \times n$ and rank $r < n$.

By Theorem I (§ 13·44), the equations possess a non-trivial solution if $r < n$, and, as before, we suppose the order of the equations and of the unknowns to be such that the leading submatrix A_1 of A, of order $r \times r$, is non-singular. The first r rows of A are then linearly independent, and the remaining $m - r$ rows are linearly dependent upon the first r rows. Thus, by subtracting suitable linear combinations of the first r rows from the last $m - r$ rows, we may reduce all these $m - r$ rows to zeros.

Hence, if $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where A_1 and A_2 have r rows, and A_1 and A_3 have r columns, then there exists a non-singular matrix P such that

$$PA = \begin{bmatrix} A_1 & A_2 \\ O & O \end{bmatrix},$$

the two zeros standing for null matrices of order $(m - r) \times r$ and $(m - r) \times (n - r)$.

Since $Ax = O$, $P(Ax) = O = (PA)\,x$, so that, if we partition the vector x into its first r and remaining $n - r$ elements, we have

$$\begin{bmatrix} A_1 & A_2 \\ O & O \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix};$$

i.e. $$A_1x^{(1)} + A_2x^{(2)} = O.$$

A_1^{-1} exists, and, multiplying by this on the left, we obtain

$$A_1^{-1}A_1x^{(1)} + A_1^{-1}A_2x^{(2)} = O,$$

or $$x^{(1)} = -\,(A_1^{-1}A_2)\,x^{(2)}.$$

We may therefore choose the $n - r$ elements of $x^{(2)}$ arbitrarily, and the r elements of $x^{(1)}$ are then given uniquely in terms of the $n - r$ elements by this equation. But there are only $n - r$

linearly independent $(n-r)$-vectors. It therefore follows that the given set of equations has exactly $n-r$ linearly independent solutions. In particular, if $r = n - 1$, the equations have only one linearly independent solution; that is to say, each solution is a multiple of any given solution.

Any particular set of $n-r$ linearly independent solutions system of homogeneous equations is called a *fundamental* Every other solution is a linear combination of this fundame set of solutions.

Example.—*Discuss the solution of the set of equations*—

$$\begin{aligned} ax - y - z &= 0, \\ -x + ay - z &= 0, \\ -x - y + az &= 0, \\ x + y + z &= 0. \end{aligned}$$

The matrix of the equations is $\begin{bmatrix} a & -1 & -1 \\ -1 & a & -1 \\ -1 & -1 & a \\ 1 & 1 & 1 \end{bmatrix}.$

If $a = -1$, the first three rows become identical and equal to minus the fourth row. Thus there is only one linearly independent row, and the matrix is of rank 1. In this case $n - r = 3 - 1 = 2$, and there are two linearly independent solutions. We may, for instance, choose x and y arbitrarily, and z is then given by the equation $x + y + z = 0$. We may take $x = 1$, $y = 0$, $z = -1$, and $x = 0$, $y = 1$, $z = -1$ as two solutions, which are clearly linearly independent. Every other solution can be expressed as a linear combination of these two.

If $a \neq -1$, the minor formed by the last three rows is

$$\begin{array}{ccc} -1 & -1 \\ -1 & a \\ 1 & 1 \end{array} \qquad \begin{array}{ccc} -1-a & a & -1 \\ 0 & -1 & a \\ 0 & 1 & 1 \end{array} = (1+a)^2 \neq 0.$$

Hence the matrix is of rank 3, and the equations have no non-trivial solution.

13·64. Reduction of a Matrix to the Form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$

Let A be a matrix of order $m \times n$ and rank r. If $r < m$ and $r < n$, we can rearrange the order of its rows and of its columns in such a way that the leading submatrix A_r of order $r \times r$ is non-singular. The first r rows are now linearly independent, and the remaining $m - r$ rows are dependent upon the first r. By a chain of elementary transformations we may therefore reduce the last

$m - r$ rows to zeros. Similarly, we may reduce the last $n - r$ columns to zeros. Hence

$$A \simeq \begin{bmatrix} A_r & O \\ O & O \end{bmatrix}, \text{ where } |A_r| \neq 0.$$

As in § 13·61, we may now reduce the non-singular matrix A_r to I_r by elementary transformations, and thus

$$A \simeq \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

It follows that there exist non-singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

13·71. The Solution of Non-homogeneous Linear Equations

We now consider equations which are no longer homogeneous, *i.e.* equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

in which not all the elements on the right-hand side vanish. These equations may be written in matrix form $Ax = b$, where A is of order $m \times n$, $x = \{x_1, x_2, \ldots, x_n\}$, $b = \{b_1, b_2, \ldots, b_m\}$. b is not a null vector.

The matrix A is called the matrix of the equations, and the matrix $[A\ b]$, formed by adjoining the column vector b to A to form an $(n + 1)$th column, is called the *augmented matrix*. We now establish the condition for the consistency of the equations.

THEOREM II.—The set of non-homogeneous linear equations $Ax = b$ is consistent if, and only if, the matrix A and the augmented matrix $[A\ b]$ have the same rank.

If A is of order $m \times n$ and of rank r, we may suppose the order of the equations and of the unknowns to be such that the leading submatrix A_r of order $r \times r$ of A is non-singular.

Thus $$A = \begin{bmatrix} A_r & C_1 \\ C_2 & C_3 \end{bmatrix}, \text{ where } |A_r| \neq 0.$$

The last $m - r$ rows of A are then linearly dependent upon the first r rows, and, by suitable elementary transformations upon the rows, we may replace these last $m - r$ rows by zeros. Thus there exists a non-singular matrix P of order $m \times m$ such that

$$PA = \begin{bmatrix} A_r & C_1 \\ O & O \end{bmatrix}.$$

Then $$(PA)\,x = P\,(Ax) = Pb = h, \text{ say.}$$

If we partition x into its first r elements $x^{(1)}$ and its last $n - r$ elements $x^{(2)}$, and similarly for h, this may be written

$$\begin{bmatrix} A_r & C_1 \\ O & O \end{bmatrix}\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \begin{bmatrix} h^{(1)} \\ h^{(2)} \end{bmatrix}.$$

This is equivalent to the two matrix equations

$$A_r x^{(1)} + C_1 x^{(2)} = h^{(1)}.$$
$$Ox^{(1)} + Ox^{(2)} = h^{(2)}.$$

If $h^{(2)} \neq 0$, we can find no x to satisfy the second of these equations. If, on the other hand, $h^{(2)} = 0$, the second is satisfied for all x, and the first equation, on pre-multiplying by A_r^{-1}, becomes

$$x^{(1)} + A_r^{-1}C_1 x^{(2)} = A_r^{-1}h^{(1)}.$$

We can therefore choose the elements of $x^{(2)}$ arbitrarily and the elements of $x^{(1)}$ are then determined by the equation.

Consequently, the equations are consistent if, and only if, $h^{(2)} = 0$.

Compare the two matrices

$$\begin{bmatrix} A_r & C_1 \\ O & O \end{bmatrix} \text{ and } \begin{bmatrix} A_r & C_1 & h^{(1)} \\ O & O & h^{(2)} \end{bmatrix}.$$

If $h^{(2)} = 0$, then every minor of order $(r + 1)$ of both matrices vanishes. Since $|A_r| \neq 0$, each is of rank r.

If $h^{(2)} \neq 0$, then it contains at least one non-zero element, say h_k. Consider the minor of order $r + 1$ of $\begin{bmatrix} A_r & C_1 & h^{(1)} \\ O & O & h^{(2)} \end{bmatrix}$ formed by elements from the first r columns and the last one, and from the first r rows and the row containing h_k. This minor is

$$\begin{vmatrix} A_r & h^{(1)} \\ 0 & h_k \end{vmatrix} = h_k\,|A_r| \neq 0.$$

All minors of order $r + 2$ clearly vanish, and the rank of the matrix is $r + 1$. Thus the two matrices above have the same rank if, and only if, $h^{(2)} = O$.

Now multiplication by a non-singular matrix (which, by § 13·61, is equivalent to a chain of elementary transformations) does not alter the rank of a matrix.

P^{-1} exists and is non-singular;

$$P^{-1}\begin{bmatrix} A_r & C_1 \\ O & O \end{bmatrix} = P^{-1}(PA) = (P^{-1}P)A = A,$$

and $$P^{-1}\begin{bmatrix} A_r & C_1 & h^{(1)} \\ O & O & h^{(2)} \end{bmatrix} = P^{-1}[PA \ \ h] = [P^{-1}(PA) \ \ P^{-1}h]$$
$$= [(P^{-1}P)A \ \ P^{-1}Pb] = [A \ \ b].$$

Hence the two matrices A, $[A \ b]$ have the same rank if, and only if, $h^{(2)} = O$.

The equations are therefore consistent if, and only if, the matrix of the equations and the augmented matrix have the same rank. This is *Rouché's Theorem.*

We have shown that the $n - r$ elements of $x^{(2)}$ can be chosen arbitrarily. To determine the set of arbitrary elements we first find a set of r linearly independent columns of the matrix A. The unknowns corresponding to the remaining $n - r$ columns can be chosen arbitrarily, and those corresponding to the r linearly independent columns are then given in terms of these $n - r$ unknowns.

Examples.—(1) *Determine the values of a for which the following equations are consistent.*

$$\begin{aligned} x + ay + az &= 1 \\ ax + y + 2az &= -4 \\ ax - ay + 4z &= 2. \end{aligned}$$

The equations may be written $Ax = b$, where

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & 2a \\ a & -a & 4 \end{bmatrix}, \quad x = \{x, y, z\}, \quad b = \{1, -4, 2\}.$$

$$|A| = (a+1)(a-2)^2.$$

If $a \neq -1$ or 2, $|A| \neq 0$. A is of rank 3, and the augmented matrix $[A \ b]$ is then also of rank 3. The equations therefore have a unique solution given by $x = A^{-1}b$.

If $a = 2$, $|A| = 0$.

The leading minor of order 2 is $\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \neq 0.$

A is therefore of rank 2.

The augmented matrix is, however, of rank 3, for the minor of order 3 obtained by deleting its third column is—

$$\begin{vmatrix} 1 & a & 1 \\ a & 1 & -4 \\ a & -a & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 2 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ 2 & -6 & 0 \end{vmatrix} = -36 \neq 0.$$

The equations are therefore inconsistent.

$$\text{If } a = -1, |A| = 0.$$

The minor of order 2 formed by the first two rows and first and third columns of A is $\begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} \neq 0$.

Hence A is of rank 2.

The augmented matrix is then $\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -2 & -4 \\ -1 & 1 & 4 & 2 \end{bmatrix}$.

Columns 2 and 3 are clearly linearly independent, and column 1 is minus column 2. The minor formed by columns 2, 3, and 4 is

$$\begin{vmatrix} -1 & -1 & 1 \\ 1 & -2 & -4 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -3 & -6 & -4 \\ 3 & 6 & 2 \end{vmatrix} = 0.$$

Hence column 4 is linearly dependent on columns 2 and 3. The augmented matrix is therefore also of rank 2, and the equations are consistent. In this case, however, there are infinitely many solutions. Since columns 2 and 3 of A are linearly independent, we may choose the unknown corresponding to column 1, viz. x, arbitrarily. y and z are then given in terms of x. The first two equations may be written—

$$-y - z = 1 - x,$$
$$y - 2z = -4 + x.$$

Hence $\qquad -3z = -3, \quad z = 1, \quad \text{and} \quad y = x - 2.$

Thus $x = \alpha$, $y = \alpha - 2$, $z = 1$ is a solution for every α.

(2) *Examine the consistency of the equations—*

$$\begin{aligned} x + y + z &= a \\ ax + by + z &= b \\ a^2x + b^2y + z &= 1, \end{aligned}$$

where a and b are real.

Let $$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & 1 \\ a^2 & b^2 & 1 \end{bmatrix}.$$

$$|A| = (b - 1)(1 - a)(a - b).$$

If $a \neq 1$, $b \neq 1$, $a \neq b$, then $|A| \neq 0$, and A and the augmented matrix are both of rank 3.

The equations are therefore consistent and have a unique solution.

If $a = b \neq 1$, $|A| = 0$, but A has a minor $\begin{vmatrix} 1 & 1 \\ b & 1 \end{vmatrix} \neq 0$.

Thus A is of rank 2.

The augmented matrix is then $\begin{bmatrix} 1 & 1 & 1 & a \\ a & a & 1 & a \\ a^2 & a^2 & 1 & 1 \end{bmatrix}$.

The first two columns are identical, and the minor of order 3 formed by the last three columns is

$$\begin{vmatrix} 1 & 1 & a \\ a & 1 & a \\ a^2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1-a & 0 & 0 \\ a & 1 & a \\ a^2 & 1 & 1 \end{vmatrix} = (1-a)^2 \neq 0.$$

Hence the augmented matrix is of rank 3, and the equations are inconsistent.

If $a = 1$, $b \neq 1$, $|A| = 0$, but A has a minor $\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = b - a \neq 0$.

A is therefore of rank 2.

The augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & b & 1 & b \\ 1 & b^2 & 1 & 1 \end{bmatrix}$.

Now $$\begin{vmatrix} 1 & 1 & 1 \\ b & 1 & b \\ b^2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1-b^2 & 0 & 0 \\ b & 1 & b \\ b^2 & 1 & 1 \end{vmatrix} = (1-b^2)(1-b) = (1+b)(1-b)^2.$$

This matrix is therefore of rank 3 if $b \neq -1$, and the equations are then inconsistent.

If $b = -1$, it is of rank 2, and the equations are consistent, with infinitely many solutions.

If $b = 1$, $a \neq 1$, A is again of rank 2.

The augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & a \\ a & 1 & 1 & 1 \\ a^2 & 1 & 1 & 1 \end{bmatrix}$.

Now $$\begin{vmatrix} 1 & 1 & a \\ a & 1 & 1 \\ a^2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & a \\ a & 1 & 1 \\ a^2 - a & 0 & 0 \end{vmatrix} = (a^2 - a)(1-a) = -a(1-a)^2.$$

The matrix is consequently of rank 3 if $a \neq 0$, and the equations are inconsistent.

If $a = 0$, it is of rank 2, and the equations are consistent, with infinitely many solutions.

If $a = b = 1$, A is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

These are both of rank 1, and the equations are consistent, with infinitely many solutions.

The results may be summarised as follows:—

(1) If $a \neq b$, $a \neq 1$, $b \neq 1$, the equations are consistent and have a unique solution.

(2) If $a = 1$, $b = -1$, or $a = 0$, $b = 1$, or $a = b = 1$, the equations are consistent and have infinitely many solutions.

(3) In all other cases the equations are inconsistent.

13·81. Latent Roots of a Matrix

Let A be a square $n \times n$ matrix, and consider the equation $Ax = \lambda x$, where λ is a scalar. We may write this

$$Ax = \lambda Ix \text{ (where } I = I_n)$$

or

$$(A - \lambda I)\, x = O.$$

$A - \lambda I$ is an $n \times n$ matrix, and, by Theorem I (§ 13·44), this equation has a non-trivial solution if, and only if, the rank of $A - \lambda I$ is less than n, i.e. $| A - \lambda I | = 0$.

Thus a solution $x \neq 0$ exists if, and only if, λ satisfies the equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0.$$

This is an equation of degree n in λ, called the *characteristic equation* of A. Its roots are called the *latent roots* (or eigenvalues) of A. Corresponding to each latent root λ is a non-trivial solution x of the equation $Ax = \lambda x$, known as a *latent column vector* (or eigenvector, or pole). If x satisfies the equation, any scalar multiple μx of x also satisfies.

Now if $\quad | A - \lambda I | = 0,\ | (A - \lambda I)' | = 0$ also.

But $\quad (A - \lambda I)' = A' - \lambda I$, so that $| A' - \lambda I | = 0$.

Hence A and A' have the same latent roots. If λ is a latent root, there exists a column vector y (not null) such that $A'y = \lambda y$. Transposing this equation gives $y'A = \lambda y'$. y' is called the *latent row vector* of A corresponding to the latent root λ.

Let us now suppose that the characteristic equation has n *distinct* roots $\lambda_1, \lambda_2, \dots, \lambda_n$ (not necessarily all real). We may then prove the following result:—

13·82. If the Latent Roots of A are Distinct, the Latent Vectors are Linearly Independent

Let $x^{(k)}$ be the latent column vector corresponding to the latent root λ_k.

Then $\qquad Ax^{(k)} = \lambda_k x^{(k)}\ (k = 1, 2, \dots, n).$

Hence $\quad A^2.x^{(k)} = A.(Ax^{(k)}) = A.(\lambda_k x^{(k)}) = \lambda_k (Ax^{(k)})$

$$= \lambda_k . \lambda_k x^{(k)} = \lambda_k^2 x^{(k)},$$

and, by the method of induction, we can show that $A^p x^{(k)} = \lambda_k{}^p x^{(k)}$, where p is any positive integer.

Let $y = c_1 x^{(1)} + c_2 x^{(2)} + \ldots + c_n x^{(n)} = O$, where $c_1, c_2, \ldots, c_n$ are scalars, and write y_i for the i-th element in the vector y.

Then $y_i = c_1 x_i{}^{(1)} + c_2 x_i{}^{(2)} + \ldots + c_n x_i{}^{(n)} = 0 \ (i = 1, 2, \ldots, n)$.

Now
$$\begin{aligned} Ay &= A\,(c_1 x^{(1)} + c_2 x^{(2)} + \ldots + c_n x^{(n)}) \\ &= c_1\,(Ax^{(1)}) + c_2\,(Ax^{(2)}) + \ldots + c_n\,(Ax^{(n)}) \\ &= c_1\lambda_1 x^{(1)} + c_2\lambda_2 x^{(2)} + \quad + c_n\lambda_n x^{(n)} \\ &= O, \text{ since } y = O. \end{aligned}$$

Similarly,
$$A^2 y = c_1\lambda_1{}^2 x^{(1)} + c_2\lambda_2{}^2 x^{(2)} + \ldots + c_n\lambda_n{}^2 x^{(n)} = O,$$
and
$$A^p y = c_1\lambda_1{}^p x^{(1)} + c_2\lambda_2{}^p x^{(2)} + \ldots + c_n\lambda_n{}^p x^{(n)} = O.$$

Equating to zero the i-th element of each of the null vectors $y, Ay, A^2y, \ldots, A^{n-1}y$, we obtain the n equations
$$\begin{aligned} c_1 x_i{}^{(1)} + c_2 x_i{}^{(2)} &+ \ldots + c_n x_i{}^{(n)} = 0 \\ \lambda_1 c_1 x_i{}^{(1)} + \lambda_2 c_2 x_i{}^{(2)} &+ \ldots + \lambda_n c_n x_i{}^{(n)} = 0 \\ \lambda_1{}^2 c_1 x_i{}^{(1)} + \lambda_2{}^2 c_2 x_i{}^{(2)} &+ \ldots + \lambda_n{}^2 c_n x_i{}^{(n)} = 0 \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_1{}^{n-1} c_1 x_i{}^{(1)} + \lambda_2{}^{n-1} c_2 x_i{}^{(2)} &+ \ldots + \lambda_n{}^{n-1} c_n x_i{}^{(n)} = 0. \end{aligned}$$

These are n homogeneous equations in the n unknowns $c_1 x_i{}^{(1)}$, $c_2 x_i{}^{(2)}, \ldots, c_n x_i{}^{(n)}$ and have a non-trivial solution if, and only if,
$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_n \\ \lambda_1{}^2 & \lambda_2{}^2 & \lambda_3{}^2 & \ldots & \lambda_n{}^2 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \lambda_1{}^{n-1} & \lambda_2{}^{n-1} & \lambda_3{}^{n-1} & \ldots & \lambda_n{}^{n-1} \end{vmatrix} = 0.$$

But this determinant vanishes if, and only if, two of the λ's are equal, which is contrary to hypothesis.

Hence $c_1 x_i{}^{(1)} = c_2 x_i{}^{(2)} = \ldots = c_n x_i{}^{(n)} = 0 \ (i = 1, 2, \ldots, n)$,

so that $c_1 x^{(1)} = c_2 x^{(2)} = \ldots = c_n x^{(n)} = O$.

But the vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are latent vectors and are not null.

Hence $c_1 = c_2 = \ldots = c_n = 0$, and the vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are consequently linearly independent.

13·83. Reduction of a Matrix to Diagonal Canonical Form

Let A be a non-singular $n \times n$ matrix whose latent roots are distinct. Let Q be the matrix whose columns $q^{(1)}, q^{(2)}, \ldots, q^{(n)}$ are the n linearly independent latent column vectors of A. Q is non-singular, so that Q^{-1} exists.

Now the equations

$$Aq^{(1)} = \lambda_1 q^{(1)},\ Aq^{(2)} = \lambda_2 q^{(2)},\ \ldots,\ Aq^{(n)} = \lambda_n q^{(n)}$$

may be written

$$AQ = Q\Lambda,$$

where Λ is the $n \times n$ diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ & & & \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Multiplying this equation on the left by Q^{-1}, we have

$$Q^{-1}AQ = Q^{-1}(Q\Lambda) = (Q^{-1}Q)\Lambda = \Lambda.$$

Similarly, if P is the matrix whose rows are the latent row vectors of A,

$$PA = \Lambda P, \text{ and } PAP^{-1} = \Lambda.$$

Thus, by means of the transformations

$$Q^{-1}AQ = \Lambda = PAP^{-1},$$

the matrix A is reduced to *diagonal canonical form* Λ.

This proof depends upon the fact that the latent vectors are linearly independent. Now this *may* be the case when the characteristic equation has multiple roots. If λ is a latent root of multiplicity k, it can be shown that there are not more than k linearly independent latent vectors corrresponding to λ. Thus there may be n linearly independent vectors, but this is not necessarily true. If there are, then the matrix Q whose columns are these vectors is non-singular, and, as before, $Q^{-1}AQ = \Lambda$. The diagonal matrix Λ no longer has distinct diagonal elements.

Examples.—(1) ***The latent roots of a unitary matrix have unit modulus.***

If A is unitary, $A\bar{A}' = I$.

If $Ax = \lambda x$, transposing and taking the conjugate gives $\bar{x}'\bar{A}' = \bar{\lambda}\bar{x}'$.

Hence $$(\bar{x}'\bar{A}')(Ax) = (\bar{\lambda}\bar{x}')(\lambda x);$$

i.e. $$\bar{x}'(\bar{A}'A)x = \lambda\bar{\lambda}\bar{x}'x,$$

or $$\bar{x}'x = \lambda\bar{\lambda}\bar{x}'x.$$

But $$\bar{x}'x = \sum_{i=1}^{n} |x_i|^2 \neq 0, \text{ since } x \neq 0.$$

Hence $\lambda\bar{\lambda} = 1$, and λ is of unit modulus.

(2) *The latent roots of a Hermitian matrix are real.*

If A is Hermitian, $\bar{A}' = A$.

If $Ax = \lambda x$, multiplying by $\bar{x}'$ on the left gives $\bar{x}'Ax = \lambda\bar{x}'x$, where each side of the equation is a scalar.

Now $\bar{x}'x = \sum_{i=1} |x_i|^2$, which is real and non-zero, since $x \neq 0$.

Transposing and taking the conjugate of $\bar{x}'Ax$ gives

$$\overline{(\bar{x}'Ax)'} = \bar{x}'\bar{A}'x = \bar{x}'Ax.$$

But transposition does not affect a scalar (regarding it as a 1×1 matrix), so that

$$\overline{(\bar{x}'Ax)} = \bar{x}'Ax, \text{ and hence } \bar{x}'Ax \text{ is real.}$$

Then
$$\lambda = \frac{\bar{x}'Ax}{\bar{x}'x} \text{ is real.}$$

(3) *For a Hermitian matrix, $P = \bar{Q}'$.*

If $Aq = \lambda q$, transposing and taking the conjugate gives

$$\bar{q}'\bar{A}' = \bar{\lambda}\bar{q}'.$$

Since $\bar{A}' = A$ and λ is real, this may be written

$$\bar{q}'A = \lambda\bar{q}'.$$

Hence, if q is a latent column vector, $\bar{q}'$ is the latent row vector corresponding to the same latent root λ.

It follows that $P = \bar{Q}'$.

(4) *If A is Hermitian, with distinct latent roots, Q is unitary.*

For
$$Aq^{(k)} = \lambda_k q^{(k)} \quad \text{and} \quad p'^{(h)}A = \lambda_h p'^{(h)}.$$

Hence
$$\lambda_h p'^{(h)}q^{(k)} = (p'^{(h)}A)\,q^{(k)} = p'^{(h)}\,(Aq^{(k)})$$
$$= p'^{(h)}\lambda_k q^{(k)} = \lambda_k p'^{(h)}q^{(k)}.$$

But if $h \neq k$, $\lambda_h \neq \lambda_k$, so that $p'^{(h)}q^{(k)} = 0$ when $h \neq k$.

But the scalar product $p'^{(h)}q^{(k)}$ is the hk-th element in the matrix product PQ, which is therefore diagonal.

As already shown, both P and Q are non-singular, so that PQ is also non-singular. It follows that the diagonal element $p'^{(k)}q^{(k)} \neq 0$ for any k, and we can choose the arbitrary scalar factor in the p's and q's in such a way that $p'^{(k)}q^{(k)} = 1$ for every k.

Then
$$PQ = I = QP.$$

By (3), since A is Hermitian, $P = \bar{Q}'$.

Hence $Q\bar{Q}' = I = \bar{Q}'Q$ and $P\bar{P}' = I = \bar{P}'P$, so that P and Q are unitary.

The equation $\bar{Q}'AQ = A$ gives a unitary reduction of A to diagonal canonical form.

(5) *If A is symmetric, Q is orthogonal.*

As in (4), $PQ = I$.

As in (3), we can show that, if A is symmetric, $P = Q'$.
Hence $PP' = I = QQ'$, and P and Q are orthogonal.

13·91. Quadratic Form

A homogeneous algebraic expression of the second degree in n real variables $x_1, x_2, \ldots, x_n$, given by

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, x_i x_j,$$

is called a *quadratic form* in n variables.

When $i \neq j$, the coefficient of $x_i x_j$ in the sum is $a_{ij} + a_{ji}$. We can associate a symmetric matrix with a quadratic form, for if we put $2b_{ij} = a_{ij} + a_{ji}$, then $b_{ij} = b_{ji}$, and the expression takes the form

$$b_{11}x_1^2 + b_{22}x_2^2 + \ldots + b_{nn}x_n^2 + 2x_1(b_{12}x_2 + b_{13}x_3 + \ldots + b_{1n}x_n) + 2x_2(b_{23}x_3 + b_{24}x_4 + \ldots + b_{2n}x_n) + \ldots + 2b_{n-1,n}x_{n-1}x_n.$$

This, in turn, may be written in the simple form $x'Bx$, where

$$x = \{x_1, x_2, \ldots, x_n\},\ B = (b_{ij}).$$

For
$$x'Bx = x'.(Bx) = \sum_{i=1} x_i (Bx)_i$$

$$= \sum_{1}^{n} x_i \sum_{j}^{n} b_{ij}x_j = \sum_{i=1}^{n} \sum_{j=1} b_{ij}x_i x_j,$$

and this is easily seen to be equal to the above expression, B being symmetric. Reverting to the original notation, we shall henceforth write a quadratic form $x'Ax$, where A is a symmetric matrix.

A sum of squares is a quadratic form whose matrix is diagonal. For

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 = [x_1, x_2, \ldots, x_n] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \end{bmatrix}$$

The equation of a central conic, referred to its centre as origin, is of the form

$$ax^2 + 2hxy + by^2 = 1.$$

This may be written

$$[x\ y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1.$$

The left-hand side is a quadratic form with matrix $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, and the equation may be written $x'Ax = 1$, where $x = \{x, y\}$.

The equation of the tangent to the conic at the point (x, y) may be written

$$x\,(ax_1 + hy_1) + y\,(hx_1 + by_1) = 1,$$

i.e.
$$[x\ y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 1,$$

or
$$x'Ax_1 = 1.$$

It may also be written $x_1'Ax = 1$.

Similarly, the equation of a central quadric referred to its centre as origin is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1,$$

i.e.
$$[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1.$$

Since $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is symmetric, the left-hand side is a quadratic form in x, y, z, and we may write the equation in the form $x'Ax = 1$, where $x = \{x, y, z\}$. Again, the tangent plane at (x_1, y_1, z_1) may be written

$$x'Ax_1 = 1 \ \ (\text{or } x_1'Ax = 1).$$

The following example will illustrate the geometrical significance of the reduction of a matrix to diagonal canonical form.

Example.—*Find the lengths of the semi-axes, equations of the principal planes, and the nature of the quadric*

$$5x^2 - 2y^2 + 5z^2 + 2zx = 2.$$

The equation may be written

$$[x\ y\ z]\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2;$$

i.e. $x'Ax = 2$, with the usual notation.

Since A is symmetric, we can find an orthogonal matrix Q such that $Q'AQ$ is diagonal.

The latent roots of A are given by the equation

$$\begin{vmatrix} 5-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix} = 0,$$

i.e.

$$-(2+\lambda)\{(5-\lambda)^2 - 1\} = 0,$$
$$-(2+\lambda)(4-\lambda)(6-\lambda) = 0,$$
$$\lambda = -2,\ 4,\ \text{or}\ 6.$$

The latent column vector corresponding to $\lambda = -2$ is given by

$$Ax = -2x,$$

i.e.

$$\begin{aligned} 5x_1 + x_3 &= -2x_1 \\ -2x_2 &= -2x_2 \\ x_1 + 5x_3 &= -2x_3. \end{aligned}$$

Thus

$$\begin{aligned} 7x_1 + x_3 &= 0 \\ x_1 + 7x_3 &= 0 \end{aligned}$$

and we must have $x_1 = x_3 = 0$. x_2 is arbitrary.

Since x is to be a column of Q, and $QQ' = I$, we must have

$$x'x = x_1^2 + x_2^2 + x_3^2 = 1.$$

Hence

$$x_2 = \pm 1.$$

Take x to be $\{0, 1, 0\}$.

Similarly, the latent column vectors corresponding to $\lambda = 4$, $\lambda = 6$ are

$$\left\{\frac{1}{\sqrt{2}},\ 0,\ \frac{-1}{\sqrt{2}}\right\},\ \left\{\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right\}$$

respectively.

Hence $Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, and the reader will easily verify that

$$Q'AQ = \Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

An orthogonal transformation corresponds to a rotation of rectangular axes, the origin remaining fixed.

Let us apply the orthogonal transformation $x = QX$, where

$$X = [X,\ Y,\ Z].$$

Then $Q'x = Q'QX = X$, so that

$$\begin{aligned} X &= y \\ Y &= \frac{1}{\sqrt{2}}(x - z) \\ Z &= \frac{1}{\sqrt{2}}(x + z). \end{aligned}$$

The equation now becomes

$$(QX)' A (QX) = 2,$$

i.e.
$$X'Q'AQX = 2,$$

i.e.
$$X'AX = 2,$$

or
$$-2X^2 + 4Y^2 + 6Z^2 = 2.$$

In standard form this is

$$-\frac{X^2}{1} + \frac{Y^2}{\frac{1}{2}} + \frac{Z^2}{\frac{1}{3}} = 1.$$

This is a hyperboloid of one sheet with semi-axes $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}$. Referred to the new axes, the principal planes are $X = 0$, $Y = 0$, $Z = 0$. Hence referred to the original axes, the principal planes are $y = 0$, $x - z = 0$, $x + z = 0$.

13·92. The Discriminant of a Quadratic Form

If A is symmetric, the *discriminant* of the quadratic form $x'Ax$ is defined to be $|A|$.

THEOREM III.—If the variables of a quadratic form undergo a linear transformation, the discriminant of the new form is the discriminant of the old, multiplied by the square of the modulus of the transformation.

Let $x'Ax$ be the given form, and let $x = By$, where $|B| \neq 0$.

Then
$$x'Ax = (By)' A (By) = y'B'ABy.$$

$B'AB$ is clearly symmetric, since A is symmetric, and $y'(B'AB)y$ is a quadratic form with matrix $B'AB$.

The discriminant of this form is $|B'AB| = |B'||A||B| = |A||B|^2$, since A, B, B' are all square and of the same order.

This proves the result.

In particular, if $|B| = 1$, the discriminant of the form is unchanged.

13·93. Positive Definite Forms

If A is real and symmetric, and if $x'Ax > 0$ for all *real* values of x other than $x = O$, then the quadratic form $x'Ax$ is said to be *positive definite*. If $x = O$, clearly $x'Ax = 0$.

Example.—*$5x^2 + y^2 + 3z^2$ is positive definite, since it takes the value 0 only when $x = y = z = 0$. On the other hand, $5x^2 - y^2 + 3z^2$ is not positive definite, since it is positive for some values of the variables* (e.g. $x = 1$, $y = 2$, $z = 3$) *and negative for others* (e.g. $x = 1$, $y = 5$, $z = 2$).

$(x-y)^2 + z^2 \geqslant 0$ for every x, y, z, but it takes the value zero for all sets of values of x, y, z of the form $x = a$, $y = a$, $z = 0$, a being any real number. The expression is therefore *not* positive definite.

We now show that, if we apply a non-singular transformation to the variables of a positive definite form, the new form is also positive definite.

For, let $\quad A' = A,\ x = By,\ |B| \neq 0.$

Then $$x'Ax = y'B'ABy.$$

Now $\quad x'Ax > 0$ if $x \neq O$ and $x'Ax = 0$ if $x = O$.

But, if $x = O$, $y = B^{-1}x = O$, and if $y = O$, $x = By = O$.

Hence, if $x = O$ then $y = O$ also, and vice versa, and we therefore have

$$y'B'ABy > 0 \text{ if } y \neq O,$$

and $$y'B'ABy = 0 \text{ if } y = O.$$

Thus $y'B'ABy$ is also positive definite.

13·94. Reduction of a Positive Definite Form to a Sum of Squares

THEOREM IV.—Every positive definite form $x'Ax$ ($A' = A$, A real) can be reduced by a transformation of unit modulus to

$$c_1y_1^2 + c_2y_2^2 + \ldots + c_ny_n^2,$$

where $\quad c_k > 0$ $(k = 1, 2, \ldots, n)$, **A being of order $n \times n$.**

For $\quad x'Ax = a_{11}x_1^2 + 2x_1(a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n) + a_{22}x_2^2 + \ldots.$

Since $x'Ax$ is positive definite, $x'Ax > 0$ when $x_1 = 1$, $x_2 = x_3 = \ldots = x_n = 0$. Hence $a_{11} > 0$, and we may write

$$x'Ax = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + \text{terms involving } x_2, x_3, \ldots, x_n \text{ only}$$

$$= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + b_{22}x_2^2 + 2x_2(b_{23}x_3 + b_{24}x_4 + \ldots + b_{2n}x_n) + b_{33}x_3^2 + \ldots, \text{ say.}$$

Now $x'Ax > 0$ when

$$x_2 = 1,\ x_3 = x_4 = \ldots = x_n = 0,\ x_1 = -\frac{a_{12}}{a_{11}}.$$

Hence $b_{22} > 0$, and we may write

$$x'Ax = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2$$
$$+ b_{22}\left(x_2 + \frac{b_{23}}{b_{22}}x_3 + \frac{b_{24}}{b_{22}}x_4 + \ldots + \frac{b_{2n}}{b_{22}}x_n\right)^2$$
$$+ c_{33}x_3^2 + 2x_3(c_{34}x_4 + c_{35}x_5 + \ldots + c_{3n}x_n)$$
$$+ c_{44}x_4^2 + \ldots, \text{ say.}$$

Again, putting $x_3 = 1$, $x_4 = x_5 = \ldots = x_n = 0$, $x_2 = -\frac{b_{23}}{b_{22}}$, $x_1 = -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}$, we see that $c_{33} > 0$.

Hence we may write

$$x'Ax = a_{11}\left(x_1 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + b_{22}\left(x_2 + \ldots + \frac{b_{2n}}{b_{22}}x_n\right)^2$$
$$+ c_{33}\left(x_3 + \ldots + \frac{c_{3n}}{c_{33}}x_n\right)^2 + \text{terms involving } x_4, x_5, \ldots, x_n.$$

Continuing in this way, we may write

$$x'Ax = a_{11}\left(x_1 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + \ldots + \kappa_{nn}x_n^2,$$

where $\qquad a_{11} > 0,\ b_{22} > 0,\ c_{33} > 0,\ \ldots,\ \kappa_{nn} > 0.$

Now write

$$\begin{aligned}
y_1 &= x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3 + \ldots + \frac{a_{1n}}{a_{11}}x_n \\
y_2 &= \qquad\quad x_2 + \frac{b_{23}}{b_{22}}x_3 + \ldots + \frac{b_{2n}}{b_{22}}x_n \\
y_3 &= \qquad\qquad\qquad x_3 + \ldots + \frac{c_{3n}}{c_{33}}x_n \\
&\cdots\cdots\cdots\cdots \\
y_n &= \qquad\qquad\qquad\qquad\qquad x_n.
\end{aligned}$$

This may be written in the form $y = Bx$, where B is an $n \times n$ matrix, and

$$|B| = \begin{vmatrix} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{a_{13}}{a_{11}} & \dots & \dfrac{a_{1n}}{a_{11}} \\ 0 & 1 & \dfrac{b_{23}}{b_{22}} & \dots & \dfrac{b_{2n}}{b_{22}} \\ 0 & 0 & 1 & \dots & \dfrac{c_{3n}}{c_{33}} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = 1.$$

Hence $x = B^{-1}y$, where $|B^{-1}| = 1$ also.

If we now replace $a_{11}, b_{22}, c_{33}, \dots, \kappa_{nn}$ by $c_1, c_2, \dots, c_n$, we have

$$x'Ax = c_1y_1^2 + c_2y_2^2 + \dots + c_ny_n^2 = y' \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & c_n \end{bmatrix} y,$$

where $c_k > 0$ for every k.

Using this result, we can now establish a set of necessary and sufficient conditions for $x'Ax$ to be positive definite, where A is a real symmetric $n \times n$ matrix.

Theorem V.—The real quadratic form $x'Ax$ is positive definite if, and only if,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{12} & \dots & a_{2n} \\ \cdot & \dots & \cdot \\ a_{n1} & \dots & a_{nn} \end{vmatrix} > 0,$$

***i.e.* every leading minor of A of order m ($m = 1, 2, \dots, n$) is positive.**

As in the previous theorem, if $x'Ax$ is positive definite, there exists a transformation $y = Bx$ of unit modulus such that

$$x'Ax = c_1y_1^2 + c_2y_2^2 + \dots + c_ny_n^2, \; c_k > 0 \text{ for every } k,$$

$$= y' \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & c_n \end{bmatrix} y.$$

The modulus of the transformation is 1, so that, by Theorem III (§ 13·92), the discriminant of the new form is equal to that of $x'Ax$.

Thus $$|A| = c_1c_2 \ldots c_n > 0.$$

If we now put $x_n = y_n = 0$, $x'Ax$ becomes a positive definite form in $(n-1)$ variables $x_1, x_2, \ldots, x_{n-1}$ with matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix}, \text{ and } x'Ax = c_1y_1^2 + c_2y_2^2 + \ldots + c_{n-1}y_{n-1}^2.$$

Putting $x_n = 0$ throughout, we see again that the modulus of the transformation from the $(n-1)$ variables $x_1, x_2, \ldots, x_{n-1}$ to the $(n-1)$ variables $y_1, y_2, \ldots, y_{n-1}$ is 1, so that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} = c_1c_2 \ldots c_{n-1} > 0.$$

Continuing the argument, we establish that the set of conditions is necessary.

We now show that it is sufficient, *i.e.* that if the conditions are satisfied, then $x'Ax$ is positive definite.

For, since $a_{11} > 0$, we may write

$$x'Ax = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + b_{22}x_2^2 + \ldots$$

as before. If we put $x_3 = x_4 = \ldots = x_n = 0$, and make the transformation $X_1 = x_1 + \dfrac{a_{12}}{a_{11}}x_1$, $X_2 = x_2$, which is clearly of unit modulus, we have

$$[x_1\ x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [X_1\ X_2]\begin{bmatrix} a_{11} & 0 \\ 0 & b_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

The discriminant of the two forms is the same, so that

$$a_{11}\ b_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \text{ by hypothesis.}$$

But $a_{11} = c_1 > 0$, and therefore $b_{22} = c_2 > 0$ also.

Consequently we may write

$$x'Ax = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + b_{22}\left(x_2 + \ldots + \frac{b_{2n}}{b_{22}}x_n\right)^2 + c_{33}x_3^2 + \ldots.$$

Now putting $x_4 = x_5 = \ldots = x_n = 0$ and making a suitable transformation of unit modulus, each side of this equation becomes a quadratic form in 3 variables. Their discriminants are equal, so that

$$a_{11}\, b_{22}\, c_{33} = \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} > 0, \text{ by hypothesis.}$$

But we have already shown that $a_{11}\, b_{22} > 0$, so that $c_{33} = c_3 > 0$.
Continuing in this way, we can show that

$$x'Ax = c_1 y_1^2 + c_2 y_2^2 + \ldots + c_n y_n^2,$$

where $c_1, c_2, \ldots, c_n$ are all positive.

The right-hand side is positive definite, and the transformation $y = Bx$ is non-singular.

Hence $x'Ax$ is also positive definite.

13·95. Reduction of a Quadratic Form of Rank r to a Sum of r Squares

The rank of the quadratic form $x'Ax$ is defined to be the rank of the matrix A.

THEOREM VI.—A real quadratic form $x'Ax$ of rank r can be transformed by a real non-singular transformation into the form

$$c_1 y_1^2 + c_2 y_2^2 + \ldots + c_r y_r^2,$$

where $c_1, c_2, \ldots c_r$, are all real and non-zero.

Since A is of rank r it contains r linearly independent columns. We can, therefore, by a chain of elementary transformations on the columns, make these r columns the first r columns of the matrix, and reduce all elements in the remaining columns to zero. Since A is symmetric, if we apply the same chain of elementary transformations to the rows, every row after the first r rows will become a zero row, and the leading submatrix A_r of order $r \times r$ will be symmetric and non-singular.

Thus there exists a real, non-singular matrix Q such that

$$Q'AQ = \begin{bmatrix} A_r & O \\ O & O \end{bmatrix}, \quad |A_r| \neq 0.$$

Again, by elementary transformations on the columns, we can reduce all elements above the leading diagonal to zeros. By symmetry, the same transformations applied to the rows reduce

all elements below the leading diagonal to zeros. Hence there exists a real non-singular matrix R such that

$$R'Q'AQR = R'\begin{bmatrix} A_r & O \\ O & O \end{bmatrix} R = \begin{bmatrix} \begin{matrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & c_r \end{matrix} & O \\ O & O \end{bmatrix}$$

A is real, and all the transformations are real, so that all the c's are real. The rank of A is unchanged by elementary transformations. Hence $c_1 c_2 \dots c_r \neq 0$; i.e. $c_k \neq 0$ for any k $(k = 1, 2, \dots, r)$.

Now write $QR = P$, $R'Q' = P'$ and make the real non-singular transformation $x = Py$.

Then $x'Ax = y'P'APy = c_1y_1^2 + c_2y_2^2 + \dots + c_ry_r^2$. This proves the theorem.

It is clear that this reduction can be performed in more than one way, but we can show that, however it is done, the number of positive and the number of negative c's will be the same.

THEOREM VII.—If a real quadratic form of rank r is reduced by one real non-singular transformation to the form

$$c_1y_1^2 + c_2y_2^2 + \dots + c_ry_r^2,$$

and by another to the form

$$\gamma_1z_1^2 + \gamma_2z_2^2 + \dots + \gamma_rz_r^2,$$

the number of positive c's is equal to the number of positive γ's.

Let $x'Ax$ be the given form, and let

$$x'Ax = c_1y_1^2 + c_2y_2^2 + \dots + c_ry_r^2, \text{ where } y = Bx, \ |B| \neq 0,$$

and $x'Ax = \gamma_1z_1^2 + \gamma_2z_2^2 + \dots + \gamma_rz_r^2$, where $z = Cx$, $|C| \neq 0$.

Let h be the number of positive c's and k be the number of positive γ's. We may suppose that the variables y and z are numbered in such a way that the positive c's and the positive γ's come first.

$$\begin{aligned} \text{Then } x'Ax &= c_1y_1^2 + \dots + c_hy_h^2 - |c_{h+1}|y_{h+1}^2 - \dots - |c_r|y_r^2 \\ &= \gamma_1z_1^2 + \dots + \gamma_kz_k^2 - |\gamma_{k+1}|z_{k+1}^2 - \dots - |\gamma_r|z_r^2. \end{aligned}$$

Suppose that $h > k$, and consider the equations

$$z_1 = 0,\ z_2 = 0,\ \dots,\ z_k = 0,\ y_{h+1} = 0,\ y_{h+2} = 0,\ \dots,\ y_n = 0.$$

There are $n - h + k < n$ homogeneous equations in the n variables $x_1, x_2, \ldots, x_n$. There is, therefore, a non-trivial solution $x = X$. Now

$$c_1y_1^2 + \ldots + c_hy_h^2 - |c_{h+1}| y_{h+1}^2 - \ldots - |c_r| y_r$$
$$= \gamma_1z_1^2 + \ldots + \gamma_kz_k^2 - |\gamma_{k+1}| z_{k+1}^2 - \quad - |\gamma_r|$$

and, when $x = X$, we have

$$c_1y_1^2 + \ldots + c_hy_h^2 = - |\gamma_{k+1}| z_{k+1}^2 - \ldots - |\gamma_r| z_r^2.$$

This is impossible unless

$$y_1 = 0, \; y_2 = 0, \; \ldots, \; y_h = 0, \; z_{k+1} = 0, \; z_{k+2} = 0, \; \ldots, \; z_r = 0,$$

when $x = X$.

But $\quad y_{h+1} = 0, y_{h+2} = 0, \ldots, y_n = 0$ when $x = X$.

Hence $y = O$ when $x = X$, and the equation $Bx = O$ has a non-trivial solution $x = X$.

But this implies that $|B| = 0$, contrary to hypothesis.

It follows that h does not exceed k.

Similarly, we can show that k does not exceed h.

Hence $\qquad h = k.$

13·96. The Signature of a Quadratic Form

Let P be the number of positive c's and N the number of negative c's. $P + N$ is equal to the rank r of the form. $P - N$ is defined to be the *signature* of the form, denoted by s. Thus $P = \frac{1}{2}(r + s)$, $N = \frac{1}{2}(r - s)$, and if two quadratic forms have the same rank and the same signature, then P and N are the same for both.

THEOREM VIII.—If $x'Ax$ and $y'By$ are two real quadratic forms with the same rank and signature, there is a real non-singular transformation which transforms $x'Ax$ into $y'By$.

As in Theorem VI, there is a real non-singular transformation $x = PX$ such that

$$x'Ax = c_1X_1^2 + \ldots + c_PX_P^2 - c_{P+1}X_{P+1}^2 - \ldots - c_rX_r^2,$$

where the c's are all positive.

If we now apply the real non-singular transformation

$$z_k = \sqrt{c_k}\,.\,X_k \; (k = 1, 2, \ldots, r), \text{ we obtain}$$

$$x'Ax = z_1^2 + \ldots + z_P^2 - z_{P+1}^2 - \ldots - z_r^2.$$

By similar real non-singular transformations, $y'By$ is transformed into

$$z_1^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_r^2.$$

The inverse transformations, which are also real and non-singular, transform

$$z_1^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_r^2 \text{ into } y'By.$$

It therefore follows that, by a real non-singular transformation, we can transform $x'Ax$ into $y'By$.

EXERCISES XIII

1. If A and B are both diagonal, prove that $AB = BA$.

2. If A is diagonal, prove that $AB = BA$ if, and only if, either B is diagonal or A is scalar.

3. If A and B are both 2×2 skew symmetric matrices, prove that $AB = BA$.

4. A is a 3×3 matrix, and $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find AJ and JA, and prove that $J^2 = I$.

5. The sum of the leading diagonal elements of a square matrix A is called the *trace* of A, written tr. A.

If AB and BA both exist, prove that tr. (AB) = tr. (BA).

6. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \\ -1 & 1 & 6 \end{bmatrix}$, find AB, BA, A^{-1}, B^{-1}, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$. Verify that $B^{-1}A^{-1} = (AB)^{-1}$ and $A^{-1}B^{-1} = (BA)^{-1}$.

7. Solve the equations $x + y + z = 4$, $2x - y + 3z = 1$, $3x + 2y - z = 1$.

8. Find the reciprocal of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix}$.

9. $A = \begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 3 & 8 \end{bmatrix}$. Find adj. A, $|A|$ and verify that

$$A \text{ adj. } A = \text{adj. } A.A = |A| I.$$

10. Prove that the reciprocal of an orthogonal matrix is orthogonal.

11. Prove that the reciprocal of a unitary matrix is unitary.

12. Prove that the reciprocal of a symmetric matrix is symmetric.

13. Prove that the reciprocal of a Hermitian matrix is Hermitian.

14. $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1-i \\ 1+i & 2 \end{bmatrix}$; find AB and BA. A and B are both Hermitian, but neither AB nor BA is Hermitian.

15. Find the rank of the matrices

$$\text{(i)} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}; \quad \text{(ii)} \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}.$$

16. Show that the vectors $x^{(1)} = \{1, 2, 3\}$, $x^{(2)} = \{-1, 2, -1\}$ and $x^{(3)} = \{0, 4, 5\}$ are linearly independent, and express the vector $\{2, 3, -2\}$ as a linear combination of $x^{(1)}, x^{(2)}, x^{(3)}$.

17. Prove that the equations $x + 2y - z = 0$, $x + y - 2z = -1$, $-x + 4y + 7z = 6$ are consistent, and solve them.

18. Show that the equations $x - y + z = 0$, $2x + y - z = 0$, $x + 5y - 5z = 0$ are consistent, and solve them.

19. Discuss the solution of

$$\begin{aligned} ax + by + z &= 1 \\ x + aby + z &= b \\ x + by + az &= 1, \end{aligned}$$

determining when the system has no solution, one solution, and infinitely many solutions. [*Lond. B.Sc.*]

20. Find the necessary and sufficient conditions for the simultaneous equations

$$\begin{aligned} x + ay + a^2z &= 0 \\ x + by + b^2z &= 0 \\ x + cy + c^2z &= 0 \\ x + y + z &= 0 \end{aligned}$$

to have solutions other than $x = y = z = 0$. [*Lond. B.Sc.*]

21. If each of the first three sets of m sets of n constants consists of n terms in arithmetic progression, prove that the m sets are linearly dependent. [*Lond. B.Sc.*]

22. If the matrices A and C are of order $n \times n$, where $|C| \neq 0$, $|A| \neq 0$, and if $D = CAC^{-1}$ and λ is any scalar, prove that

$$|D + \lambda I| = |A + \lambda I|.$$

23. If A is a non-singular square matrix, and C is any matrix of the same order as A, prove that there is a unique matrix X such that $AX = C$, and a unique matrix Y such that $YA = C$. Find X and Y if

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 17 \\ 14 & 47 \end{bmatrix}.$$

[*Lond. B.Sc.*]

24. Determine the nature of the quadric

$$6x^2 + 3y^2 - 2z^2 + 4xy = 14.$$

Find the lengths of its semi-axes, and the equations of its principal planes.

25. Discuss the solution of

$$\begin{aligned} x + a^2y + a^4z &= a \\ x + b^2y + b^4z &= b \\ x + c^2y + c^4z &= c, \end{aligned}$$

determining when the system has no solution, one solution, and infinitely many solutions. [*Lond. B.Sc.*]

26. Find all the 2×2 matrices X which satisfy the equation

$$X^2 + 4X + 3I = O,$$

I being the unit matrix and O the null matrix. [*Lond. B.Sc.*]

27. A is a square matrix of order 3×3. Prove that AA' is a unit matrix if, and only if, the rows of A are the direction cosines of three perpendicular lines.

If P is the matrix $\begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$ and X is a square matrix of order 3×3

such that $XX' = P^2$, show that the columns of X are coordinates of the extremities of a triad of conjugate diameters of the quadric

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1.$$

[*Lond. B.Sc.*]

28. Find a matrix U such that $U^{-1}AU$ is diagonal, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

[*Lond. B.Sc.*]

29. If $A = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}$ find a matrix P of the form $\begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}$ such that $P^{-1}AP$ is a diagonal matrix. Hence express A^r, where r is an integer, in the form of a 2×2 matrix. [*Lond. B.Sc.*]

30. Find the rank of the matrix

$$A = \begin{bmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{bmatrix}.$$

Obtain a formula for the complete set of solutions of the equations $AX = O$, where X is the column vector $\{x_1, x_2, x_3, x_4\}$. Which, if any, of these solutions (*a*) satisfy the equations $x_1 + x_2 + x_3 + x_4 + 1 = 0$ and $x_1 - x_2 - x_3 - x_4 - 3 = 0$; (*b*) are linearly dependent on the pair $\{0, 1, 2, 3\}$, $\{3, 2, 1, 0\}$? [*Lond. B.Sc.*]

31. Determine the exact range of λ for which the form

$$\lambda(x^2 + y^2 + z^2) + 2xy - 2yz + 2zx + t^2$$

is positive definite in (x, y, z, t). Discuss the case $\lambda = 2$. [*Lond. B.Sc.*]

32. If $A = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & 0 \\ 0 & 0 & \end{bmatrix}$, find a set of numbers α, β, γ and an orthogonal matrix B such that $BA = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} B$.

[*Lond. B.Sc.*]

33. If I is the unit $n \times n$ matrix, and A is skew symmetric of order $n \times n$ such that $I + A$ is non-singular, prove that the matrix $B = (I + A)^{-1}(I - A)$ is orthogonal.

Find B in the case $n = 2$, when $A = \begin{bmatrix} 0 & m \\ -m & 0 \end{bmatrix}$, and verify the result by showing that

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ where } m = \tan(\theta/2).$$

[*Lond. B.Sc.*]

34. A is an $m \times n$ matrix. Prove that if A is of rank 1 it can be represented as a product $A = BC$, where B is an $m \times 1$ matrix, and C is a $1 \times n$ matrix. [*Lond. B.Sc.*]

35. Find a real orthogonal transformation of the variables x_1, x_2, x_3, x_4, x_5 which reduces the quadratic form $2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5)$ to one of the type $a_1y_1^2 + a_2y_2^2 + a_3y_3^2 + a_4y_4^2 + a_5y_5^2$, where $y_1, \ldots, y_5$ are the new variables. [*Lond. B.Sc.*]

36. Find a non-trivial solution for the equations

$$\frac{x_1}{a + \lambda_1} + \frac{x_2}{a + \lambda_2} + \frac{x_3}{a + \lambda_3} = 0$$

$$\frac{}{b + \lambda_1} + \frac{x_2}{b + \lambda_2} + \frac{x_3}{b + \lambda_3} = 0,$$

where none of the denominators is zero. [*Lond. B.Sc.*]

37. A is a square $n \times n$ matrix, whose elements are all 0, 1 or -1, and in each row or column there is exactly one element which is not zero. Prove that A^2, A^3, ... are of the same type, and hence show that $A^h = I$ for some positive integer h, where I denotes the unit matrix. [*Lond. B.Sc.*]

38. If A is a real orthogonal matrix, show that its latent roots all have absolute value unity.

Find these latent roots when A is the matrix

$$\begin{bmatrix} 0 & \cos\theta & \sin\theta \\ \cos\phi & -\sin\theta\sin\phi & \cos\theta\sin\phi \\ \sin\phi & \sin\theta\cos\phi & -\cos\theta\cos\phi \end{bmatrix}.$$

39. Prove that the form $14x^2 + 6y^2 + 3z^2 - 4yz - 10xy$ is positive definite, but that $6x^2 + 4y^2 + z^2 + 8zx - 18xy$ is not.

40. Prove that the quadratic form $2x^2 + 4y^2 + 9z^2 + 6yz + 8zx + 6xy$ has rank 3 and signature 1.

CHAPTER XIV

ELIMINATION

IN this chapter we consider the application of determinants to the general problem of elimination.

14·1. The Meaning of Elimination

Suppose we are given a system of $n + 1$ equations, involving n variables. Then n of these $n + 1$ equations would in general be sufficient to determine the values of n variables. If we substitute the values of the variables in the $(n + 1)$th equation which has not been used we would obtain an equation containing only the coefficients of the given $n + 1$ equations. Such a relation when expressed in a rational integral form is called the **eliminant** of the given equations and the process of obtaining it, is called *eliminating* the variables. What the eliminant asserts is that the given equations are *consistent*, i.e. they are satisfied by definite finite values of the variables.

The cumbersome method suggested above of first solving the equations, followed by substitution, is obviously very limited in its application. Thus, *e.g.* in the case of one variable it is not in general possible to solve an equation of higher degree than the fourth. Hence it is necessary to devise methods which do not require the equations to be solved.

Again some methods of elimination may give rise to an extraneous factor in the eliminant.

As an example of the direct method of elimination consider the problem of eliminating x between the two equations

$$ax^2 + 2bx + c = 0, \text{ and } a'x^2 + 2b'x + c' = 0.$$

Solving the equations

$$x = \{-b \pm \sqrt{(b^2 - ac)}\}/a \text{ or } x = \{-b' \pm \sqrt{(b'^2 - a'c')}\}/a'.$$

Hence in order that the equations be simultaneously true:

$$\frac{1}{a}\{-b \pm \sqrt{(b^2 - ac)}\} = \frac{1}{a'}\{-b' \pm \sqrt{(b'^2 - a'c')}\}.$$

It is now necessary to express this result in rational integral form. The result may be written

$$ab' - a'b = \pm a\sqrt{(b'^2 - a'c')} \pm a'\sqrt{(b^2 - ac)}.$$

Squaring both sides,

$$(ab' - a'b)^2 = a^2(b'^2 - a'c') + a'^2(b^2 - ac)$$
$$\pm 2aa'\sqrt{\{(b'^2 - a'c')(b^2 - ac)\}},$$

i.e. $a^2a'c' + a'^2ac - 2aa'bb' = \pm 2aa'\sqrt{\{(b'^2 - a'c')(b^2 - ac)\}}$.

The factor aa' is clearly extraneous so dividing throughout by this factor and squaring

$$(ac' + a'c - 2bb')^2 = 4(b'^2 - a'c')(b^2 - ac).$$

This is the required eliminant.

14·2. Linear Homogeneous Equations

Let $x_1, x_2, x_3, \ldots x_n$ be n variables which satisfy the n homogeneous linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = 0.$$

Divide each of the equations by x_1 and write $y_r = x_r/x_1$. Then

$$a_{12}y_2 + a_{13}y_3 + \ldots + a_{1n}y_n = -a_{11}$$
$$a_{22}y_2 + a_{23}y_3 + \ldots + a_{2n}y_n = -a_{21}$$
$$a_{32}y_2 + a_{33}y_3 + \ldots + a_{3n}y_n = -a_{31}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{n2}y_2 + a_{n3}y_3 + \ldots + a_{nn}y_n = -a_{n1}.$$

Take the last $n - 1$ of these n equations and solve them for $y_2, y_3, \ldots y_n$ as in Chapter XI, § 12·91. Then

$y_2 = (-a_{21}\,a_{33}\,a_{44}\ldots a_{nn})/\Delta$, $y_3 = (a_{22}\cdot - a_{31}\,a_{44}\ldots a_{nn})/\Delta, \ldots$

$y_n = (a_{22}\,a_{33}\,a_{44}\ldots\cdot - a_{n1})/\Delta$, where $\Delta = (a_{22}\,a_{33}\,a_{44}\ldots a_{nn})$.

Substituting in the first equation in the last group of n equations we obtain

$$a_{11}\Delta + a_{12}(-a_{21}\,a_{33}\,a_{44}\ldots a_{nn}) + a_{13}(a_{22}\cdot - a_{31}\,a_{44}\ldots a_{nn}) + \ldots$$
$$+ a_{1n}(a_{22}\,a_{33}\,a_{44}\ldots\cdot - a_{n1}) = 0,$$

i.e. $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \ldots + a_{1n}A_{1n} = 0,$

where $A_{11}, A_{12}, A_{13}, \ldots A_{1n}$ are the cofactors of $a_{11}, a_{12}, a_{13}, \ldots, a_{1n}$ in the determinant $(a_{11}\, a_{22}\, a_{33} \ldots a_{nn})$. Hence the last equation may be written in the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot\cdot & \cdot\cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot\cdot & \cdot\cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot\cdot & \cdot\cdot & a_{3n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot\cdot & \cdot\cdot & a_{nn} \end{vmatrix} = 0.$$

What has been proved above is that *the result of eliminating n variables between n equations which are homogeneous and linear in the variables is that the determinant formed by their coefficients is zero.*

Examples.—(1) *By means of a determinant eliminate x, y, z and w from the equations*

$$\begin{aligned} tx + a(y + z + w) &= 0, \\ ty + b(z + w + x) &= 0, \\ tz + c(w + x + y) &= 0, \\ tw + d(x + y + z) &= 0, \end{aligned}$$

and show that the coefficient of t^2 in the result is

$$-(ab + ac + ad + bc + bd + cd). \quad \text{[}\textit{Lond. B.Sc. Eng.}\text{]}$$

The equations may be written in the form

$$\begin{aligned} tx + ay + az + aw &= 0, \\ bx + ty + bz + bw &= 0, \\ cx + cy + tz + cw &= 0, \\ dx + dy + dz + tw &= 0. \end{aligned}$$

Eliminating x, y, z, w,

$$\Delta = \begin{vmatrix} t & a & a & a \\ b & t & b & b \\ c & c & t & c \\ d & d & d & t \end{vmatrix} = 0.$$

Subtracting the second column from the third and fourth,

$$\Delta = \begin{vmatrix} t & a & 0 & 0 \\ b & t & b-t & b-t \\ c & c & t-c & 0 \\ d & d & 0 & t-d \end{vmatrix} = t\begin{vmatrix} t & b-t & b-t \\ c & t-c & 0 \\ d & 0 & t-a \end{vmatrix}$$

$$-a\begin{vmatrix} b & b-t & b-t \\ c & t-c & 0 \\ d & 0 & t-d \end{vmatrix}$$

In the first of these determinants add the first column to the second and third, and in the other determinant add the sum of the second and third rows on to the first row. Thus

$$\Delta = t\begin{vmatrix} t & b & b \\ c & t & c \\ d & d & t \end{vmatrix} - a\begin{vmatrix} b+c+d & b-c & b-d \\ c & t-c & 0 \\ d & 0 & t-d \end{vmatrix}$$

The coefficient of t^2 is now easily seen to be

$$-cd - bc - bd - a(b + c + d) = -(ab + ac + ad + bc + bd + cd).$$

(2) *Eliminate x' and y' from the equations*

$$ax' + by' + g = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$
$$hx' + by' + f = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$
$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0 \quad \ldots\ldots\ldots \quad (3).$$

Equation (3) may be written

$$x'(ax' + by' + g) + y'(hx' + by' + f) + gx' + fy' + c = 0,$$

whence
$$gx' + fy' + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Eliminating x' and y' from (1), (2), add (4), we have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

[This is the condition that the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent a pair of straight lines. *Cf.* Chapter XI., § 12·71.]

(3) *Eliminate x' and y' from the equations*

$$\frac{ax' + by' + g}{l} = \frac{hx' + by' + f}{m} = \frac{gx' + fy' + c}{n},$$

and
$$lx' + my' + n = 0.$$

Let each fraction be equal to k. Then

$$\begin{aligned} ax' + hy' + g - lk &= 0 \\ hx' + by' + f - mk &= 0 \\ gx' + fy' + c - nk &= 0 \\ lx' + my' + n &= 0. \end{aligned}$$

Treating these as four simultaneous equations in x', y', unity, and k, and eliminating, we have

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

The determinant when expanded may be expressed in the form

$$-(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm),$$

where $A, B, C, \ldots$ are the cofactors of $a, b, c, \ldots$ in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

[The vanishing of the above determinant or the equation

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + Gnl + 2Hlm = 0$$

expresses the condition that the line $lx + my + n = 0$ should touch the conic given by the general equation of the second degree, *i.e.*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.]$$

14·3. Euler's Method

This method is concerned with the elimination of one variable between two equations. Let $f(x)$, $\phi(x)$ be two polynomials in x

of degrees p and q respectively, and consider the problem of eliminating x between the equations $f(x) = 0$, $\phi(x) = 0$.

Let λ be any common root of the equations so that $x - \lambda$ is a factor of $f(x)$ and $\phi(x)$. Thus

$$f(x) = (x - \lambda) f_1(x), \quad \phi(x) = (x - \lambda) \phi_1(x),$$

where $f_1(x)$ and $\phi_1(x)$ are polynomials of degrees $p - 1$, $q - 1$ respectively. The coefficients in $f_1(x)$, $\phi_1(x)$ will depend on λ since these functions are obtained from $f(x)$ and $\phi(x)$ by dividing by $x - \lambda$. Hence

$$\frac{f(x)}{\phi(x)} \equiv \frac{f_1(x)}{\phi_1(x)}, \text{ i.e. } f(x)\,\phi_1(x) \equiv f_1(x)\,\phi(x).$$

This is an identity of degree $p + q - 1$ in x. Equating corresponding coefficients we obtain $p + q$ equations which are homogeneous and of the first degree in the $p + q$ coefficients of $f_1(x)$ and $\phi_1(x)$. Eliminating these coefficients by the method of § 14·2 we obtain the required eliminant in the form of a determinant.

Example.—*Prove that the equations*
$a(x) \equiv a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ *and* $b(x) \equiv b_0x^3 + b_1x^2 + b_2x + b_3 = 0$ *will have a common root if*

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \end{vmatrix} = 0.$$

If $a(x) \equiv (x - \lambda)(a_0'x^2 + a_1'x + a_2')$ *and*
$b(x) \equiv (x - \lambda)(b_0'x^2 + b_1'x + b_2')$ *and*

$$\Delta_1 \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 & b_3 \end{vmatrix},$$

prove that

$$\Delta_1 \times \begin{vmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \\ 0 & 1 & \lambda & \lambda^2 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_0' & a_1' & a_2' & 0 \\ 0 & a_0' & a_1' & a_2' \\ 0 & b_0' & b_1' & b_2' \\ b_0' & b_1' & b_2' & 0 \end{vmatrix}$$ [*Camb. Sch.*]

If λ be the common root, then $\dfrac{a(x)}{b(x)} \equiv \dfrac{a_0'x^2 + a_1'x + a_2'}{b_0'x^2 + b_1'x + b_2'}$,

i.e. $(a_0x^3 + a_1x^2 + a_2x + a_3)(b_0'x^2 + b_1'x + b_2')$
$\equiv (a_0'x^2 + a_1'x + a_2')(b_0x^3 + b_1x^2 + b_2x + b_3)$.

Equating corresponding coefficients on both sides of the identity, we have

$$\begin{aligned} a_0b_0' &= a_0'b_0 \\ a_0b_1' + a_1b_0' &= a_0'b_1 + a_1'b_0 \\ a_0b_2' + a_1b_1' + a_2b_0' &= a_2'b_0 + a_1'b_1 + a_0'b_2 \\ a_1b_2' + a_2b_1' + a_3b_0' &= a_0'b_3 + a_1'b_2 + a_2'b_1 \\ a_2b_2' + a_3b_1' &= a_1'b_3 + a_2'b_2 \\ a_3b_2' &= a_2'b_3. \end{aligned}$$

Arranging these six equations as linear equations in the quantities a_0', a_1', a_2', b_0', b_1', b_2', we obtain

$$\begin{array}{ccccccccccccc} b_0a_0' & + & 0 & + & 0 & - & a_0b_0' & + & 0 & + & 0 & = & 0 \\ b_1a_0' & + & b_0a_1' & + & 0 & - & a_1b_0' & - & a_0b_1' & + & 0 & = & 0 \\ b_2a_0' & + & b_1a_1' & + & b_0a_2' & - & a_2b_0' & - & a_1b_1' & - & a_0b_2' & = & 0 \\ b_3a_0' & + & b_2a_1' & + & b_1a_2' & - & a_3b_0' & - & a_2b_1' & - & a_1b_2' & = & 0 \\ 0 & + & b_3a_1' & + & b_2a_2' & + & 0 & - & a_3b_1' & - & a_2b_2' & = & 0 \\ 0 & + & 0 & + & b_3a_2' & + & 0 & + & 0 & - & a_3b_2' & = & 0. \end{array}$$

Eliminating a_0', a_1', a_2', b_0', b_1', b_2' we obtain

$$\begin{vmatrix} b_0 & 0 & 0 & -a_0 & 0 & 0 \\ b_1 & b_0 & 0 & -a_1 & -a_0 & 0 \\ b_2 & b_1 & b_0 & -a_2 & -a_1 & -a_0 \\ b_3 & b_2 & b_1 & -a_3 & -a_2 & -a_1 \\ 0 & b_3 & b_2 & 0 & -a_3 & -a_2 \\ 0 & 0 & b_3 & 0 & 0 & -a_3 \end{vmatrix} = 0.$$

Taking out common factors and interchanging rows and columns this eliminant is clearly equivalent to the $\Delta = 0$.

Since $a(x) = (x - \lambda)(a_0'x^2 + a_1'x + a_2')$

$$= a_0'x^3 + (a_1' - \lambda a_0')x^2 + (a_2' - \lambda a_1')x - \lambda a_2',$$

it follows that $a_0 = a_0'$, $a_1 = a_1' - \lambda a_0'$, $a_2 = a_2' - \lambda a_1'$, $a_3 = -\lambda a_2'$.

Similarly $b_0 = b_0'$, $b_1 = b_1' - \lambda b_0'$, $b_2 = b_2' - \lambda b_1'$, $b_3 = -\lambda b_2'$.

$$\text{Hence } \Delta_1 \quad \begin{array}{cccc} a_0' & a_1' - \lambda a_0' & a_2' - \lambda a_1' & -\lambda a_2' \\ 0 & a_0' & a_1' - \lambda a_0' & a_2' - \lambda a_1' \\ 0 & b_0' & b_1' - \lambda b_0' & b_2' - \lambda b_1' \\ b_0' & b_1' - \lambda b_0' & b_2' - \lambda b_1' & -\lambda b_2' \end{array}$$

Using the product rule for two determinants

$$\Delta_1 \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ \lambda^2 & \lambda & 1 & 0 \\ \lambda^3 & \lambda^2 & \lambda & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_0' + \lambda a_1' - \lambda^2 a_0' + \lambda^2 a_2 - \lambda^3 a_1' - \lambda^4 a_2 & \lambda a_0' + \lambda^2 a_1' - \lambda^3 a_0' + \lambda^3 a_2' - \lambda_4 a_1' & \lambda b_0' + \lambda^2 b_1' - \lambda^3 b_0' + \lambda^3 b_2' - \lambda^4 b_1' & b_0' + \lambda b_1' - \lambda^2 b_0' + \lambda^2 b_2' - \lambda^3 b_1' - \lambda^4 b_2' \\ a_1' - \lambda a_0' + \lambda a_2' - \lambda^2 a_1' - \lambda^3 a_2' & a_0' + \lambda a_1' - \lambda^2 a_0' + \lambda^2 a_2' - \lambda^3 a_1' & b_0' + \lambda b_1' - \lambda^2 b'_0 + \lambda^2 b_2' - \lambda^3 b_1' & b_1' - \lambda b_0' + \lambda b_2' - \lambda^2 b_1' - \lambda^3 b_2' \\ a_2' - \lambda a_1' - \lambda^2 a_2' & a_1' - \lambda a_0' + \lambda a_2' - \lambda^2 a_1' & b_1' - \lambda b_0' + \lambda b_2' - \lambda^2 b_1' & b_2' - \lambda b_1' - \lambda^2 b_2' \\ -\lambda a_2' & a_2' - \lambda a_1' & b_2' - \lambda b_1' & -\lambda b_2' \end{vmatrix}$$

Multiply the second row by λ and subtract from the first; the third row by λ and subtract from the second, the fourth row by λ and subtract from the third. We obtain

$$\begin{vmatrix} a_0' & 0 & 0 & b_0' \\ a_1' - \lambda a_0' & a_0' & b_0' & b_1' - \lambda b_0' \\ a_2' - \lambda a_1' & a_1' - \lambda a_0' & b_1' - \lambda b_0' & b_2' - \lambda b_1' \\ -\lambda a_2' & a_2' - \lambda a_1' & b_2' - \lambda b_1' & -\lambda b_2' \end{vmatrix}$$

Multiply the first row by λ and add to the second. Thus we obtain

$$\begin{vmatrix} a_0' & 0 & 0 & b_0' \\ a_1' & a_0' & b_0' & b_1' \\ a_2' - \lambda a_1' & a_1' - \lambda a_0' & b_1' - \lambda b_0' & b_2' - \lambda b_1' \\ -\lambda a_2' & a_2' - \lambda a_1' & b_2' - \lambda b_1' & -\lambda b_2' \end{vmatrix}$$

Multiply the second row by λ and add to the third. In the resulting determinant multiply the third row by λ and subtract from the fourth. It then follows that

$$\Delta_1 \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ \lambda^2 & \lambda & 1 & 0 \\ \lambda^3 & \lambda^2 & \lambda & 1 \end{vmatrix} = \begin{vmatrix} a_0' & 0 & 0 & b_0' \\ a_1' & a_0' & b_0' & b_1' \\ a_2' & a_1' & b_1' & b_2' \\ 0 & a_2' & b_2' & 0 \end{vmatrix}$$

Interchanging rows and columns we obtain the required form for the result.

14·4. Sylvester's Method

Another method which leads to the same determinants as those obtained by Euler's method, is known as *Sylvester's Dialytic Method of Elimination.* It has the advantage that it may sometimes be used to obtain the eliminant in the case of more than one variable.

The procedure is as follows. Using the notation of § 14·3 we write

$$f(x) \equiv a_0x^p + a_1x^{p-1} + a_2x^{p-2} + \ldots + a_p$$
$$\phi(x) \equiv b_0x^q + b_1x^{q-1} + b_2x^{q-2} + \ldots + b_q.$$

The $p + q$ equations required are obtained by multiplying $f(x)$ by $1, x, x^2, \ldots, x^{q-1}$ in succession, by multiplying $\phi(x)$ by $1, x, x^2, \ldots, x^{p-1}$ in succession and equating each to zero. The highest power of x involved is x^{p+q-1}. Regarding the powers of x as distinct variables we have $p + q$ equations from which x^{p+q-1}, $x^{p+q-2}, \ldots, x^1, x^0$ may be eliminated by the method of § 14·2.

Example.—*If the equations $x^3 + px^2 + qx + r = 0$ and $x^2 + ax + b = 0$ have a common root, prove that*

$$\begin{vmatrix} 1 & 0 & 1 & a-p \\ a & 1 & p & b-q \\ b & a & q & -r \\ 0 & b & r & 0 \end{vmatrix} = 0.$$

If the equations $x^3 + 2x^2 + 3x + 6 = 0$ and $x^2 + ax - 6 = 0$ have a common root, find the possible values of a and the corresponding common roots. [Cambs. Sch.]

Since the given equations are simultaneously true we obtain on multiplying by powers of x

$$\begin{aligned} 0 + x^3 + px^2 + qx + r &= 0 \\ x^4 + px^3 + qx^2 + rx + 0 &= 0 \\ 0 + 0 + x^2 + ax + b &= 0 \\ 0 + x^3 + ax^2 + bx + 0 &= 0 \\ x^4 + ax^3 + bx^2 + 0 + 0 &= 0. \end{aligned}$$

Eliminating x^4, x^3, x^2, x^1, x^0 we obtain

$$\begin{vmatrix} 0 & 1 & p & q & r \\ 1 & p & q & r & 0 \\ 0 & 0 & 1 & a & b \\ 0 & 1 & a & b & 0 \\ 1 & a & b & 0 & 0 \end{vmatrix} = 0.$$

Subtracting the last row from the second,

$$\begin{vmatrix} 0 & 1 & p & q & r \\ 0 & p-a & q-b & r & 0 \\ 0 & 0 & 1 & a & b \\ 0 & 1 & a & b & 0 \\ 1 & a & b & 0 & 0 \end{vmatrix} = 0.$$

Expanding along the first column it follows that

$$\begin{vmatrix} 1 & p & q & r \\ p-a & q-b & r & 0 \\ 0 & 1 & a & b \\ 1 & a & b & 0 \end{vmatrix} = 0.$$

Interchanging rows and column it easily follows that this condition is equivalent to

$$\begin{matrix} 1 & 0 & 1 & a-p \\ a & 1 & p & b-q \\ b & a & q & -r \\ 0 & b & r & 0 \end{matrix} = 0.$$

In the particular case given $p \quad 2$, $q = 3$, $r = 6$, $b = -6$ and the determinant becomes

$$\Delta = \begin{vmatrix} 1 & 0 & 1 & a-2 \\ a & 1 & 2 & -9 \\ -6 & a & 3 & -6 \\ 0 & -6 & 6 & 0 \end{vmatrix} = 0.$$

Expanding Δ by the usual methods it is easily seen that this condition gives $a^3 + a^2 + 27a + 27 = 0$, i.e. $(a + 1)(a^2 + 27) = 0$.

Hence $a = -1$ or $\pm 3i\sqrt{3}$.

Taking $a = -1$, $x^2 - x - 6 = 0$, $x = 3$ or -2 and the common root is $x = -2$.

Taking $a = +3i\sqrt{3}$ the quadratic becomes $x^2 + 3i\sqrt{3}x - 6 = 0$. The roots are $-i\sqrt{3}$, $-2i\sqrt{3}$, and it is easily verified that $-i\sqrt{3}$ is a root of the given cubic.

Taking $a = -3i\sqrt{3}$, the roots of the quadratic equation are found to be $2i\sqrt{3}$ and $i\sqrt{3}$. The latter is a root of the cubic equation.

14·5. Bezout's Method

The principles involved may be understood by a study of the two examples given below. In the first the equations are of the same degree and in the second they are of different degrees.

Examples.—(1) *Eliminate x between the equations*

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad b_0x^3 + b_1x^2 + b_2x + b_3 = 0.$$

Multiply the first equation by b_0, $b_0x + b_1$, $b_0x^2 + b_1x + b_2$ in succession, and the second by a_0, $a_0x + a_1$, $a_0x^2 + a_1x + a_2$ in succession and subtract the corresponding terms. Thus

$$b_0(a_0x^3 + a_1x^2 + a_2x + a_3) - a_0(b_0x^3 + b_1x^2 + b_2x + b_3) = 0.$$

$$(b_0x + b_1)(a_0x^3 + a_1x^2 + a_2x + a_3) - (a_0x + a_1)(b_0x^3 + b_1x^2 + b_2x + b_3) = 0,$$

$$(b_0x^2 + b_1x + b_2)(a_0x^3 + a_1x^2 + a_2x + a_3) \\ - (a_0x^2 + a_1x + a_2)(b_0x^3 + b_1x^2 + b_2x + b_3) = 0.$$

These equations reduce to

$$\begin{aligned}
&(a_1b_0 - a_0b_1)x^2 + (a_2b_0 - a_0b_2)x + (a_3b_0 - a_0b_3) &= 0\\
&(a_2b_0 - a_0b_2)x^2 + (a_3b_0 - a_0b_3 + a_3b_1 - a_1b_2)x + (a_3b_1 - a_1b_3) &= 0.\\
&(a_3b_0 - a_0b_3)x^2 + (a_3b_1 - a_1b_3)x + (a_3b_2 - a_2b_3) &= 0.
\end{aligned}$$

Regarding different powers of x as distinct variables we obtain an elimination:

$$\begin{vmatrix}
a_1b_0 - a_0b_1 & a_2b_0 - a_0b_2 & a_3b_0 - a_0b_3\\
a_2b_0 - a_0b_2 & a_3b_0 - a_0b_3 + a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3\\
a_3b_0 - a_0b_3 & a_3b_1 - a_1b_3 & a_3b_2 - a_2b_3
\end{vmatrix} = 0.$$

(2) *Eliminate x between the equations*

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad b_0x^2 + b_1x + b_2 = 0.$$

Multiply the first equation by b_0, $b_0x + b_1$ in succession and second by a_0x, $(a_0x + a_1)x$ in succession and subtract the corresponding equations. Thus

$$b_0(a_0x^3 + a_1x^2 + a_2x + a_3) - a_0x(b_0x^2 + b_1x + b_2) = 0.$$

$$(b_0x + b_1)(a_0x^3 + a_1x^2 + a_2x + a_3) - x(a_0x + a_1)(b_0x^2 + b_1x + b_2) = 0,$$

i.e.

$$(a_1b_0 - a_0b_1)x^2 + (a_2b_0 - a_0b_2)x + a_3b_0 = 0.$$

$$(a_2b_0 - a_0b_2)x^2 + (a_3b_0 + a_2b_1 - a_1b_2)x + a_3b_1 = 0.$$

Combining these two equations with $b_0x^2 + b_1x + b_2 = 0$ and eliminating x^2, x^1, x^0 we obtain

$$\begin{vmatrix}
b_0 & b_1 & b_2\\
a_1b_0 - a_0b_1 & a_2b_0 - a_0b_2 & a_3b_0\\
a_2b_0 - a_0b_2 & a_3b_0 + a_2b_1 - a_1b_2 & a_3b_1
\end{vmatrix} = 0.$$

It will be observed that Bezout's method has the advantage of expressing the eliminant as a determinant of lower order than either of the determinants obtained by Euler's or Sylvester's methods.

14·6. Use of Known Identities

We give some trigometrical examples to illustrate the method.

Examples.—(1) ONE VARIABLE. *Eliminate θ from the equations*

$$\sin 3\,(\tfrac{1}{4}\pi + \theta) + 3 \sin (\tfrac{1}{4}\pi + \theta) = 2a,$$
$$\sin 3\,(\tfrac{1}{4}\pi - \theta) + 3 \sin (\tfrac{1}{4}\pi - \theta) = 2b. \qquad \text{[Camb. Sch.]}$$

In this example we make use of the following trigonometrical identities:

(i) $\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$;

(ii) $\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$;

(iii) $\cos 3A = 4 \cos^3 A - 3 \cos A, \; \sin 3A = 3 \sin A - 4 \sin^3 A$;

(iv) $\cos^2 A + \sin^2 A = 1$.

In these identities A, B denote any angles.

Adding and subtracting the two given equations, we have

$$\sin 3\,(\tfrac{1}{4}\pi + \theta) + \sin 3\,(\tfrac{1}{4}\pi - \theta) + 3 \sin (\tfrac{1}{4}\pi - \theta) + 3 \sin (\tfrac{1}{4}\pi + \theta) = 2a + 2b,$$
$$\sin 3\,(\tfrac{1}{4}\pi + \theta) - \sin 3\,(\tfrac{1}{4}\pi - \theta) + 3 \sin (\tfrac{1}{4}\pi + \theta) - 3 \sin (\tfrac{1}{4}\pi - \theta) = 2a - 2b.$$

From (i) and (ii) we obtain

$$2 \sin \tfrac{3}{4}\pi \cos 3\theta + 6 \sin \tfrac{1}{4}\pi \cos \theta = 2a + 2b,$$
$$2 \cos \tfrac{3}{4}\pi \sin 3\theta + 6 \cos \tfrac{1}{4}\pi \sin \theta = 2a - 2b.$$

Substituting $\sin \tfrac{3}{4}\pi = \sin \tfrac{1}{4}\pi = 1/\sqrt{2}, \quad \cos \tfrac{3}{4}\pi = -\cos \tfrac{1}{4}\pi = -1/\sqrt{2}$; the equations take the form

$$\cos 3\theta + 3 \cos \theta = \sqrt{2}\,(a + b), \text{ and } -\sin 3\theta + 3 \sin \theta = \sqrt{2}\,(a - b).$$

From (iii) it follows that $4 \cos^3 \theta = \sqrt{2}\,(a + b)$, $4 \sin^3 \theta = \sqrt{2}\,(a - b)$.

$$\text{Hence } 4^{\frac{2}{3}} \cos^2 \theta = 2^{\frac{1}{3}}\,(a + b)^{\frac{2}{3}}$$
$$4^{\frac{2}{3}} \sin^2 \theta = 2^{\frac{1}{3}}\,(a - b)^{\frac{2}{3}}.$$

Adding, using (iv) and dividing both sides by $2^{\frac{1}{3}}$,

$$2 = (a + b)^{\frac{2}{3}} + (a - b)^{\frac{2}{3}}.$$

This result is readily expressed in rational form. Thus writing

$$(a + b)^2 = u, \quad (a-b)^2 = v, \quad \text{then } 2 = u^{\frac{1}{3}} + v^{\frac{1}{3}}.$$

Cubing both sides of this equation we have

$$8 = u + v + 3u^{\frac{1}{3}}v^{\frac{1}{3}}\,(u^{\frac{1}{3}} + v^{\frac{1}{3}}) = u + v + 6u^{\frac{1}{3}}v^{\frac{1}{3}},$$
$$\text{i.e. } 8 - u - v = 6u^{\frac{1}{3}}v^{\frac{1}{3}}.$$

Cubing again, $(8 - u - v)^3 = 216uv$, i.e. $(4 - a^2 - b^2)^3 = 27\,(a^2 - b^2)^2$.

(2) TWO VARIABLES. *Eliminate θ and ϕ from the equations*

$$\sin \theta + \sin \phi = a, \quad \cos \theta + \cos \phi = b, \quad \tan \theta + \tan \phi = c,$$

obtaining a result in rational form. [*Camb. Sch.*]

In addition to (i) and (iv) of Ex. 1 we use the following identities:

(v) $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$,

$\qquad \sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$;

(vi) $\cos 2A = 2 \cos^2 A - 1$, $\sin 2A = 2 \sin A \cos A$;

(vii) $\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$.

The given equations are

$$\sin\theta + \sin\phi = a \quad \ldots\ldots (1)$$
$$\cos\theta + \cos\phi = b \quad \ldots\ldots (2)$$
$$\tan\theta + \tan\phi = c \quad \ldots\ldots (3)$$

Squaring (1) and (2), adding and using (iv) and (v), we obtain

$$2\{1 + \cos(\theta - \phi)\} = a^2 + b^2.$$

Using (vi) we have

$$4\cos^2\tfrac{1}{2}(\theta - \phi) = a^2 + b^2 \quad \ldots\ldots (4)$$

Applying (i) and (vii) to (1) and (2) we have

$$2\sin\tfrac{1}{2}(\theta + \phi)\cos\tfrac{1}{2}(\theta - \phi) = a \quad \ldots\ldots (5)$$
$$2\cos\tfrac{1}{2}(\theta + \phi)\cos\tfrac{1}{2}(\theta - \phi) = b \quad \ldots\ldots (6)$$

From (3), $c = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\phi}{\cos\phi}$

$= \{\sin\theta\cos\phi + \cos\theta\sin\phi\}/\cos\theta\cos\phi,$

$= \sin(\theta + \phi)/\cos\theta\cos\phi,$ from (v),

$= 2\sin\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta + \phi)/\cos\theta\cos\phi,$ from (vi),

$= ab/2\cos^2\frac{1}{2}(\theta - \phi)\cos\theta\cos\phi,$ from (5) and (6),

$= 2ab/(a^2 + b^2)\cos\theta\cos\phi,$ from (4).

Hence $\cos\theta\cos\phi = 2ab/c(a^2 + b^2) \quad \ldots\ldots (7)$

Combining this result with (2),

$$(\cos\theta - \cos\phi)^2 = (\cos\theta + \cos\phi)^2 - 4\cos\theta\cos\phi$$
$$= b^2 - \frac{8ab}{c(a^2 + b^2)} = \frac{a^2b^2c + b^4c - 8ab}{c(a^2 + b^2)} = \lambda^2, \text{ say.}$$

Taking the positive square root,

$$\cos\theta = \tfrac{1}{2}(b + \lambda) = p, \text{ say, } \cos\phi = \tfrac{1}{2}(b - \lambda) = q, \text{ say.}$$

If the negative square root is taken p and q are interchanged. Squaring (1),

$$\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi = a^2,$$

i.e. $1 - p^2 + 1 - q^2 + 2\sin\theta\sin\phi = a^2$, from (iv),

i.e. $2 - p^2 - q^2 - a^2 = -2\sin\theta\sin\phi.$

Squaring again, $(2 - p^2 - q^2 - a^2)^2 = 4\sin^2\theta\sin^2\phi$

$= 4(1 - p^2)(1 - q^2)$, from (iv).

Since λ^2 is rational it follows that in order to show the eliminant is in rational form it is only necessary to prove that $p^2 + q^2$ and p^2q^2 involve even powers of λ only. This is obviously the case.

(3) THREE VARIABLES. *If*

$$l = \cos\alpha + \cos\beta + \cos\gamma \quad \ldots\ldots (1)$$
$$m = \sin\alpha + \sin\beta + \sin\gamma \quad \ldots\ldots (2)$$
$$p = \cos 2\alpha + \cos 2\beta + \cos 2\gamma \quad \ldots\ldots (3)$$
$$q = \sin 2\alpha + \sin 2\beta + \sin 2\lambda \quad \ldots\ldots (4)$$

prove that $(p - l^2 + m^2)^2 + (q - 2lm)^2 = 4(l^2 + m^2)$. [*Camb. Sch.*]

In this example the identities (i)-(vii) given in Exs. 1, 2 are sufficient. Squaring and adding (1) and (2) and using (iv), (v)

$$l^2 + m^2 = 3 + 2\cos(\alpha - \beta) + 2\cos(\beta - \gamma) + 2\cos(\gamma - \alpha) \quad \ldots\ldots (5)$$

Next squaring and subtracting (2) from (1), and using (v), (vi),

$$l^2 - m^2 = \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 2\cos(\alpha+\beta) + 2\cos(\beta+\gamma) + 2\cos(\gamma+\alpha)$$

$$= p + 2\cos(\alpha+\beta) + 2\cos(\beta+\gamma) + 2\cos(\gamma+\alpha) \text{ from (3).}$$

Multiplying (1) and (2) and using (v), (vi),

$$2lm = \sin 2\alpha + \sin 2\beta + \sin 2\gamma + 2\sin(\alpha+\beta) + 2\sin(\beta+\gamma) + 2\sin(\gamma+\alpha)$$

$$= q + 2\sin(\alpha+\beta) + 2\sin(\beta+\gamma) + 2\sin(\gamma+\alpha), \text{ from (4).}$$

Hence $\frac{1}{4}(p - l^2 + m^2)^2 + \frac{1}{4}(q - 2lm)^2$

$$= \Sigma\cos^2(\alpha+\beta) + \Sigma\sin^2(\alpha+\beta) + 2\Sigma\cos(\alpha+\beta)\cos(\beta+\gamma) + 2\Sigma\sin(\alpha+\beta)\sin(\beta+\gamma)$$

$$= 3 + 2\Sigma\{\cos(\alpha+\beta)\cos(\beta+\gamma) + \sin(\alpha+\beta)\sin(\beta+\gamma)\}$$

$$= 3 + 2\Sigma\cos(\alpha+\beta-\beta-\gamma), \text{ from (v)}$$

$$= 3 + l^2 + m^2 - 3, \text{ from (5).}$$

Thus the required eliminant is

$$(p - l^2 + m^2)^2 + (q - 2lm)^2 = 4(l^2 + m^2).$$

Alternative method, using complex numbers. The four given equations are equivalent to

$$l + im = e^{i\alpha} + e^{i\beta} + e^{i\gamma} \quad \ldots\ldots\ldots\ldots \quad (6)$$

$$p + iq = e^{2i\alpha} + e^{2i\beta} + e^{2i\gamma} \quad \ldots\ldots\ldots\ldots \quad (7)$$

since $\cos\theta + i\sin\theta = e^{i\theta}$. Squaring (6), we obtain

$$l^2 - m^2 + 2ilm = e^{2i\alpha} + e^{2i\beta} + e^{2i\gamma} + 2\Sigma e^{i(\alpha+\beta)}.$$

$$\text{i.e. } l^2 - m^2 - p + i(2lm - q) = 2\Sigma e^{i(\alpha+\beta)} \quad \ldots\ldots\ldots \quad (8)$$

Using the conjugate function

$$l^2 - m^2 - p - i(2lm - q) = 2\Sigma e^{-i(\alpha+\beta)} \quad \ldots\ldots\ldots\ldots \quad (9)$$

Multiplying (8) and (9) together gives

$$(l^2 - m^2 - p)^2 + (2lm - q)^2 = 4\left(\Sigma e^{i(\alpha+\beta)}\right)\left(\Sigma e^{-i(\alpha+\beta)}\right)$$

$$= 4\left\{3 + \Sigma e^{\pm i(\alpha-\beta)}\right\}.$$

Combining (6) with the conjugate equation

$$l - im = e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma}$$

it follows that

$$l^2 + m^2 = \left(\Sigma e^{i\alpha}\right)\left(\Sigma e^{-i\alpha}\right) = 3 + \Sigma e^{\pm i(\alpha-\beta)}.$$

Hence $(l^2 - m^2 - p)^2 + (2lm - q)^2 = 4(l^2 + m^2)$, as before.

14·7. Change of Variables

Sometimes an elimination can be carried out more simply by first changing the given variables.

Example.—*Eliminate* (i) x, y, z and (ii) a, b, c *from the equations*

$$\frac{by}{z} + \frac{cz}{y} = a, \quad \frac{cz}{x} + \frac{ax}{z} = b, \quad \frac{ax}{y} + \frac{by}{x} = c. \qquad \text{[Camb. Sch.]}$$

a, b, c can be eliminated immediately since the three equations are linear and homogeneous in a, b, c. Thus:

$$-a+\frac{y}{z}b+\frac{z}{y}c=0$$

$$\frac{x}{z}a-b+\frac{z}{x}c=0$$

$$\frac{x}{y}a+\frac{y}{x}b-c=0.$$

The required eliminant is $\begin{vmatrix} -1 & \frac{y}{z} & \frac{z}{y} \\ \frac{x}{z} & -1 & \frac{z}{x} \\ \frac{x}{y} & \frac{y}{x} & -1 \end{vmatrix}=0.$

In order to eliminate x, y, z it is convenient to replace them by variables u, v, w where $u=y/z,\ v=z/x,\ w=x/y$. Then

$$bu+c/u=a \quad\dots\dots\dots\dots\dots\dots \quad (1)$$

$$cv+a/v=b \quad\dots\dots\dots\dots\dots\dots \quad (2)$$

$$aw+b/w=c \quad\dots\dots\dots\dots\dots\dots \quad (3)$$

$$uvw=1 \quad\dots\dots\dots\dots\dots\dots \quad (4)$$

Multiplying (1), (2) and (3) together and using (4),

$$2abc+ac^2.\frac{1}{u^2}+a^2b.\frac{1}{v^2}+b^2c.\frac{1}{w^2}+a^2c.w^2+bc^2.v^2+ab^2.u^2=abc,$$

i.e. $a\left(\frac{c^2}{u^2}+b^2u^2\right)+b\left(\frac{a^2}{v^2}+c^2v^2\right)+c\left(\frac{b^2}{w^2}+a^2w^2\right)=-abc \dots\dots$ (5)

Squaring (1), (2) and (3),

$$b^2u^2+\frac{c^2}{u^2}=a^2-2bc$$

$$c^2v^2+\frac{a^2}{v^2}=b^2-2ac$$

$$a^2w^2+\frac{b^2}{w^2}=c^2-2ab.$$

Substituting in (5),

$$a(a^2-2bc)+b(b^2-2ac)+c(c^2-2ab)=-abc,$$

i.e. $a^3+b^3+c^3=5abc.$

14·8. Use of Ratios

Theorems on ratios can sometimes be used to effect an elimination. The method may be seen by a study of the following examples.

Examples.—(1) *Show that if* $x^3+y^3+z^3=3mxyz$ *and*

$$ax^2+by^2+cz^2=0$$

$$ayz+bzx+cxy=0,$$

then $a^3+b^3+c^3=3mabc.$ [*Camb. Sch.*]

The second and third equations are linear and homogeneous in the letters a, b, c. Using the rule of cross-multiplication (Vol. I, Chap. IV, § 4·7),

$$\frac{a}{x(y^3 - z^3)} = \frac{b}{y(z^3 - x^3)} = \frac{c}{z(x^3 - y^3)} = k \text{ (say)}.$$

Then $\quad a^3 + b^3 + c^3 = k^3\, \Sigma x^3 (y^3 - z^3)^3,$

$$3mabc = 3mk^3xyz\,(y^3 - z^3)(z^3 - x^3)(x^3 - y^3)$$
$$= k^3 (y^3 - z^3)(z^3 - x^3)(x^3 - y^3)(x^3 + y^3 + z^3).$$

$$(a^3 + b^3 + c^3 - 3mabc)/k^3$$
$$= \Sigma x^3 (y^3 - z^3)^3 - \Sigma x^3 (y^3 - z^3)(z^3 - x^3)(x^3 - y^3)$$
$$= \Sigma[x^3 (y^3 - z^3)\{(y^3 - z^3)^2 - (z^3 - x^3)(x^3 - y^3)\}]$$
$$= \Sigma\,[x^3 (y^3 - z^3)(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)]$$
$$= (x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)\, \Sigma x^3 (y^3 - z^3) = 0.$$

Hence $a^3 + b^3 + c^3 - 3mabc = 0$.

(2) *Eliminate x and y from the equations*

$$\frac{a^4}{x^2} + \frac{b^4}{y^2} = (a + b)^2,$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{p^2}.$$

From the first two, by multiplication we have

$$a^2 + b^2 + \frac{a^4}{b^2}\cdot\frac{y^2}{x^2} + \frac{b^4}{a^2}\cdot\frac{x^2}{y^2} = (a + b)^2$$

$$\text{i.e. } \left(\frac{a^2}{b}\cdot\frac{y}{x} - \frac{b^2}{a}\cdot\frac{x}{y}\right)^2 = 0, \quad \text{or} \quad \frac{x^2}{a^3} = \frac{y^2}{b^3}.$$

$$\text{Hence } \frac{x^2}{a^3} = \frac{y^2}{b^3} = \frac{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}{\dfrac{a^3}{a^2} + \dfrac{b^3}{b^2}} = \frac{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}}{\dfrac{a^3}{a^4} + \dfrac{b^3}{b^4}},$$

$$\text{that is } \frac{1}{a + b} = \frac{\dfrac{1}{p^2}}{\dfrac{1}{a} + \dfrac{1}{b}} \quad \text{or } p^2 = ab.$$

(3) *Eliminate λ from the equations*

$$\frac{\dfrac{h}{a^2 + \lambda}}{l} = \frac{\dfrac{k}{b^2 + \lambda}}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$\frac{h^2}{a^2 + \lambda} + \frac{k^2}{b^2 + \lambda} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

The second equation may be written

$$k^2 (a^2 + \lambda) + h^2 (b^2 + \lambda) = (a^2 + \lambda)(b^2 + \lambda) \quad \ldots\ldots\ldots \text{(3)}$$

From (1) we have

$$\frac{a^2+\lambda}{\frac{h}{l}} = \frac{b^2+\lambda}{\frac{k}{m}} = \frac{k^2(a^2+\lambda)+h^2(b^2+\lambda)}{\frac{k^2h}{l}+\frac{h^2k}{m}}$$

$$= \frac{\sqrt{\{(a^2+\lambda)(b^2+\lambda)\}}}{\sqrt{\frac{hk}{lm}}} = \frac{a^2-b^2}{\frac{h}{l}-\frac{k}{m}}.$$

Thus $\dfrac{(a^2+\lambda)(b^2+\lambda)}{\frac{hk}{lm}} = \dfrac{\{k^2(a^2+\lambda)+h^2(b^2+\lambda)\}\{a^2-b^2\}}{\left(\frac{k^2h}{l}+\frac{h^2k}{m}\right)\left(\frac{h}{l}-\frac{k}{m}\right)}$

Using (3) we have $lm\left(\dfrac{k}{l}+\dfrac{h}{m}\right)\left(\dfrac{h}{l}-\dfrac{k}{m}\right) = a^2-b^2$

i.e. $(km+hl)(hm-kl) = (a^2-b^2)\,lm.$

14·9. Special Methods

The general methods of elimination which have been considered are sometimes advantageous, but in many cases it will be necessary to employ some special artifice which may be suggested by the nature of the particular equations involved. We now give some miscellaneous examples.

Examples.—(1) *Show that the result of eliminating x and y from the three equations*

$$\frac{1}{x-a}+\frac{1}{y-a}=\frac{1}{a} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{1}{x-b}+\frac{1}{y-b}=\frac{1}{b} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$x^2+y^2=2(a^2+b^2) \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

is $a^2+b^2-6ab=0.$ [*M.T.*]

From (i) and (ii) we have

$$2x+2y=\frac{xy}{a}+3a \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

$$2x+2y=\frac{xy}{b}+3b \quad \ldots\ldots\ldots\ldots \text{(v)}$$

Adding (iv) and (v)

$$4(x+y)=xy\left(\frac{1}{a}+\frac{1}{b}\right)+3(a+b),$$

$$\text{i.e. } 4(x+y)=(a+b)\left(\frac{xy}{ab}+3\right) \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

Subtraction of (iv) from (v) gives $\dfrac{xy}{ab}(a-b)=3(a-b).$

From (i) and (ii) it is clear that $a \neq b$, for $a=b$ implies that the equations are identical. Hence

$$\frac{xy}{ab}=3 \quad \ldots\ldots\ldots\ldots \text{(vii)}$$

Substituting in (vi), it follows that

$$x + y = \tfrac{3}{2}(a + b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(viii)}$$

Squaring both sides of (viii) and subtracting corresponding sides of (iii),

$$8xy = a^2 + b^2 + 18ab.$$

From (vii), $8xy = 24ab.$

Subtracting, we obtain the eliminant, $0 = a^2 + b^2 - 6ab.$

(2) *If the equations*

$$\begin{aligned} axy + bx + cy + d &= 0 \\ ayz + by + cz + d &= 0 \\ azw + bz + cw + d &= 0 \\ awx + bw + cx + d &= 0, \end{aligned}$$

are satisfied by values of x, y, z, w which are all different, show that $b^2 + c^2 = 2ad.$

[*Camb. Sch.*]

We first eliminate x between the first and fourth equations, and then z between the second and third equations.

$$x(ay + b) + (cy + d) = 0, \quad x(aw + c) + (bw + d) = 0.$$

$$\text{Hence} \quad \frac{cy + d}{ay + b} = \frac{bw + d}{aw + c}.$$

Similarly from the other pair of equations we obtain

$$\frac{by + d}{ay + c} = \frac{cw + d}{aw + b}.$$

On simplification these equations reduce to

$$(ca - ab)\,yw + (c^2 - ad)\,y - w\,(b^2 - ad) + cd - bd = 0$$

$$(ca - ab)\,yw + (ad - b^2)\,y - w\,(ad - c^2) + cd - bd = 0.$$

Subtracting, $y\,(b^2 + c^2 - 2ad) - w\,(b^2 + c^2 - 2ad) = 0.$

Hence either $b^2 + c^2 - 2ad = 0$ or $y = w$. From the question $y \neq w$. Hence $b^2 + c^2 - 2ad = 0.$

(3) *Eliminate x, y, z from the equations*

$$\frac{y}{z} - \frac{z}{y} = a, \quad \frac{z}{x} - \frac{x}{z} = b, \quad \frac{x}{y} - \frac{y}{x} = c.$$

Substitute $u = y/z$, $v = z/x$, $w = x/y$. Then the equations become

$$u - \frac{1}{u} = a, \quad v - \frac{1}{v} = b, \quad w - \frac{1}{w} = c, \quad uvw = 1.$$

Adding the first three equations,

$$u + v + w - \left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) = a + b + c.$$

i.e. $uvw\,(u + v + w) - vw - uw - uv = a + b + c$, since $uvw = 1$,

i.e. $1 - vw - uw - uv + uvw\,(u + v + w) - u^2v^2w^2 = a + b + c$,

i.e. $(1 - vw)(1 - uw)(1 - uv) = a + b + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

If in this equation we change the signs of v, w, the effect is to change b into $-b$ and c into $-c$, leaving a unaltered with $uvw = 1$ as before.

Hence $(1 - vw)(1 + uw)(1 + uv) = a - b - c$ .... (2)

Similarly $(1 + vw)(1 - uw)(1 + uv) = -a + b - c$ ...... (3)

$(1 + vw)(1 + uw)(1 - uv) = -a - b + c$.......... (4)

Multiplying (1), (2), (3) and (4) together, we have

$$(1 - v^2w^2)^2 (1 - u^2w^2)^2 (1 - u^2v^2)^2$$
$$= (a + b + c)(a - b - c)(-a + b - c)(-a - b + c).$$

Remembering that $uvw = 1$ it follows that the left-hand side is

$$\left(\frac{1}{vw} - vw\right)^2 \left(\frac{1}{uw} - uw\right)^2 \left(\frac{1}{uv} - uv\right)^2 = \left(u - \frac{1}{u}\right)^2 \left(v - \frac{1}{v}\right)^2 \left(w - \frac{1}{w}\right)^2$$
$$= a^2b^2c^2.$$

Hence required eliminant is

$$(a + b + c)(a - b - c)(-a + b - c)(-a - b + c) = a^2b^2c^2.$$

(4) *Prove that if*

$y^2 + yz + z^2 = a^2$ .... (i)

$z^2 + zx + x^2 = b^2$ .... (ii)

$x^2 + xy + y^2 = c^2$ .... (iii)

$yz + zx + xy = 0$ .... (iv)

then $a \pm b \pm c = 0$. [*Camb. Sch.*]

Adding (i), (ii), (iii) and using (iv)

$2x^2 + 2y^2 + 2z^2 = a^2 + b^2 + c^2$ .... (v)

Squaring both sides of (v),

$$(a^2 + b^2 + c^2)^2 = 4(x^4 + y^4 + z^4 + 2x^2y^2 + 2y^2z^2 + 2z^2x^2).$$

Multiply (i), (ii), (iii) together in pairs and add. Then

$$a^2b^2 + b^2c^2 + c^2a^2$$
$$= x^4 + y^4 + z^4 + 3(y^2z^2 + z^2x^2 + x^2y^2) + yz^3 + y^3z + x^3y + xy^3$$
$$+ xz^3 + x^3z + 3(x^2yz + xy^2z + 3xyz^2)$$
$$= \tfrac{1}{4}(a^2 + b^2 + c^2)^2 + (y^2z^2 + z^2x^2 + x^2y^2 + 2xy^2z + 2x^2yz + 2xyz^2)$$
$$+ yz(x^2 + y^2 + z^2) + zx(x^2 + y^2 + z^2) + xy(x^2 + y^2 + z^2)$$
$$= \tfrac{1}{4}(a^2 + b^2 + c^2)^2 + (xy + yz + zx)^2 + (x^2 + y^2 + z^2)(xy + yz + zx)$$
$$= \tfrac{1}{4}(a^2 + b^2 + c^2)^2.$$

Thus $a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = 0$,

i.e. $(a^2 - b^2 - c^2)^2 = 4b^2c^2$.

Taking the square root of both sides,

$a^2 - b^2 - c^2 = \pm 2bc$, i.e $a^2 = (b \pm c)^2$.

Hence $a = \pm(b \pm c)$, which is the required form.

(5) *By eliminating x and y from the equations*

$$a_1x + b_1y = 0, \text{ and } a_2x + b_2y = 0,$$

prove the identity

$$\begin{vmatrix} a_1^3 & 3a_1^2b_1 & 3a_1b_1^2 & b_1^3 \\ a_1^2a_2 & a_1^2b_2 + 2a_1a_2b_1 & 2a_1b_1b_2 + a_2b_1^2 & b_1^2b_2 \\ a_1a_2^2 & a_2^2b_1 + 2a_1a_2b_2 & 2a_2b_1b_2 + a_1b_2^2 & b_1b_2^2 \\ a_2^3 & 3a_2^2b_2 & 3a_2b_2^2 & b_2^3 \end{vmatrix} \quad (a_1b_2 - a_2b_1)^6.$$

The eliminant of the two given equations is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$,

i.e. $a_1b_2 - a_2b_1 = 0$.

Since $a_1x + b_1y = 0$, $a_2x + b_2y = 0$, it follows that

$$(a_1x + b_1y)^3 = 0, \quad (a_1x + b_1y)^2 (a_2x + b_2y) = 0,$$
$$(a_1x + b_1y)(a_2x + b_2y)^2 = 0, \quad (a_2x + b_2y)^3 = 0.$$

These equations may be written in the form

$$\left.\begin{aligned} &a_1^3x^3 + 3a_1^2b_1x^2y + 3a_1b_1^2xy^2 + b_1^3y^3 = 0 \\ &a_1^2a_2x^3 + (a_1^2b_2 + 2a_1a_2b_1)x^2y + (2a_1b_1b_2 + a_2b_1^2)xy^2 + b_1^2b_2y^3 = 0 \\ &a_1a_2^2x^3 + (a_2^2b_1 + 2a_1a_2b_2)x^2y + (2a_2b_1b_2 + a_1b_2^2)xy^2 + b_1b_2^2y^3 = 0 \\ &a_2^3x^3 + 3a_2^2b_2x^2y + 3a_2b_2^2xy^2 + b_2^3y^3 = 0. \end{aligned}\right.$$

Regarding x^3, x^2y, xy^2, y^3 as variables the eliminant is

$$\begin{vmatrix} a_1^3 & 3a_1^2b_1 & 3a_1b_1^2 & b_1^3 \\ a_1^2a_2 & a_1^2b_2 + 2a_1a_2b_1 & 2a_1b_1b_2 + a_2b_1^2 & b_1^2b_2 \\ a_1a_2^2 & a_2^2b_1 + 2a_1a_2b_2 & 2a_2b_1b_2 + a_1b_2^2 & b_1b_2^2 \\ a_2^3 & 3a_2^2b_2 & 3a_2b_2^2 & b_2^3 \end{vmatrix} = 0.$$

Clearly this determinant must be some power of $(a_1b_2 - a_2b_1)$. Consider $a_1^6b_2^6$, which is the highest power of a_1b_2, and which occurs as part of the principal term. It is clearly seen that this can occur in no other term in the determinant. It follows that the value of the determinant is $(a_1b_2 - a_2b_1)^6$.

(6) *Eliminate a, b, c from the equations*

$$(c + a - b)(a + b - c) = bcx,$$
$$(a + b - c)(b + c - a) = cay,$$
$$(b + c - a)(c + a - b) = abz.$$

Multiplying the three equations together,

$$(c + a - b)^2 (a + b - c)^2 (b + c - a)^2 = a^2b^2c^2xyz.$$

Again $abc(x + y + z) = \Sigma a(c + a - b)(a + b - c)$.

$= (a + b - c)\{a(c + a - b) + b(b + c - a)\} + c(b + c - a)(c + a - b)$.. (i)

Now $(b + c - a)(c + a - b) = \{c + (b - a)\}\{c - (b - a)\}$
$= c^2 - (b - a)^2$.

Hence $(b + c - a)(c + a - b) - 4ab = c^2 - (b + a)^2$
$= (c - b - a)(c + b + a)$.

From (i) it follows that

$abc(x + y + z) + 4abc$
$= (a + b - c)\{a(c + a - b) + b(b + c - a)\} + c(c - b - a)(c + b + a)$
$= (a + b - c)\{a(c + a - b) + b(b + c - a) - c(c + b + a)\}$
$= (a + b - c)\{(a^2 - 2ab + b^2) - c^2\}$
$= (a + b - c)(a - b - c)(a - b + c)$.

$\therefore a^2b^2c^2(x + y + z + 4)^2 = (a + b - c)^2(b + c - a)^2(c + a - b)^2$
$= a^2b^2c^2xyz$.

Thus the required eliminant is $(x + y + z + 4)^2 = xyz$.

EXERCISES XIV

1. Eliminate x, y, z from the equations

$$(b - c)x + (c - a)y + (a - b)z = 0.$$
$$(c - a)x + (a - b)y + (b - c)z = 0.$$
$$(a - b)x + (b - c)y + (c - a)z = 0.$$

2. Find all the values of t for which the equations

$$(t - 1)x + (3t + 1)y + 2tz = 0,$$
$$(t - 1)x + (4t - 2)y + (t + 3)z = 0,$$
$$2x + (3t + 1)y + 3(t - 1)z = 0,$$

are compatible, and find the ratios $x:y:z$ when t has the smallest of these values. What happens when t has the greatest of these values?

[*Lond., B.Sc. Eng.*]

3. Prove that if $\dfrac{A}{1 - ax} + \dfrac{B}{1 - bx} + \dfrac{C}{1 - cx}$ be expanded in ascending powers of x, where $|x| < 1$, no three consecutive terms can vanish, given that a, b, c are all different. [*Madras, B.Sc.*]

4. Prove that if there are three numbers x_1, x_2, x_3 not all zero, such that $\sum_{r=1}^{3} a_{ir}x_r = 0$, $(i = 1, 2, 3)$, then there are three numbers y_1, y_2, y_3 not all zero, such that $\sum_{r=1}^{3} a_{ri}y_r = 0$, $(i = 1, 2, 3)$. [*M.T.*]

5. Evaluate the determinant $\begin{vmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 0 & 1 & a' & b' \\ 1 & a' & b' & 0 \end{vmatrix}$

Show that the result of equating this determinant to zero is the same as the result of eliminating x from the equations,

$$x^2 + ax + b = 0, \quad x^2 + a'x + b' = 0.$$

[*Lond., B.Sc. Eng.*]

6. Obtain the eliminant of the equations $ax^2 + bx + c = 0$, $x^3 = k$ in the form

$$\begin{vmatrix} a & b & c \\ b & c & ak \\ c & ak & bk \end{vmatrix} = 0$$

7. Use the equations $x^3 + px^2 + qx + r = 0$, $x^2 + bx + c = 0$ to express

$$(r + bc - pc)^2 - (cq - c^2 - br)(bp - b^2 - q + c)$$

as a determinant of the fifth order. [*Camb. Sch.*]

8. Eliminate x and y from the equations

$$a_1x^3 + b_1x^2y + c_1xy^2 + d_1y^3 = 0, \quad a_2x^3 + b_2x^2y + c_2xy^2 + d_2y^3 = 0,$$

obtaining the result as a sixth order determinant.

9. Eliminate θ from the equation

$$a^2 + b^2 = \frac{a\cos 3\theta + b\sin 3\theta}{\cos^3\theta} = \frac{b\cos 3\theta - a\sin 3\theta}{\sin^3\theta}.$$

[*Camb. Sch.*]

10. Given that $a \sin 2\theta + b \sin \theta = c$ and that $a \cos 2\theta + b \cos \theta = d$, eliminate θ and thus find an equation connecting a, b, c and d. [*Camb. Sch.*]

11. Eliminate θ and ϕ from the equations

$$\sin \theta + \sin \phi = a$$
$$\cos \theta + \cos \phi = b$$
$$\tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\phi = c.$$

12. Eliminate θ from the equations

$$x(1 + \sin^2\theta - \cos\theta) - y \sin\theta (1 + \cos\theta) = c(1 + \cos\theta),$$
$$y(1 + \cos^2\theta) - x \sin\theta \cos\theta = c \sin\theta.$$ [*Camb. Sch.*]

13. Eliminate α and β from the equations

$$x \cos \alpha + y \sin \alpha = a,$$
$$x \cos \beta + y \sin \beta = a,$$
$$2 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta = 1.$$

14. Eliminate A, H, B from the equations

$$Ax^2 + 2Hxy + By^2 = 0,$$
$$Ab - 2Hh + Ba = 0,$$
$$Ab' - 2Hh' + Ba' = 0.$$

15. Prove that the result of eliminating α', β', γ' from the equations

$$\frac{a\alpha' + h\beta' + g\gamma'}{l} = \frac{h\alpha' + b\beta' + f\gamma'}{m} = \frac{g\alpha' + f\beta' + c\gamma'}{n},$$
$$l\alpha' + m\beta' + n\gamma' = 0,$$

is $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$, where $A, B, C, \ldots$, are cofactors of $a, b, c, \ldots$ in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

16. If a, b, c are the sides of a triangle and

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0,$$
$$a\alpha + b\beta + c\gamma = 0,$$
$$l\alpha + m\beta + n\gamma = 0.$$

then $l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0$.

17. Given that

$ax + by = 1$, $a'x + b'y = 1$, $ab = a'b'$, and $a + b + a' + b' = c$,

show that in general $x + y = cxy$. [*Camb. Sch.*]

18. When x and y are eliminated from the equations

$$x^2 - y^2 = ax - by; \quad 4xy = bx + ay; \quad x^2 + y^2 = 1,$$

prove that $(a + b)^{\frac{2}{3}} + (a - b)^{\frac{2}{3}} = 2$. [*Camb. Sch.*]

19. Eliminate x and y from the equations

$$\frac{1}{x - a} + \frac{1}{y - a} = \frac{1}{a}, \quad \frac{1}{x - b} + \frac{1}{y - b} = \frac{1}{b}, \quad x^2 + y^2 = 2(a^2 + b^2).$$

[*Camb. Sch.*]

20. Eliminate x, y, z from the equations

$$y+z=a, \quad z+x=b, \quad x+y=c, \quad x^2+y^2+z^2=d^2.$$

21. Eliminate x, y, z from the equations

$$\frac{y}{z}+\frac{z}{y}=a, \quad \frac{z}{x}+\frac{x}{z}=b, \quad \frac{x}{y}+\frac{y}{x}=c.$$

22. Eliminate x, y, z from the equations

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=a,$$

$$\frac{x}{z}+\frac{y}{x}+\frac{z}{y}=b,$$

$$\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)=c.$$

[*Camb. Sch.*]

23. Prove that if

$$\frac{a}{l^2}+\frac{b}{m^2}+\frac{c}{n^2}=\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=\frac{al}{x^3}+\frac{bm}{y^3}+\frac{cn}{z^3}=0,$$

where a, b, c are not zero and no two of $\frac{x}{l}, \frac{y}{m}, \frac{z}{n}$ are equal, then

$$\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=0.$$

[*Camb. Sch.*]

24. Prove that if the equations

$$cy^2-2fyz+bz^2=0, \quad az^2-2gzx+cx^2=0, \quad bx^2-2hxy+ay^2=0,$$

are satisfied by all values of x, y, z different from zero, then

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Show that when this condition is satisfied, the equations are, in general, satisfied by two sets of values of x, y, z not proportional to one another.

[*Camb. Sch.*]

CHAPTER XV

THE THEORY OF EQUATIONS

IN this and the following chapter we consider properties of the equation $f(x) = 0$, where $f(x)$ is a polynomial in x.

15·1. Some Elementary Properties

Let $f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$ denote a polynomial of the nth degree in x in which the coefficients $p_0, p_1, p_2, \ldots, p_{n-1}, p_n$ are all *real*, $p_0 > 0$; $p_n \neq 0$. Suppose further that $f(a_i) = 0$, $i = 1, 2, 3, \ldots n$. Then $a_1, a_2, \ldots, a_n$ are the roots of $f(x) = 0$.

The quantities a_i may not be real and need not all be distinct.

Thus if

$$f(x) = (x - a)^r \phi(x),$$

where $\phi(a) \neq 0$ then $x = a$ is an *r-multiple** root of $f(x) = 0$. In particular when $r = 2$, $f(x) = 0$ is said to have a double root at $x = a$.

In Vol. I, Chapter XVIII the following theorems have been proved.

THEOREM I.—$f(x) \equiv p_0 \prod_{i=1}^{n} (x - a_i)$.

THEOREM II.—Assuming $f(x) = 0$ has one root, then the equation has n and only n roots.

THEOREM III.—The imaginary (or complex) roots of $f(x) = 0$ occur in pairs.

THEOREM IV.—If the coefficients $p_0, p_1, p_2, \ldots, p_n$ are rational, the irrational roots of $f(x) = 0$ occur in pairs.

In the same chapter the following transformations have been considered.

(*a*) *The roots of* $f(-x) = 0$ *are* $-a_1, -a_2, -a_3, \ldots, -a_n$.

(*b*) *The roots of* $f(x/k) = 0$ *are* $ka_1, ka_2, ka_3, \ldots, ka_n$.

(*c*) *The roots of* $f(1/x) = 0$ *are* $1/a_1, 1/a_2, 1/a_3, \ldots, 1/a_n$.

(*d*) *The roots of* $f(\sqrt{x}) = 0$, *when expressed in rational form are* $a_1^2, a_2^2, a_3^2, \ldots a_n^2$.

* *Multiple* roots are frequently referred to as *repeated* roots.

(*e*) *The roots of* $f(x+k) = 0$ *are* $a_1 - k,\ a_2 - k,\ a_3 - k, \ldots$ $a_n - k$.

If $k > 0$ the last transformation is called *reducing the roots of an equation by k.*

We now pass to the consideration of theorems which will determine the nature of the roots, and their position when real. The nature of the roots is determined when we know how many are imaginary and how many are real.

The position of a real root is determined by its position in the scale of real numbers. Thus, *e.g.* as a first approximation we might determine two consecutive integers between which a root lies. For a complete discussion on the separation of the roots of an equation Sturm's theorem* is necessary. This provides a definite method of separation. It will be seen, however, that the theorems considered in this chapter do provide us in many cases with a good deal of information.

15·21. Descartes' Rule of Signs

Consider a polynomial whose terms are arranged in the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n,$$

i.e. in *descending powers of x.* Then a *change of sign* is said to occur when any particular term has the opposite sign to the term which immediately precedes it. Thus, *e.g.* in the polynomial

$$2x^4 + 3x^3 - 4x^2 + 5x - 1$$

we can represent the sequence of signs as

$$+ \quad + \quad - \quad + \quad -.$$

Here there are three changes in sign, for $-4x^2$ is of opposite sign to $3x^3$, $5x$ is of opposite sign from $-4x^2$, and -1 is of opposite sign to $5x$. The three changes are $+$ to $-$, $-$ to $+$, and $+$ to $-$.

Now let a be any positive real number, $f(x)$ any polynomial. Then *the product* $(x - a) f(x)$ *when expanded has at least one more change of sign than* $f(x)$.

Suppose in the first instance that there are no terms missing from $f(x)$, i.e. $p_r \neq 0$, $r = 0, 1, 2, \ldots n$, and consider any arbitrary arrangement of signs corresponding to $f(x)$, say

$$+++--++-+-$$

* See, *e.g.* Burnside and Panton, *Theory of Equations*, Vol. I.

Multiply by $x - a$, whose signs are $+ -$. Then considering only the signs we may represent the working as follows:

$$\begin{array}{ccccccccccc}
+ & + & + & - & - & + & + & - & + & - & \\
 & & & & & & & & + & - & \\
\hline
+ & + & + & - & - & + & + & - & + & - & \\
 & - & - & - & + & + & - & - & + & - & + \\
\hline
+ & \pm & \pm & - & \pm & + & \pm & - & + & - & +
\end{array}$$

The $\pm$ signs together indicate that in the total the actual sig is ambiguous.

Now in $f(x)$ there are five changes of sign. Consider now the product $(x - a) f(x)$. It will be observed that the ambiguous signs occur when two or more similar signs come together in $f(x)$. It follows that omitting for the moment the consideration of the last term in $(x - a) f(x)$ there are *at least* as many changes in sign in $(x - a) f(x)$ as there are in $f(x)$. But the last term introduces one change in sign. This is obvious in the above example where $f(x)$ finishes with a change in sign. If $f(x)$ does not have this property it is easily seen that there is one additional change in sign. For consider, *e.g.*

$$\begin{array}{cccccc}
- & + & + & + & + & \\
 & & & + & - & \\
\hline
- & + & + & + & + & \\
 & + & - & - & - & - \\
\hline
- & + & \pm & \pm & \pm & -
\end{array}$$

The most unfavourable case occurs when three ambiguous terms all have the same sign. If the sign is $+$ then the last term provides the additional change in sign. If the sign is $-$ the term fifth from the end and an ambiguous term provides the additional change.

It follows that if $a > 0$, $(x - a) f(x)$ has at least one more change of sign than $f(x)$.

The fact that no changes of sign are lost because some of the coefficients in $f(x)$ are zero is easily seen by considering examples.

We may now deduce the rule of signs.

THEOREM V.—The equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$ or more negative roots than there are changes of sign in $f(-x)$.

Consider first the case of positive roots. Let $\phi(x)$ be the polynomial formed of the factors corresponding to the negative and imaginary roots of $f(x)$, and suppose that $a_1, a_2, \ldots a_r$ are the positive real roots.

Let $\phi(x)$ have s changes in sign when $s \geqslant 0$. Then

$(x - a_1)\,\phi(x)$ has at least $(s + 1)$ changes in sign,

$(x - a_1)(x - a_2)\,\phi(x)$ at least $(s + 2)$ changes,

. . . .

$$f(x) = (x - a_1)(x - a_2) \ldots (x - a_r)\,\phi(x)$$

has at least $s + r$ changes.

Also r, the number of real positive roots, is less than or equal to $s + r$, since $s \geqslant 0$.

The result for negative roots follows from the transformation (*a*) of § 15·1. For the real positive roots of $f(-x) = 0$ are the real negative roots of $f(x) = 0$.

15·22. Detection of Imaginary Roots from Descartes' Rule

Let $f(x)$ be a polynomial of degree n and suppose that $f(x)$ has p changes and $f(-x)$ has q changes in sign. Then if

$$p + q < n$$

we conclude that $f(x) = 0$ has at least $n - p - q$ imaginary roots.

It will be seen in what follows that the method is of service only if some of the terms are missing from $f(x)$, *i.e.* some of the coefficients are zero.

Example.—*Prove that the equation $x^7 - 2x^4 + 3x^3 - 1 = 0$ has at least four imaginary roots.*

Write $f(x) \equiv x^7 - 2x^4 + 3x^3 - 1$. The number of changes in sign in $f(x)$ are 3. Hence there are at most three positive roots.

Again $f(-x) = (-x)^7 - 2(-x)^4 + 3(-x)^3 - 1$
$$= -x^7 - 2x^4 - 3x^3 - 1.$$

Since there are no changes in sign in $f(-x)$ it follows that there are no negative roots.

Thus there are at most three real roots, *i.e.* at least four imaginary roots.

Let p denote the number of changes in sign in $f(x)$, q the number of changes in sign of $f(-x)$, h the number of positive roots, k the number of negative roots of $f(x) = 0$.

$$\text{Now } f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n.$$

$$(-1)^n f(-x) = p_0x^n - p_1x^{n-1} + p_2x^{n-2} - \ldots + (-1)^n p_n.$$

Then if *none of the coefficients are zero* it is clear that the number of changes in sign in $f(x)$ plus the number of changes in sign in $f(-x)$ is exactly n. In this case $p+q=n$.

Again, if *some of the coefficients are zero* it is clear that the number of changes in sign cannot exceed n. Hence in general

$$p+q \leqslant n.$$

Also from Descartes' rule, $h \leqslant p$, $k \leqslant q$. So that

$$h+k \leqslant p+q \leqslant n \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

We now prove that *if all the roots of the equation* $f(x)=0$ *are real then* $h=p$, $k=q$.

In this case $h+k=n$ and hence from (i) $h+k=p+q=n$.

Suppose that $h<p$ so that $k>q$. This implies that the number of negative roots is greater than the number of changes in sign of $f(-x)$, a result which contradicts Descartes' Rule.

Since $h \leqslant p$ it follows that $h=p$. Similarly $k=q$.

15·31. Some General Theorems

As before $f(x)$ denotes the polynomial $\sum_{r=0}^{n} p_r x^{n-r}$, $p_0>0$, while the letters a, b will denote real numbers.

THEOREM VI.—**If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one real root of $f(x)=0$ between a and b.**

THEOREM VII.—**If n is odd the equation $f(x)=0$ has at least one real root whose sign is opposite to that of p_n, the last term.**

THEOREM VIII.—**If n is even and $p_n<0$ then $f(x)=0$ has at least two real roots, one positive and one negative.**

THEOREM VI.—Since $f(x)$ is a polynomial it is continuous for all values of x. (Chapter II., § 2·72.) Hence $f(x)$ can only change sign* by passing through the value zero, *i.e.* by passing through a real root of $f(x)=0$.

* It should be noted that $f(x)$ *need not* change sign as $f(x)$ passes through a root of $f(x)=0$. For if a is a r-multiple root, then $f(x)=(x-a)^r\phi(x)$, where $\phi(a)\neq 0$. Since $\phi(x)$ is continuous, it follows that for $|x-a|$ sufficiently small, $\phi(x)$ will keep the same sign. Hence the changes in sign of $f(x)$, as x passes through the value $x=a$, will be determined entirely by $(x-a)^r$. If r is *odd*, $(x-a)^r$ will change sign, while if r be *even*, no change in sign takes place as x passes through $x=a$.

The property that is being asserted is that *if $f(x)$ does change sign*, then *x must have passed through* a root of $f(x)=0$.

Since $f(a)$ and $f(b)$ have opposite signs it follows that there is at least one real root intermediate in value between a and b.

The result can be stated more generally as follows. *If $f(a)$ and $f(b)$ have opposite signs then there are an odd number of real roots between a and b; if $f(a)$ and $f(b)$ have the same sign then either there is an even number of real roots between a and b or there are no real roots at all between a and b.*

It should be observed that in this statement multiple roots are counted according to their degree of multiplicity. Thus, *e.g.*

if $f(x) = (x - 2)^2 (x^2 + 1)$, then $f(1) = 2$, $f(3) = 10$.

We conclude that there is an even number of roots between 1 and 3 or else there are no real roots in this range. Actually the two roots are $x = 2, 2$, the values being *coincident*, $x = 2$ being a double root.

We can deduce from the theorem that *if there exists no real value of x for which $f(x) = 0$ then $f(x)$ is always positive.*

It is convenient to make use of the notation $\phi(\infty)$, where $\phi(x)$ denotes any function of x. In accordance with the definition adopted in connexion with limits

$$\phi(\infty) = \lim_{x \to \infty} \phi(x).$$

If $\phi(x) \to +\infty$ as $x \to \infty$ we write

$$\phi(\infty) = +\infty,$$

while if $\phi(x) \to -\infty$ as $x \to \infty$ we write

$$\phi(\infty) = -\infty.$$

A similar form is used for $\phi(-\infty)$.

In the present case $f(\infty) = +\infty$; for when x is large the *highest power* of x is the dominating term and $p_0 > 0$.

Since $f(x)$ is continuous and never zero it must always be positive. If $f(x) < 0$ for any value of x there would also be a value of x such that $f(x) = 0$, which is contrary to the hypothesis.

THEOREM VII.—This follows immediately from Theorem VI. For since $p_0 > 0$ and n is odd,

$$f(-\infty) = -\infty, \quad f(0) = p_n, \quad f(+\infty) = +\infty.$$

If $p_n < 0$, there is at least one real root between $x = 0$ and $x = +\infty$, *i.e.* a positive root.

If $p_n > 0$, there is a real root between $x = 0$ and $x = -\infty$, *i.e.* a negative root.

THEOREM VIII.—In this case, $f(-\infty) = +\infty$, $f(0) = p_n$, $f(+\infty) = +\infty$. Arranging the corresponding values of x opposite the corresponding signs for $f(x)$ we have since $p_n < 0$,

$$\begin{array}{cc} -\infty & + \\ 0 & - \\ +\infty & + \end{array}$$

Hence there is a real root between $-\infty$ and 0, and another between 0 and $+\infty$.

Examples.—(1) *Prove that if n is even the equation $x^n = 1$ has two and only two real roots, one positive and one negative.*

Write $f(x) = x^n - 1$, Then since n is even,

$$f(+\infty) = +\infty,$$
$$f(0) = -1,$$
$$f(-\infty) = +\infty.$$

It follows that there are at least two real roots, one positive and the other negative.

Since $f(x) = f(-x)$ and $f(x)$ has only one change in sign it follows that $f(x) = 0$ has at most one positive and one negative root.

(2) *By means of the equation $(x + b)(x + c) - f^2 = 0$, prove that the equation in x*

$$\begin{vmatrix} x+a & h & g \\ h & x+b & f \\ g & f & x+c \end{vmatrix} = 0$$

has three roots which are separated by the two roots of the first equation. It may be assumed that a, b, c, f, g, h are all real and different from zero.

[*Camb. Sch.*]

If $\Delta(x)$ denote the determinant, we have on expansion

$$\Delta(x) = (x+a)(x+b)(x+c) - f^2(x+a) - g^2(x+b) - h^2(x+c) + 2fgh.$$

Let α and β, $\alpha < \beta$ be the roots of the equation

$$(x+b)(x+c) = f^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Then, writing $\lambda^2 = (b-c)^2 + 4f^2$, $\lambda > 0$, we have

$$\alpha = -\tfrac{1}{2}(b + c + \lambda), \quad \beta = -\tfrac{1}{2}(b + c - \lambda).$$

We may suppose without loss of generality that $b > c$. Then

$$\lambda > b - c > 0.$$

Again $2\alpha = -b - c - \lambda < -b - c - (b - c)$, i.e. $\alpha < -b$.

Again $2\beta = -b - c + \lambda > -b - c + (b - c)$, i.e. $\beta > -c$.

Thus $\alpha < -b < -c < \beta$.

Substitute the values $-\infty, \alpha, \beta, +\infty$ in $\Delta(x)$. Clearly

$$\Delta(-\infty) = -\infty, \quad \Delta(+\infty) = +\infty,$$
$$\Delta(\alpha) = 2fgh - g^2(\alpha + b) - h^2(\alpha + c).$$

Since $\alpha + b < 0$, and $b > c$, $\alpha + c < 0$.

Write $a + b = -\mu^2$, $a + c = -\nu^2$, where $\mu, \nu > 0$, $\mu^2\nu^2 = f^2$. Then

$$\Delta(a) = g^2\mu^2 \pm 2\mu\nu gh + h^2\nu^2 = (g\mu \pm h\nu)^2 > 0.$$

Similarly it may be proved that $\Delta(\beta) < 0$.

Arranging the values of x against the corresponding signs for $\Delta(x)$:

$-\infty$	$-$
a	$+$
β	$-$
$+\infty$	$+$

we see that the three roots of $\Delta(x) = 0$ are real and are separated by the roots of (i).

15·32. Zero Coefficients

We now consider a property of equations in which some of the coefficients are zero.

THEOREM IX.—*If r consecutive coefficients in $f(x)$ are zero then*

(i) if r is even, the equation $f(x) = 0$ has at least r imaginary roots;

(ii) if r is odd, there are at least $r + 1$ or at least $r - 1$ imaginary roots according as the terms which immediately succeed and precede the group of terms with zero coefficients, have the same or opposite signs.

Suppose that the group of consecutive zero coefficients is

$$p_s,\ p_{s+1},\ p_{s+2},\ \ldots\ p_{s+r-1},$$

so that $f(x) = p_0x^n + p_1x^{n-1} + \ldots + p_{s-1}x^{n-s+1}$
$$+ p_{s+r}x^{n-s-r} + \ldots + p_n.$$

Write $F(x) = p_0x^n + p_1x^{n-1} + \ldots + p_{s-1}x^{n+s-1}$
$$+ q_sx^{n-s} + \ldots + q_{s+r-1}x^{n-s-r+1} + p_{s+r}x^{n-s-r} + \ldots + p_n,$$

where $q_s, q_{s+1}, \ldots q_{s+r-1}$ are different from zero. Let p, q be the changes in sign in $f(x)$ and $f(-x)$, respectively, P, Q and p', q' the corresponding quantities for $F(x)$ and $p_{s-1}x^{n-s+1} + p_{s+r}x^{n-s-r}$ respectively. Now

$$p_{s-1}x^{n-s+1} + q_sx^{n-s} + \ldots + q_{s+r-1}x^{n-s-r+1} + p_{s+r}x^{n-s-r} = x^{n-s-r}\phi(x),$$

where $\phi(x)$ is a polynomial of degree $r + 1$ which has no zero coefficients. Hence the number of changes in sign in $\phi(x)$ together with the number of changes in sign in $\phi(-x)$ is $(r + 1)$. [See § 15·22.] Hence

$$P + Q = (p + q) - (p' + q') + (r + 1).$$

Now $P + Q \leqslant n$, since $F(x)$ is a polynomial of degree n. Thus

$$(r + 1) - (p' + q') \leqslant n - (p + q) \quad \ldots\ldots\ldots\ (I)$$

Again from Descartes' rule of signs the number of real roots of $f(x) = 0$ cannot exceed $p + q$, i.e. $f(x) = 0$ has at least $n - (p + q)$ imaginary roots. It follows from (I) that the equation has at least $(r + 1) - (p' + q')$ imaginary roots.

CASE (I).—(*r even*): $p_{s-1}x^{n-s+1} + p_{s+r}x^{n-s-r}$
$$= x^{n-s-r}(p_{s-1}x^{r+1} + p_{s+r}) \text{ and}$$
$$(-x)^{r+1} = -x^{r+1}.$$

If p_{s-1}, p_{s+r} have the same sign, $p' = 0, q' = 1$, so that $p' + q' = 1$.

If p_{s-1}, p_{s+r} have opposite signs, $p' = 1, q' = 0$. As before $p' + q' = 1$.

$$\text{Hence } (r + 1) - (p' + q') = r.$$

Thus when r is even there are at least r imaginary roots.

CASE II.—(*r odd*): $p_{s-1}(-x)^{r+1} + p_{s+r} = p_{s-1}x^{r+1} + p_{s+r}$.

If p_{s-1}, p_{s+r} have the same sign, $p' = 0, q' = 0$ so that $p' + q' = 0$ and $(r + 1) - (p' + q') = r + 1$. Hence there are $r + 1$ imaginary roots.

If p_{s-1}, p_{s+r} have opposite signs, $p' = 1, q' = 1$, so that $(r + 1) - (p' + q') = r - 1$. Hence when the coefficients have opposite signs there are at least $r - 1$ imaginary roots.

Example.—*Prove that if* $(n - 1)a_1^2 - 2na_2$ *is negative the roots of the equation*
$$f(x) = x^n + a_1x^{n-1} + \ldots + a_n = 0$$
are not all real. [*Camb. Sch.*]

$f(x + h)$
$$= (x + h)^n + a_1(x + h)^{n-1} + a_2(x + h)^{n-2} + \ldots + a_n$$
$$= x^n + nhx^{n-1} + \frac{n(n-1)}{2!}h^2x^{n-2} + \ldots$$
$$+ a_1x^{n-1} + (n - 1)a_1hx^{n-2} + \ldots$$
$$+ a_2x^{n-2} + \ldots$$
$$\ldots$$
$$= x^n + \{nh + a_1\}x^{n-1} + \left\{\frac{n(n-1)}{2!}h^2 + (n-1)a_1h + a_2\right\}x^{n-2} + \ldots$$

Write $h = -a_1/n$. Then
$$f(x + h) = x^n + \frac{1}{2n}\{-(n - 1)a_1^2 + 2na_2\}x^{n-2} + \ldots$$

If the coefficients of x^n and x^{n-2} have the same sign the equation $f(x + h) = 0$ will have at least two imaginary roots. The condition is
$$-(n - 1)a_1^2 + 2na_2 > 0, \text{ i.e. } (n - 1)a_1^2 - 2na_2 < 0.$$

Now the roots of $f(x+h) = 0$ are those of $f(x) = 0$ increased by a_1/n. [§ 13·1.] Since a_1/n is real it follows that if $f(x+h) = 0$ has at least two imaginary roots, the equation $f(x) = 0$ possesses the same property.

EXERCISES XV

1. Prove that if b, c and d are positive, the equation
$$x^4 + bx^2 + cx - d = 0$$
has one positive, one negative, and two imaginary roots.

2. Prove that if $q > 0$, $r > 0$ the cubic equation
$$x^3 + qx + r = 0$$
has one negative and two imaginary roots.

3. Determine the nature of the roots of the equation
$$x^4 + 18x^3 + 40x + 12 = 0.$$

4. Prove that the equation $x^n + 1 = 0$ has no real roots when n is even, and only one real root, which is negative, when n is odd.

5. Prove that the roots of the equation $\sum_{r=0}^{n} x^{2r} = 0$ are all imaginary.

6. Prove that the equation $x^7 - 2x^1 + 3x^3 - 1 = 0$ has no negative roots and at least four imaginary roots.

7. Find the nature of the roots of the equation
$$4x^4 + 13x^3 + 5x - 4 = 0.$$

8. Prove that the equation $x^9 + x^5 + x^4 + x^2 + 1 = 0$ has one real root, which is negative, and eight imaginary roots.

9. Prove that if λ_r, a_r, k are real, the roots of the equation
$$\sum_{r=1}^{n} \frac{\lambda_r^2}{x - a_r} = x - k \text{ are all real.}$$

15·41. Relations between the Coefficients and the Roots

It is clear that we may assume without loss of generality that p_0, the coefficient of x^n, is unity. For if a given equation is not of this form it may be reduced to it by dividing both sides by a constant factor. Thus

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = \prod_{r=1}^{n} (x - a_r),$$

where $a_1, a_2, a_3, \ldots a_n$ are the n roots of $f(x) = 0$.

Let S_r denote the sum of the product of the roots taken r at a time. Thus $S_1 = \sum_{r=1}^{n} a_r$; $S_2 = \Sigma a_r a_s$, $r = 1, 2, 3 \ldots n$,

$s = 1, 2, 3 \ldots n, r \neq s; \ldots; S_n = a_1 a_2 a_3 \ldots a_n$. Then

$$\prod_{r=1}^{n} (x - a_r) = x^n - S_1 x^{n-1} + S_2 x^{n-2} + \ldots + (-1)^r S_r x^{n-r} + \ldots + (-1)^n S_n$$

Equating corresponding coefficients,

$$p_1 = -S_1, \ p_2 = S_2, \ \ldots, \ p_r = (-1)^r S_r, \ldots, p_n = (-1)^n S_n.$$

From this result it follows that:

(i) every root of $f(x) = 0$ is a factor of p_n, *i.e.* the absolute term;

(ii) if all the roots are positive, the coefficients $p_1, p_2 \ldots, p_n$ must be alternately positive and negative;

(iii) if all the roots are negative, the coefficients $p_1, p_2, \ldots p_n$ must all be positive.

It is easily seen that (ii) and (iii) may also be deduced from Descartes' rule of signs.

It will be observed that the above discussion gives n distinct relations between the roots. These n relations, however, do not facilitate the general solution of an equation. For if we attempt to determine a particular root a it will be found that the original equation will be reached with a instead of x.

Example.—*If α, β, γ are three of the roots of the quartic*

$$ax(1 - x^2) + b(1 - x^4) = c(1 + x^2)^2,$$

prove that $(\beta\gamma + \gamma\alpha + \alpha\beta)^2 + (\alpha\beta\gamma)^2 = (\alpha + \beta + \gamma)^2 + 1$. [*Camb. Sch.*]

The given equation may be written in the form

$$(b + c)x^4 + ax^3 + 2cx^2 - ax - b + c = 0.$$

Let δ be the fourth root of the quartic. Then

$$\alpha + \beta + \gamma + \delta = -a/(b + c),$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 2c/(b + c),$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = a/(b + c),$$
$$\alpha\beta\gamma\delta = (c - b)/(b + c).$$

Writing $u = \alpha + \beta + \gamma$, $v = \beta\gamma + \gamma\alpha + \alpha\beta$, $w = \alpha\beta\gamma$ these equations become

$$u + \delta = -a/(b + c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$v + u\delta = 2c/(b + c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$w + v\delta = a/(b + c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (iii)$$
$$w\delta = (c - b)/(b + c) \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

From (i) and (iii), $\delta = -(u + w)/(1 + v)$.

Subtracting (iv) from (ii) $v + u\delta - w\delta = 1$.

Hence $v - \dfrac{u(u+w)}{1+v} + \dfrac{(u+w)w}{1+v} = 1.$

This reduces to $w^2 + v^2 = 1 + u^2$, which is the required result.

(2) *If the roots of the equation* $x^3 + lx^2 + mx + n = 0$ *are the cosines of the angles of a triangle, prove that* $l^2 = 2m + 2n + 1$. [*Camb. Sch.*]

Let the roots of the equation be $\cos\alpha, \cos\beta, \cos\gamma$ where α, β, γ are the angles of a triangle. Then

$$\Sigma\cos\alpha = -l,\quad \Sigma\cos\alpha\cos\beta = m,\quad \cos\alpha\cos\beta\cos\gamma = -n.$$

$$l^2 = (\Sigma\cos\alpha)^2 = \Sigma\cos^2\alpha + 2\,\Sigma\cos\alpha\cos\beta = \Sigma\cos^2\alpha + 2m$$
$$= 2m + 2n + \Sigma\cos^2\alpha + 2\cos\alpha\cos\beta\cos\gamma.$$

Now $\Sigma\cos^2\alpha + 2\cos\alpha\cos\beta\cos\gamma$

$$= \cos\alpha(\cos\alpha + \cos\beta\cos\gamma) + \cos\beta(\cos\beta + \cos\alpha\cos\gamma) + \cos^2\gamma$$
$$= \cos\alpha\{-\cos(\beta+\gamma) + \cos\beta\cos\gamma\}$$
$$+ \cos\beta\{-\cos(\alpha+\gamma) + \cos\alpha\cos\gamma\} + \cos^2\gamma,\quad \alpha+\beta+\gamma = \pi,$$
$$= \cos\alpha\sin\beta\sin\gamma + \cos\beta\sin\alpha\sin\gamma + \cos^2\gamma$$
$$= \sin\gamma(\cos\alpha\sin\beta + \cos\beta\sin\alpha) + \cos^2\gamma$$
$$= \sin\gamma\sin(\alpha+\beta) + \cos^2\gamma = \sin^2\gamma + \cos^2\gamma = 1.$$

Hence $l^2 = 2m + 2n + 1$.

(3) *The roots of the equation*

$$x^n - p_{n-1}x^{n-1} + p_{n-2}x^{n-2} - p_{n-3}x^{n-3} + \dots + (-1)^{n-1}p_1x + (-1)^n p_0 = 0$$

are all positive and not all equal. Prove that

$$p^2_{n-1} > \frac{2np_{n-2}}{n-1} > n^2 p_0^{\frac{2}{n}},\ \textit{and}\ p_1 > np_0^{1-\frac{1}{n}}.\quad [\textit{Camb. Sch.}]$$

Write $f(x) = \sum_{r=0}^{n} (-1)^r p_{n-r}x^{n-r},\ p_n = 1.$

Since all the roots are positive there must be n changes in sign in $f(x)$. As the signs are alternately positive and negative, this can only occur if $p_0, p_1, \dots, p_{n-1}$ are all positive.

Consider $f(x+h) = \sum_{r=0}^{n} (-1)^r p_{n-r}(x+h)^{n-r}$

Proceeding as in the example of § 15·32 and writing $h = p_{n-1}/n$, we have

$$f(x+h) = x^n + \frac{1}{2n}\{-(n-1)p^2_{n-1} + 2np_{n-2}\}x^{n-2} + \dots$$

Now the roots of the equation $f(x+h) = 0$ are those of $f(x) = 0$ decreased by p_{n-1}/n [§ 15·1]. Thus the roots of $f(x+h) = 0$ are real.

If the coefficient of x^{n-2} is positive or zero then the equation has at least two imaginary roots. [§ 15·32]. Hence

$$-(n-1)p^2_{n-1} + 2np_{n-2} < 0,$$
i.e. $p^2_{n-1} > 2np_{n-2}/(n-1)$ .... (i)

Now if a_r, $r = 1, 2, \ldots n$ are the n roots of $f(x) = 0$ then $\Sigma\, a_r a_s = p_{n-2}$. Also $\Sigma a_r a_s$ contains ${}_nC_2 = \frac{1}{2}n(n-1)$ terms, all of which are positive and which are not all equal. Hence the arithmetic mean of the quantities $a_r a_s$ is greater than their geometric mean [Vol. I., § 15·8]. Thus

$$\frac{\Sigma a_r a_s}{\frac{1}{2}n(n-1)} > (\Pi a_r a_s)^{\frac{1}{n}} = (\Pi a_r)^{\frac{2}{n}}.$$

Now $\Pi a_r = p_0$. Hence

$$\frac{p_{n-2}}{\frac{1}{2}n(n-1)} > p_0^{\frac{2}{n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Combining (i) and (ii) the first of the required inequalities is obtained.

Again, if σ denote the sum of the products of the roots taken $n-1$ at time, then $\sigma = p_1$. Also the sum σ contains ${}_nC_{n-1}$, i.e. n terms. Thus

$$\frac{p_1}{p_0} = \frac{\sigma}{\Pi a_r} = \sum_{r=1}^{n} \frac{1}{a_r},$$

the quantities $1/a_r$ being positive and not all equal. Applying the theorem on arithmetic and geometric means,

$$\frac{\sum_{r=1}^{n} \frac{1}{a_r}}{n} > \left\{ \prod_{r=1}^{n} 1/a_r \right\}^{\frac{1}{n}}, \text{ i.e. } \frac{p_1}{np_0} > \left(\frac{1}{p_0}\right)^{\frac{1}{n}}$$

$$\text{i.e. } p_1 > np_0^{1-\frac{1}{n}}.$$

15·42: Some Applications

Although the n equations obtained in § 15·41 are of no help in the *general* solution of an equation, they are frequently helpful in dealing with particular equations in which some special relation is known to exist among the roots. The method is illustrated in the examples given below.

Examples.—(1) *Solve the equation* $6x^4 - 3x^3 + 8x^2 - x + 2 = 0$ *having given that it has a pair of roots whose sum is zero.* [*M.T.*.]

Let $\alpha, \beta, \gamma, \delta$ be the roots of $6x^4 - 3x^3 + 8x^2 - x + 2 = 0$. Then

$$\alpha + \beta + \gamma + \delta = \tfrac{1}{2}$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \tfrac{4}{3}$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \tfrac{1}{6}$$
$$\alpha\beta\gamma\delta = \tfrac{1}{3}.$$

We may also write $\alpha + \beta = 0$.

Substituting $\alpha = -\beta$ the equations become

$$\gamma + \delta = \tfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$
$$-\alpha^2 + \gamma\delta = \tfrac{4}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$
$$-\alpha^2(\gamma + \delta) = \tfrac{1}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$
$$-\alpha^2\gamma\delta = \tfrac{1}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

From (i) and (iii) $\alpha^2 = -\frac{1}{3}$, $\alpha = \pm i/\sqrt{3}$. From (iv) $\gamma\delta = 1$ ...... (v)

From (i) and (v) we obtain $\gamma = \frac{1}{4}(1 \pm i\sqrt{15})$.

Thus the four roots are $\pm i/\sqrt{3}$, $\frac{1}{4}(1 \pm i\sqrt{15})$.

(2) *The sum of two of the roots of the equation*

$$x^4 + 6x^3 + 13x^2 + 12x - 5 = 0$$

is equal to the sum of the other two; solve the equation. [*N. Sc.*]

Let $\alpha, \beta, \gamma, \delta$ be the four roots of the equation

$$x^4 + 6x^3 + 13x^2 + 12x - 5 = 0.$$

Then $\alpha + \beta = \gamma + \delta$.

The fundamental relations between the roots and the coefficients are

$$\alpha + \beta + \gamma + \delta = -6 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 13 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -12 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

$$\alpha\beta\gamma\delta = -5 \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

Write $\alpha + \beta = \gamma + \delta = u$. Then from (i), $u = -3$.

From (ii), $\alpha\beta + \gamma\delta + u^2 = 13$, i.e. $\alpha\beta + \gamma\delta = 4 \ldots\ldots\ldots\ldots$ (v)

From (iv) and (v) we obtain the values of $\alpha\beta$, $\gamma\delta$, viz. $5, -1$. Taking $\alpha\beta = 5$, $\gamma\delta = -1$ we have

$$\alpha + \beta = -3, \quad \alpha\beta = 5, \quad \text{and} \quad \gamma + \delta = -3, \quad \gamma\delta = -1.$$

Solving these equations we find that the four roots of the quartic equation are $\frac{1}{2}(-3 \pm i\sqrt{11})$ and $\frac{1}{2}(-3 \pm \sqrt{13})$.

(3) *Show that if the roots of the equation* $ax^4 + bx^3 + cx^2 + dx + e = 0$ *are in harmonic progression, then* $d^3 = 4cde - 8be^2$, *and*

$$25\,ad^2e = (cd - eb)(11eb - cd).$$

Verify these conditions in the case of $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$ *and solve for* x. [*Camb. Sch.*]

Let $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$ be the four roots of the given equation. Then $\alpha, \beta, \gamma, \delta$ are in arithmetic progression and may be represented in the form

$$\lambda - 3k, \quad \lambda - k, \quad \lambda + k, \quad \lambda + 3k.$$

The equation whose roots are $\alpha, \beta, \gamma, \delta$ is

$$ex^4 + dx^3 + cx^2 + bx + a = 0. \qquad [\S\ 15\cdot1.]$$

The four equations relating the roots to the coefficients are

$$\alpha + \beta + \gamma + \delta = -\frac{d}{e},$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{e},$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{b}{e},$$

$$\alpha\beta\gamma\delta = \frac{a}{e}.$$

Expressing these equations in terms of λ, k we obtain

$$\lambda - 3k + \lambda - k + \lambda + k + \lambda + 3k = -d/e.$$

$$\text{i.e. } \lambda = -d/4e \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$(\lambda - 3k)(\lambda - k) + (\lambda - 3k)(\lambda + k) + (\lambda - 3k)(\lambda + 3k) + (\lambda - k)(\lambda + k) + (\lambda - k)(\lambda + 3k) + (\lambda + k)(\lambda + 3k) = c/e,$$

$$\text{i.e. } 3\lambda^2 - 5k^2 = c/2e \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Again $(\lambda - 3k)(\lambda - k)(\lambda + 3k) + (\lambda - 3k)(\lambda - k)(\lambda + k)$

$$+ (\lambda - 3k)(\lambda + k)(\lambda + 3k) + (\lambda - k)(\lambda + k)(\lambda + 3k) = -b/e,$$

$$\text{i.e.}\quad \lambda^3 - 5k^2\lambda = -b/4e \quad\text{(iii)}$$

Finally, $(\lambda - 3k)(\lambda - k)(\lambda + k)(\lambda + 3k) = a/e$,

$$\text{i.e.}\quad (\lambda^2 - k^2)(\lambda^2 - 9k^2) = a/e \quad\text{(iv)}$$

From (i) and (ii), $3d^2 - 8ce = 80e^2k^2$.

Substituting for λ and k in (iii), this equation reduces to

$$d^3 = 4cde - 8be^2 \quad\text{(v)}$$

From (iv) we obtain in a similar way

$$(d^2 + 4ce)(-11d^2 + 36ce) = 1600\,ae^3,$$

$$\text{i.e.}\quad (d^3 + 4\,cde)(-11d^3 + 36\,cde) = 1600\,ad^2e^3.$$

Substituting from (v), $(8\,cde - 8\,be^2)(88\,be^2 - 8\,cde) = 1600\,ad^2e^3$,

$$\text{i.e.}\quad (cd - be)(11\,be - cd) = 25\,ad^2e \quad\text{(vi)}$$

In the particular case again, $a = 40$, $b = -22$, $c = -21$, $d = 2$, $e = 1$. It is easily verified that (v) and (vi) are satisfied. Then

$$\lambda = -d/4e = -\tfrac{1}{2},\quad k^2 = 9/4,\quad k = \pm 3/2.$$

Hence the roots are the reciprocals of

$$-\tfrac{1}{2} - \tfrac{9}{2},\ -\tfrac{1}{2} - \tfrac{3}{2},\ -\tfrac{1}{2} + \tfrac{3}{2},\ -\tfrac{1}{2} + \tfrac{9}{2},\ \text{i.e. of } -5,\ -2,\ 1,\ 4.$$

Thus the roots are $-\tfrac{1}{5},\ -\tfrac{1}{2},\ 1,\ \tfrac{1}{4}$.

(4) *The roots of the equation* $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$ *are known to be in arithmetical progression; solve the equation.* [*M.T.*]

Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation. Then

$$\alpha + \beta + \gamma + \delta = 4 \quad\text{(i)}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \tfrac{7}{2} \quad\text{(ii)}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -1 \quad\text{(iii)}$$

$$\alpha\beta\gamma\delta = -\tfrac{15}{16} \quad\text{(iv)}$$

Since the roots are in A.P. we may write β, γ, δ in the form $\beta = \alpha + k$, $\gamma = \alpha + 2k$, $\delta = \alpha + 3k$. Thus

$$\alpha + \delta = \beta + \gamma \quad\text{(v)}$$

Hence (i) may be written in the form

$$\beta + \gamma = 2 \quad\text{(vi)}$$

From (ii), $\alpha(\beta + \gamma) + \delta(\beta + \gamma) + \alpha\delta + \beta\gamma = \tfrac{7}{2}$,

i.e. $(\alpha + \delta)(\beta + \gamma) + \alpha\delta + \beta\gamma = \tfrac{7}{2}$.

$$\text{From (v) and (vi),}\quad \alpha\delta + \beta\gamma = -\tfrac{1}{2} \quad\text{(vii)}$$

$$\text{From (iv),}\quad (\alpha\delta)(\beta\gamma) = -\tfrac{15}{16} \quad\text{(viii)}$$

From (vii) and (viii) we find $\alpha\delta = \tfrac{3}{4}$ or $-\tfrac{5}{4}$, $\beta\gamma = -\tfrac{5}{4}$ or $\tfrac{3}{4}$.

Taking $\beta\gamma = \tfrac{3}{4}$ and combining with (v) we obtain $\beta = \tfrac{1}{2}$ or $\tfrac{3}{2}$, $\gamma = \tfrac{3}{2}$ or $\tfrac{1}{2}$.

Taking $\alpha\delta = -\tfrac{5}{4}$, $\alpha + \delta = 2$ we obtain $\alpha = -\tfrac{1}{2}$ or $\tfrac{5}{2}$, $\delta = \tfrac{5}{2}$ or $-\tfrac{1}{2}$.

It follows that the roots of the given equation are $-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}$.

(5) *If one root of the equation $x^3 + ax + b = 0$ is twice the difference of the other two, prove that one root is $13b/3a$.* [*Camb. Sch.*]

Let α, β, γ be the three roots so that

$$\alpha = 2\beta - 2\gamma \quad \text{(i)}$$

Since the coefficient of x^2 is zero

$$\alpha + \beta + \gamma = 0 \quad \text{(ii)}$$

From (i) and (ii) it follows that $\beta = -\tfrac{1}{4}\alpha$, $\gamma = -\tfrac{3}{4}\alpha$.

Again, since $\alpha\beta\gamma = -b$, we have

$$\tfrac{3}{16}\alpha^3 = -b \quad \text{(iii)}$$

Substituting $\beta + \gamma = -\alpha$ and $\beta\gamma = \frac{3}{16}\alpha^2$ in $\alpha\beta + \alpha\gamma + \beta\gamma = a$ we obtain

$$\tfrac{13}{16}\alpha^2 = -a \quad \text{(iv)}$$

From (iii) and (iv) $\frac{3}{13}\alpha = \dfrac{b}{a}$, i.e. $\alpha = 13b/3a$.

(6) *Solve the equation* $162x^4 + 135x^3 - 135x^2 - 30x + 8 = 0$, *whose roots are in geometric progression.*

Let the roots be a/r^3, a/r, ar, ar^3 so that r^2 is the common ratio. Then

$$\frac{a}{r^3}\cdot\frac{a}{r} + \frac{a}{r^3}.ar + \frac{a}{r^3}.ar^3 + \frac{a}{r}.ar + \frac{a}{r}.ar^3 + ar.ar^3 = -\tfrac{135}{162},$$

$$\frac{a}{r^3}\cdot\frac{a}{r}.ar.ar^3 = \tfrac{8}{162}.$$

These equations give on simplification

$$a^2\left\{\frac{1}{r^4} + \frac{1}{r^2} + 2 + r^2 + r^4\right\} = -\tfrac{5}{6} \quad \text{(i)}$$

$$a^4 = \tfrac{4}{81} \quad \text{(ii)}$$

From (ii), $a^2 = \pm 2/9$.

Taking $a^2 = -2/9$, (i) becomes $\frac{1}{r^4} + \frac{1}{r^2} + 2 + r^2 + r^4 = \frac{15}{4}$.

Substituting $y = r^2 + 1/r^2$, this equation gives

$$y^2 + y - 15/4 = 0, \text{ i.e. } y = 3/2 \text{ or } -5/2.$$

The value $r^2 + 1/r^2 = 3/2$ makes r^2 imaginary.

Taking $r^2 + 1/r^2 = -5/2$ we obtain $r^2 = -2$ or $-\frac{1}{2}$. If $r^2 = -2$, $a^2r^2 = \frac{4}{9}$; if $r^2 = -\frac{1}{2}$, $a^2r^2 = \frac{1}{9}$.

In the former case $ar = \pm\frac{2}{3}$. Taking the positive square root it follows that

$$\frac{a}{r^3} = \frac{ar}{r^4} = \tfrac{1}{6},\ \frac{a}{r} = -\tfrac{1}{3},\ ar = \tfrac{2}{3},\ ar^3 = -\tfrac{4}{3},$$

and these values satisfy the equation. If we take $ar = -\frac{2}{3}$,

$$\frac{a}{r^3} = -\tfrac{1}{6},\ \frac{a}{r} = \tfrac{1}{3},\ ar = -\tfrac{2}{3},\ ar^3 = \tfrac{4}{3}.$$

These values are the roots of the equations:

$$162x^4 - 135x^3 - 135x^2 + 30x + 8 = 0 \quad \text{(iii)}$$

which is obtained from the given equation by changing x into $-x$.

We must expect to obtain the roots of this equation for in the argument we have used only the coefficients of x^4, x^2 and the constant term, which are the same in each equation, as the transformation x into $-x$ does not alter the coefficients of the even powers of x.

Next consider $a^2r^2 = 1/9$, i.e. $ar = \pm 1/3$, $r^2 = -1/2$.

If $ar = +1/3$, we have $\dfrac{a}{r^3} = \frac{4}{3}$, $\dfrac{a}{r} = -\frac{2}{3}$, $ar = \frac{1}{3}$, $ar^3 = -\frac{1}{6}$. and these are the roots of equation (iii). If $ar = -1/3$, we have

$$\frac{a}{r^3} = -\tfrac{4}{3},\ \frac{a}{r} = \tfrac{2}{3},\ ar = -\tfrac{1}{3},\ ar^3 = \tfrac{1}{6}.$$

These are the roots of the given equation.

This example shows that it is sometimes necessary to consider whether the method adopted has introduced extraneous values.

15·43. Reduction of the Degree of an Equation

The examples of the previous section show that *when a relation between two of the roots is known the degree of the equation may be reduced by two dimensions.* This may be proved quite generally as follows:

Let α and β be two roots of the equation

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$$

which satisfy the relation $\beta = \phi(\alpha)$.

Let $\psi(x)$ be the function obtained from $f(x)$ when x is changed into $\phi(x)$. Thus

$$\psi(x) \equiv f\{\phi(x)\}.$$

Now $\psi(\alpha) = f\{\phi(\alpha)\} = f(\beta) = 0$. It follows that the equations $f(x) = 0$, $\psi(x) = 0$ have a root in common, i.e. $f(x)$ and $\psi(x)$ have a common factor $x - \alpha$. This may be determined by the method of Chapter IX., § 9·11. Thus α and hence β are found. Division of $f(x)$ by the quadratic factor $(x-\alpha)(x-\beta)$ will reduce the degree of the equation by 2.

Example.—*Given that* $x^4 - 10x^2 + 9x - 2 = 0$ *has the product of two of its roots equal to unity, find them.* [*Madras, B.Sc.*]

Let α, β be the two roots such that $\alpha\beta = 1$. Substituting $1/x$ for x in the given equation, we obtain

$$2x^4 - 9x^3 + 10x^2 - 1 = 0.$$

Thus the polynomials $x^4 - 10x^2 + 9x - 2$, $2x^4 - 9x^3 + 10x^2 - 1$ must have a common factor.

Proceeding as in Chapter IX, § 9·11, we have the following:

$$\begin{array}{r|l} x^4 - 10x^2 + 9x - 2) & 2x^4 - 9x^3 + 10x^2 \quad\quad - 1\,(2 \\ & \underline{2x^4 \quad\quad - 20x^2 + 18x - 4} \\ & \quad - 9x^3 + 30x^2 - 18x + 3 \end{array}$$

Multiply the divisor by 3, and divide the remainder by -3.

$$
\begin{array}{l}
3x^3 - 10x^2 + 6x - 1)3x^4 \qquad\qquad - 30x^2 + 27x - 6(x \\
\qquad\qquad\qquad\qquad\quad \underline{3x^4 - 10x^3 + \ 6x^2 - \quad x} \\
\qquad\qquad\qquad\qquad\qquad + 10x^3 - 36x^2 + 28x - 6 \ \times \tfrac{3}{2} \\
\\
\qquad\qquad\qquad\qquad\qquad\quad 15x^3 - 54x^2 + 42x - 9(5 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{15x^3 - 50x^2 + 30x - 5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\ - \ 4x^2 + 12x - 4 \quad \div -4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^2 - \ 3x + 1 \\
x^2 - 3x + 1)3x^3 - 10x^2 + 6x - 1(3x - 1 \\
\qquad\qquad\quad\ \underline{3x^3 - \ 9x^2 + 3x} \\
\qquad\qquad\qquad - \ x^2 + 3x - 1 \\
\qquad\qquad\qquad \underline{- \ x^2 + 3x - 1} \\
\qquad\qquad\qquad\quad \cdot \qquad \cdot \qquad \cdot
\end{array}
$$

Hence the common factor is $x^2 - 3x + 1$. The roots of $x^2 - 3x + 1 = 0$ are $\frac{1}{2}(3 \pm \sqrt{5})$. One of these roots must be α. But since the product of the roots is clearly equal to unity it follows that $x^2 - 3x + 1 = 0$ gives both the required roots.

15·44. Symmetric Functions of the Roots

Let the equation be

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0,$$

and its roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$. By a *symmetric function of the roots* is meant a function in which all the roots are involved and such that *the function is unaltered in value if any two of the roots are interchanged.* Now using the formulae

$$S_1 = \Sigma\alpha_1 = -p_1,$$
$$S_2 = \Sigma\alpha_1\alpha_2 = p_2,$$
$$S_3 = \Sigma\alpha_1\alpha_2\alpha_3 = -p_3,$$

we can express any symmetric functions of the roots in terms of the coefficients of $f(x)$.

It is usual to denote the symmetric function by attaching the sign Σ to any one of the terms of this function. Thus, *e.g.* if $f(x)$ is a polynomial of the third degree, then

$$\Sigma\alpha_1^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2,$$
$$\Sigma\alpha_1^2\alpha_2 = \alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2,$$
$$\Sigma\alpha_1^2\alpha_2\alpha_3 = \alpha_1^2\alpha_2\alpha_3 + \alpha_2^2\alpha_3\alpha_1 + \alpha_3^2\alpha_1\alpha_2.$$

Example.—*If α, β, γ are roots of the cubic equation*

$$x^3 + px^2 + qx + r = 0$$

evaluate (i) $\Sigma\alpha^2$, (ii) $\Sigma\alpha^3$, (iii) $\Sigma\alpha^2\beta$.

(i) Since $\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$,

$$p^2 = (\Sigma\alpha)^2 = \Sigma\alpha^2 + 2\,\Sigma\alpha\beta, \text{ i.e. } \Sigma\alpha^2 = p^2 - 2q.$$

(ii) Since

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

it follows that

$$\Sigma\alpha^3 = 3\alpha\beta\gamma + (\Sigma\alpha)(\Sigma\alpha^2 - \Sigma\alpha\beta)$$
$$= -3r - p(p^2 - 2q - q) = -p^3 + 3pq - 3r.$$

(iii) Consider $\Sigma\alpha^2\beta$. The product $(\Sigma\alpha)(\Sigma\alpha\beta)$ will contain $\Sigma\alpha^2\beta$ together with terms of the form $\alpha\beta\gamma$. The terms of $\Sigma\alpha^2\beta$ will occur only *once* while the term $\alpha\beta\gamma$ will occur *three* times, viz. as the product of any one of α, β, γ and a term from $\Sigma\alpha\beta$. Hence

$$(\Sigma\alpha)(\Sigma\alpha\beta) = \Sigma\alpha^2\beta + 3\alpha\beta\gamma,$$
$$\text{i.e. } \Sigma\alpha^2\beta = (-p)(q) - 3(-r) = -pq + 3r.$$

The procedure adopted in (iii) of the Ex. indicates the method of dealing with the general equation. We now prove a number of results for this equation. It is important to observe that the statement of the results *is independent of the degree of the equation*, and so is the same for equations of all degrees. There is, of course, the obvious restriction that the number of different roots which occur in a typical term of the symmetric function cannot exceed the degree of the equation.

If $\alpha_1, \alpha_2, \alpha_3, \ldots$ *are the roots of the equation* $f(x) = 0$, *then*

(*a*) $\Sigma\alpha_1^2 = p_1^2 - 2p_2$

(*b*) $\Sigma\alpha_1^2\alpha_2 = 3p_3 - p_1p_2$

(*c*) $\Sigma\alpha_1^3 = -p_1^3 + 3p_1p_2 - 3p_3$

(*d*) $\Sigma\alpha_1^2\alpha_2\alpha_3 = p_1p_3 - 4p_4$

(*e*) $\Sigma\alpha_1^2\alpha_2^2 = p_2^2 - 2p_1p_3 + 2p_4$

(*f*) $\Sigma\alpha_1^4 = p_1^4 - 4p_1^2p_2 + 4p_1p_3 + 2p_2^2 - 4p_4$.

It is clear that there is an infinite variety for symmetric functions of the roots. The methods adopted in the proofs of (*a*) to (*f*) will provide the reader with a technique which should enable him to deal with any particular case.

(*a*) $\Sigma\alpha_1^2 = (\Sigma\alpha_1)^2 - 2\,\Sigma\alpha_1\alpha_2 = p_1^2 - 2p_2$.

(*b*) $\Sigma\alpha_1^2\alpha_2$ will be obtained by considering the product $(\Sigma\alpha_1)(\Sigma\alpha_1\alpha_2)$. In addition to terms of the form $\alpha_1^2\alpha_2$ this product will include terms of the form $\alpha_1\alpha_2\alpha_3$. Terms of the form $\alpha_1^2\alpha_2$ can only occur once in the product. On the other hand terms

of the form $\alpha_1\alpha_2\alpha_3$ will occur three times. For $\alpha_1\alpha_2\alpha_3$ can be obtained in the following three, and only three, ways.

$$\alpha_1 \times \alpha_2\alpha_3, \quad \alpha_2 \times \alpha_1\alpha_3, \quad \alpha_3 \times \alpha_1\alpha_2.$$

It follows that every term of $\Sigma\alpha_1\alpha_2\alpha_3$ occurs three, and only three, times. Thus

$$\begin{aligned}\Sigma\alpha_1^2\alpha_2 &= (\Sigma\alpha_1)(\Sigma\alpha_1\alpha_2) - 3\,\Sigma\alpha_1\alpha_2\alpha_3\\ &= (-p_1)(p_2) - 3(-p_3) = 3p_3 - p_1p_2.\end{aligned}$$

(c) Consider the product $(\Sigma\alpha_1)(\Sigma\alpha_1^2)$. This will contain only terms of the form α_1^3, $\alpha_1^2\alpha_2$. Clearly α_1^3 can be obtained in only one way. Further, $\alpha_1^2\alpha_2$ occurs only once in the product, viz. as the product of α_2 and α_1^2. Hence

$$\begin{aligned}\Sigma\alpha_1^3 &= (\Sigma\alpha_1)(\Sigma\alpha_1^2) - \Sigma\alpha_1^2\alpha_2\\ &= (-p_1)(p_1^2 - 2p_2) - (3p_3 - p_1p_2) = -p_1^3 + 3p_1p_2 - 3p_3.\end{aligned}$$

(d) Consider the product $(\Sigma\alpha_1)(\Sigma\alpha_1\alpha_2\alpha_3)$ which will contain terms of the forms $\alpha_1^2\alpha_2\alpha_3$, $\alpha_1\alpha_2\alpha_3\alpha_4$. The term $\alpha_1^2\alpha_2\alpha_3$ can be obtained in only one way, viz. from the product $\alpha_1 \times \alpha_1\alpha_2\alpha_3$. On the other hand $\alpha_1\alpha_2\alpha_3\alpha_4$ can be obtained in four ways, viz. $\alpha_1 \times \alpha_2\alpha_3\alpha_4$, $\alpha_2 \times \alpha_1\alpha_3\alpha_4$, $\alpha_3 \times \alpha_1\alpha_2\alpha_4$, $\alpha_4 \times \alpha_1\alpha_2\alpha_3$. Hence

$$\begin{aligned}\Sigma\alpha_1^2\alpha_2\alpha_3 &= (\Sigma\alpha_1)(\Sigma\alpha_1\alpha_2\alpha_3) - 4\,\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\\ &= (-p_1)(-p_3) - 4(p_4) = p_1p_3 - 4p_4.\end{aligned}$$

(e) Consider $(\Sigma\alpha_1\alpha_2)^2$. This square will contain terms of the form $\alpha_1^2\alpha_2^2$, $\alpha_1^2\alpha_2\alpha_3$, $\alpha_1\alpha_2\alpha_3\alpha_4$.

$\alpha_1^2\alpha_2^2$ occurs only once as $\alpha_1\alpha_2 \times \alpha_1\alpha_2$.

$\alpha_1^2\alpha_2\alpha_3$ occurs twice, $\alpha_1\alpha_2 \times \alpha_1\alpha_3$, $\alpha_1\alpha_3 \times \alpha_1\alpha_2$.

$\alpha_1\alpha_2\alpha_3\alpha_4$ occurs six times, viz. $\alpha_1\alpha_2 \times \alpha_3\alpha_4$, $\alpha_1\alpha_3 \times \alpha_2\alpha_4$, $\alpha_1\alpha_4 \times \alpha_2\alpha_3$, $\alpha_2\alpha_3 \times \alpha_1\alpha_4$, $\alpha_2\alpha_4 \times \alpha_1\alpha_3$, $\alpha_3\alpha_4 \times \alpha_1\alpha_2$.

Hence
$$\begin{aligned}\Sigma\alpha_1^2\alpha_2^2 &= (\Sigma\alpha_1\alpha_2)^2 - 2\,\Sigma\alpha_1^2\alpha_2\alpha_3 - 6\,\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\\ &= p_2^2 - 2(p_1p_3 - 4p_4) - 6\,p_4\\ &= p_2^2 - 2p_1p_3 + 2p_4.\end{aligned}$$

(f) Now $(\Sigma\alpha_1^2)^2$ contains terms of the form α_1^4, $\alpha_1^2\alpha_2^2$. The first occurs once and second twice. Hence

$$\begin{aligned}\Sigma\alpha_1^4 &= (\Sigma\alpha_1^2)^2 - 2\,\Sigma\alpha_1^2\alpha_2^2\\ &= (p_1^2 - 2p_2)^2 - 2(p_2^2 - 2p_1p_3 + 2p_4)\\ &= p_1^4 - 4p_1^2p_2 + 4p_1p_3 + 2p_2^2 - 4p_4.\end{aligned}$$

Examples.—(1) *If a, b, c are the roots of the equation* $px^3 + qx^2 + rx + s = 0$ *find the value of the expression* $\Sigma a^2 + \Sigma ab + 3abc$. [*Madras, B.Sc.*]

$$\Sigma a = -q/p, \quad \Sigma ab = r/p, \quad abc = -s/p.$$

$$\Sigma a^2 = (\Sigma a)^2 - 2\,\Sigma ab = \frac{q^2}{p^2} - \frac{2r}{p}.$$

$$\therefore\ \Sigma a^2 + \Sigma ab + 3abc = \frac{q^2}{p^2} - \frac{2r}{p} + \frac{r}{p} - \frac{3s}{p}$$
$$= (q^2 - rp - 3sp)/p^2.$$

(2) *If* α, β, γ *are the roots of* $x^3 + px + q = 0$, *prove that*

$$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3}{3} \cdot \frac{\alpha^2 + \beta^2 + \gamma^2}{2}.$$

[*Camb. Sch.*]

Since $x^3 + px + q \equiv (x - \alpha)(x - \beta)(x - \gamma)$ it follows that

$$(1 - \alpha x)(1 - \beta x)(1 - \gamma x) \equiv 1 + px^2 + qx^3.$$

Taking logarithms,

$$\log(1 + px^2 + qx^3) = \log(1 - \alpha x) + \log(1 - \beta x) + \log(1 - \gamma x).$$

Expanding by the logarithmic series,

$$\sum_{n=1}^{\infty} (-1)^{n-1}(px^2 + qx^3)^n/n = -\sum_{n=1}^{\infty} \alpha^n x^n/n - \sum_{n=1}^{\infty} \beta^n x^n/n - \sum_{n=1}^{\infty} \gamma^n x^n/n,$$

the expansions being absolutely convergent for $|x|$ sufficiently small.

Since the series are absolutely convergent we may arrange them in ascending powers of x. Thus

$$px^2 + qx^3 - \tfrac{1}{2}p^2x^4 - pqx^5 + \ldots = -\sum_{n=1}^{\infty} (\alpha^n + \beta^n + \gamma^n)\,x^n/n.$$

From the theorem on the identical equality of power series [Chapter I, § 1·8] we may equate coefficients of corresponding powers of x. Equating coefficients of x^2, x^3 and x^5, we have

$$-\tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) = p,$$
$$-\tfrac{1}{3}(\alpha^3 + \beta^3 + \gamma^3) = q,$$
$$-\tfrac{1}{5}(\alpha^5 + \beta^5 + \gamma^5) = -pq.$$

Hence $$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^2 + \beta^2 + \gamma^2}{2} \cdot \frac{\alpha^3 + \beta^3 + \gamma^3}{3}.$$

(3) *If* $\alpha + \beta + \gamma = a$, $\alpha^2 + \beta^2 + \gamma^2 = b$, $\alpha^3 + \beta^3 + \gamma^3 = c$, *find* $\alpha\beta\gamma$ *and* $\alpha^4 + \beta^4 + \gamma^4$ *in terms of a, b, c. Verify that when* $a = 0$, *they are respectively* $\frac{1}{3}c$ *and* $\frac{1}{2}b^2$. [*Camb. Sch.*]

Now $(\Sigma\alpha)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$. Hence $\Sigma\alpha\beta = \frac{1}{2}(a^2 - b)$.

Again, $(\Sigma\alpha)^3 = \Sigma\alpha^3 + 3\Sigma\alpha^2\beta + 6\alpha\beta\gamma$

$$= \Sigma\alpha^3 + 3(\Sigma\alpha)(\Sigma\alpha\beta) - 9\alpha\beta\gamma + 6\alpha\beta\gamma.$$

Hence $3\alpha\beta\gamma = c - a^3 + \frac{3}{2}a(a^2 - b)$,

i.e. $\alpha\beta\gamma = (2c + a^3 - 3ab)/6$.

In particular when $a = 0$, $\alpha\beta\gamma = \frac{1}{3}c$.

Again, $\Sigma\alpha^4 = (\Sigma\alpha^2)^2 - 2\Sigma\alpha^2\beta^2$

$= (\Sigma\alpha^2)^2 - 2\{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\Sigma\alpha\}$

$= b^2 - 2\{\frac{1}{4}(a^2 - b)^2 - \frac{1}{3}a(2c + a^3 - 3ab)\}$

$= \{a^4 + 3b^2 + 6a^2b + 8ac\}/6.$

In particular, if $a = 0$, $\Sigma\alpha^4 = \frac{1}{2}b^2$.

(4) *If* $(x + a_1)(x + a_2)(x + a_3)\ldots(x + a_n)$

$$\equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n,$$

prove that $a_1^3 + a_2^3 + a_3^3 + \ldots = p_1^3 - 3p_1p_2 + 3p_3$.

Prove also that the sum of the products r at a time of the $(n - 1)$ quantities $a_2, a_3, \ldots, a_n$ is equal to

$$p_r - a_1p_{r-1} + a_1^2p_{r-2} - \ldots + (-a_1)^r. \qquad \text{[Camb. Sch.]}$$

Now $\Sigma a_1 = p_1$, $\Sigma a_1a_2 = p_2$, $\Sigma a_1a_2a_3 = p_3$.

Then $(\Sigma a_1)^3 = \Sigma a_1^3 + 3\Sigma a_1^2a_2 + 6\Sigma a_1a_2a_3$

Again $\Sigma a_1^2a_2 = (\Sigma a_1)(\Sigma a_1a_2) - 3\Sigma a_1a_2a_3$.

Hence $\Sigma a_1^3 = (\Sigma a_1)^3 - 6\Sigma a_1a_2a_3 - 3\Sigma a_1^2a_2$

$= (\Sigma a_1)^3 - 6\Sigma a_1a_2a_3 - 3(\Sigma a_1)(\Sigma a_1a_2) + 9\Sigma a_1a_2a_3$

$= p_1^3 - 6p_3 - 3p_1p_2 + 9p_3$

$= p_1^3 - 3p_1p_2 + 3p_3$.

Let $x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \ldots + q_rx^{n-r-1} + \ldots$ be the quotient when $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots$ is divided by $x + a_1$.

Then the sum of the products r at a time of the $(n - 1)$ quantities $a_2, a_3, \ldots a_n$ is q_r. The value of q_r may be found in the following way.

Write $f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_rx^{n-r} + \ldots + p_n$.

$$\text{Then } \frac{f(x)}{x + a_1} = \frac{f(x)}{x}\left(1 + \frac{a_1}{x}\right)^{-1}$$

$$= \frac{f(x)}{x}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a_1}{x}\right)^r, \qquad \left|\frac{a_1}{x}\right| < 1.$$

Now q_r will be the coefficient of x^{n-r-1} in $f(x)/(x + a_1)$, i.e. the coefficient of x^{n-r} in

$$(x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_rx^{n-r} + \ldots + p_n)$$
$$\left(1 - \frac{a_1}{x} + \frac{a_1^2}{x^2} - \ldots + (-1)^r\frac{a_1^r}{x^r} + \ldots\right).$$

This coefficient is $p_r - a_1p_{r-1} + a_1^2p_{r-2} - a_1^3p_{r-3} + \ldots + (-1)^ra_1^r$.

It is clear that any symmetric function of the roots can be expressed in terms of functions of the type $\Sigma a_1^ra_2^s\ldots a_p^t$, where $r, s, \ldots t$ are positive integers.

Suppose that $\Sigma a_1^ra_2^s\ldots a_p^t$ has been expressed in terms of the coefficients $p_1, p_2, p_3\ldots$ and denote this function by $F(p_1, p_2, \ldots)$. Then *the sum of the indices* $r + s + \ldots + t$ *is equal to the sum*

of the suffixes in each term of F $(p_1, p_2 \ldots)$. Thus, *e.g.* consider

$$\Sigma a_1^4 = p_1^4 - 4p_1^2p_2 + 4p_1p_3 + 2p_2^2 - 4p_4.$$

$$\Sigma a_1^2a_2a_3 = p_1p_3 - 4p_4.$$

In each of these the sum of the indices is 4, while the sum of the suffixes in each term in the expression in terms of $p_1, p_2, \ldots$ is 4. Note that if a coefficient p_μ is raised to any power, say m, then the sum of the suffixes corresponding to $p_\mu{}^m$ is $m\mu$. The truth of the result in general follows from the fundamental equations stated at the beginning of this section.

EXERCISES XV

10. The numbers $u_0, u_1, u_2, \ldots$ satisfy the relation $u_n = u_1u_{n-1} - u_{n-2}$ for all values of n greater than 1. If $u_0 = 2$, $u_1 = 2\cos\theta$, show that $u_n = 2\cos n\theta$. By means of the given relation express u_5 as a polynomial in u_1, and show that one root of the equation

$$u_1^4 + u_1^3 - 4u_1^2 - 4u_1 + 1 = 0$$

is $2\cos\frac{\pi}{15}$. What are the other three roots? [*M.T.*]

11. If $\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$, $\frac{x}{a+\mu} + \frac{y}{b+\mu} + \frac{z}{c+\mu} = 1$, $\frac{x}{a+\nu} + \frac{y}{b+\nu} + \frac{z}{c+\nu} = 1$, prove that for all values of ξ (except $-a$, $-b$ and $-c$),

$$\frac{x}{a+\xi} + \frac{y}{b+\xi} + \frac{z}{c+\xi} = 1 + \frac{(\lambda-\xi)(\mu-\xi)(\nu-\xi)}{(a+\xi)(b+\xi)(c+\xi)}.$$

[*Camb. Sch.*]

12. Given that two of the roots of

$$45x^4 - 54x^3 - 98x^2 + 150x - 75 = 0$$

are equal in absolute magnitude but opposite in sign, complete the solution of the equation. [*M.T.*]

13. Prove that $x^6 - 7x^4 + 15x^2 - 9$ is divisible by $x^2 - 1$ and that the quotient is a perfect square. Hence solve the equation

$$x^6 - 7x^4 + 15x^2 - 9 = 0$$

and write down the equation whose roots are the reciprocals of the roots of this equation. [*Lond. Inter. Econ.*]

14. The equation $4x^5 - 57x^3 + 64x^2 + 108x - 144 = 0$ has two roots which are equal in magnitude and opposite in sign. Solve it completely. [*Camb. Sch.*]

15. Show that if $c^2 = a^2d$, then the product of two of the roots of the equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

is equal to the product of the other two. Hence, or otherwise solve the equation

$$x^4 + x^3 + 2x^2 + 2x + 4 = 0.$$

[*Camb. Sch.*]

16. If one of the roots of the equation

$$ax^3 + bx^2 + cx + d = 0$$

is the geometric mean of the other two roots, prove that $ac^3 = b^3d$.

Show that $x^3 + 4x^2 - 12x - 27 = 0$ is such an equation and solve it completely. [*Lond. Inter. Econ.*]

17. Solve $81x^4 + 54x^3 - 189x^2 - 66x + 40 = 0$, given that the roots are in arithmetical progression. [*Camb. Sch.*]

18. Find for what values of a and b the roots of the equation

$$x^4 - 4x^3 + ax^2 + bx - \tfrac{7}{4} = 0$$

are in arithmetical progression. [*Camb. Sch.*]

19. Prove that, if the sum of two roots of the quintic equation

$$x^5 + ax + b = 0$$

is equal to the sum of two other roots, b must vanish. [*Lond. B.Sc.*]

20. Denoting by x_1, x_2, x_3 the roots of the equation $x^3 + px + q = 0$, find the value of the sum

$$x_1 (x_2^2 + x_3^2) + x_2 (x_3^2 + x_1^2) + x_3 (x_1^2 + x_2^2).$$ [*Camb. Sch.*]

21. If α, β, γ are the roots of $x^3 + px + q = 0$, find $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$, and deduce that if $l + m + n = 0$, then $l^3 + m^3 + n^3 = 3lmn$ [*N.Sc. Prelim.*]

22. Given that $x^3 + y^3 + z^3 = a^3$, $x^2 + y^2 + z^2 = b^2$, $x + y + z = c$, express $yz + zx + xy$ and xyz in terms of a, b, c; hence show that x, y, z are the roots of the equation

$$6\lambda^3 - 6c\lambda^2 - 3(b^2 - c^2)\lambda - (2a^3 - 3b^2c + c^3) = 0.$$

[*Lond. Inter. Econ.*]

23. Solve the equation $2 \sin \theta = \dfrac{1 + \tan^2\theta}{3 - \tan^2\theta}$, and show from it that $\sin 10° . \sin 50° . \sin 70° = \tfrac{1}{8}$. [*Camb. Sch.*]

24. If $\alpha = \cos \tfrac{2}{7}\pi + i \sin \tfrac{2}{7}\pi$ show that $\alpha + \alpha^2 + \alpha^4$ and $\alpha^3 + \alpha^5 + \alpha^6$ are the roots of $x^2 + x + 2 = 0$. Hence show that

$$\sin \tfrac{2}{7}\pi + \sin \tfrac{4}{7}\pi - \sin \tfrac{1}{7}\pi = \tfrac{1}{2}\sqrt{7}.$$ [*Camb. Sch.*]

25. If α, β are the roots of the equation $ax^2 + 2bx + c = 0$, prove that

$$ax^2 + 2bx + c \equiv a(x - \alpha)(x - \beta)$$

and evaluate $(b^4 - a^4\alpha^4)(b^4 - a^4\beta^4)$. [*Lond. B.Sc.*]

26. If the roots of the equation $x^2 + px + q = 0$, and $x^2 + p_1x + q_1 = 0$, are α, β and α^2, β respectively, prove that

$$q_1^2 + pq_1q + q^3 = 0, \text{ and } q_1^2 - qq_1 + q^2(p_1 - p) = 0.$$

Hence, or otherwise, prove that

$$q(p - p_1)^2 + p(p - p_1)(q_1 - q) + (q_1 - q)^2 = 0.$$

[*Lond. B.Sc.*]

27. If the roots of the equation $x^2 + px + q = 0$ are α, β and the roots of the equation $x^2 + ax + b = 0$ are $1/\alpha, \gamma$, show that

$$(p - aq)(a - pb) = (1 - bq)^2.$$

Show also that β and γ are the roots of the quadratic equation

$$x^2(1-bq)-x\{(a+p)bq-(aq+bp)\}+bq(1-bq)=0.$$

28. Obtain a cubic equation whose roots are the values of x, y, z gi

$$x+y+z=3,\quad x^2+y^2+z^2=5,\quad \quad +y^3+z^3=7.$$

Prove that $x^4+y^4+z^4=9$. [*Camb. Sch.*]

29. Prove that if x be a root of the equation

$$x^3-3x(a^2+a+1)+(a^2+a+1)(2a+1)=0,$$

then the other roots are $\phi(x)$ and $\phi\{\phi(x)\}$, where

$$\phi(x)=x^2+ax-2(a^2+a+1).$$ [*Camb. Sch.*]

30. Prove that the sum of the squares of two of the roots of the equation $8x^3+8ax^2-3a^3=0$ is equal to their product.

[*N.Sc., Prelim.*]

15·51. Transformation of Equations

We first consider some typical examples of the determination of equations whose roots are functions of the roots of a given equation and whose coefficients can be expressed in terms of symmetric functions of the roots of this equation.

Examples.—(1) *If α and β are the roots of the equation $px^2+qx+r=0$, find the equation whose roots are $(p\alpha+q)/\beta$, $(p\beta+q)/\alpha$.*

[*N.Sc., Prelim.*]

$$\text{Now } \frac{p\alpha+q}{\beta}+\frac{p\beta+q}{\alpha}=\frac{p\alpha^2+q\alpha+p\beta^2+q\beta}{\alpha\beta}$$

$$=\{p(\alpha^2+\beta^2)+q(\alpha+\beta)\}/\alpha\beta.$$

Since $\alpha+\beta=-q/p$, $\alpha\beta=r/p$ we have

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{q^2}{p^2}-\frac{2r}{p}=\frac{q^2-2rp}{p^2}.$$

$$\text{Hence } \frac{p\alpha+q}{\beta}+\frac{p\beta+q}{\alpha}=\left\{\frac{q^2-2rp}{p}-\frac{q^2}{p}\right\}\Big/\frac{r}{p}=-2p \quad\ldots\ldots\ldots \text{(i)}$$

$$(p\alpha+q)(p\beta+q)/\alpha\beta=\{p^2\alpha\beta+pq(\alpha+\beta)+q^2\}/\alpha\beta=p^2.$$

Hence the required equation is $x^2+2px+p^2=0$, i.e. $(x+p)^2=0$.

This equation implies that $\dfrac{p\alpha+q}{\beta}=\dfrac{p\beta+q}{\alpha}=-p$.

The fact that this is true follows immediately from the given equation and from (i). Thus $p\alpha^2+q\alpha=-r=p\beta^2+q\beta$. Dividing by $\alpha\beta$,

$$\frac{p\alpha+q}{\beta}=\frac{p\beta+q}{\alpha}.$$

(2) *Find the equation whose roots are the squares of the roots of*

$$x^3+bx^2+cx+d=0,$$

and find the condition that the squares of roots of the given equation shall be in arithmetic progression. [*M.T.*]

In the given equation $x^3 + bx^2 + cx + d = 0$ write $x = \sqrt{y}$ so that $y = x^2$. Then $y\sqrt{y} + by + c\sqrt{y} + d = 0$,

$$\text{i.e. } \sqrt{y}\,(y + c) = -by - d.$$

Squaring both sides of the equation

$$y(y + c)^2 = (by + d)^2, \text{ i.e. } y^3 + y^2(2c - b^2) + y(c^2 - 2bd) - d^2 = 0.$$

The roots of this equation in y are the squares of the roots of the cubic in x. We require the condition that the roots of cubic in y be in arithmetic progression.

Let the roots be α, β, γ. Then if the roots are in A.P. we can write $2\beta = \alpha + \gamma$.

$$\text{Now } \alpha + \beta + \gamma = b^2 - 2c,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = c^2 - 2bd,$$
$$\alpha\beta\gamma = d^2.$$
$$\text{Thus } 3\beta = b^2 - 2c, \quad \ldots\ldots\ldots \text{(i)}$$
$$2\beta^2 + \alpha\gamma = c^2 - 2bd,$$
$$\alpha\gamma = d^2/\beta.$$
$$\text{Hence } 2\beta^2 + d^2/\beta = c^2 - 2bd. \quad \ldots\ldots\ldots \text{(ii)}$$

The required condition will be obtained by eliminating β between (i) and (ii). On simplification this gives

$$2(b^2 - 2c)^3 - 9(b^2 - 2c)(c^2 - 2bd) + 27d^2 = 0.$$

(3) *Find the equation the roots of which are the sums of the roots of*

$$x^3 + ax^2 + bx + c = 0$$

taken in pairs. [*Camb. Sch.*]

Let the roots be α, β, γ. Then we require the equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. Write $y = \alpha + \beta$.

Then since $\alpha + \beta + \gamma = -a$, $y = -a - \gamma$, i.e. $\gamma = -a - y$.

Since $\gamma^3 + a\gamma^2 + b\gamma + c = 0$, it follows that

$$(-a - y)^3 + a(-a - y)^2 + b(-a - y) + c = 0,$$
$$\text{i.e. } y^3 + 2ay^2 + (a^2 + b)y + ab - c = 0,$$

and this is the required equation.

(4) *If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$ find the equation whose roots are*

$$\beta + \gamma - 2\alpha, \quad \gamma + \alpha - 2\beta, \quad \alpha + \beta - 2\gamma. \qquad \text{[Camb. Sch.]}$$

Write $y = \beta + \gamma - 2\alpha = \beta + \gamma + \alpha - 3\alpha = -p - 3\alpha$.

Since $\alpha^3 + p\alpha^2 + q\alpha + r = 0$ we obtain the required equation by substituting $\alpha = -\frac{1}{3}(y + p)$. This gives on simplification

$$y^3 + (9q - 3p^2)y - 2p^3 + 9pq - 27r = 0.$$

(5) *Form the equation whose roots are $(\alpha^2 - \beta\gamma)/\alpha$, $(\beta^2 - \gamma\alpha)/\beta$, $(\gamma^2 - \alpha\beta)/\gamma$ where α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$.*

Write $y = (\alpha^2 - \beta\gamma)/\alpha = \alpha - \alpha\beta\gamma/\alpha^2 = \alpha + r/\alpha^2$.

$$\text{Now } a^3 + pa^2 + qa + r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$\text{i.e. } a + p + \frac{q}{a} + \frac{r}{a^2} = 0, \; a \neq 0,$$

$$\text{Hence } a + p + \frac{q}{a} + y - a = 0, \text{ i.e. } a = -q\,(p + y)^{-1}.$$

Substituting in (i) and simplifying we obtain

$$-q^3 + pq^2\,(p + y) - q^2\,(p + y)^2 + r\,(p + y)^3 = 0.$$

This is the required equation.

(6) *Find the equation whose roots are given by the formula*

$$y_i = x_i - x_i^2 + (x_1^2 + x_2^2 + x_3^2), \; (i = 1, 2, 3)$$

where x_1, x_2, x_3 *are the roots of the equation* $x^3 - x^2 + 4 = 0$. [*M.T.*]

Now $x_1 + x_2 + x_3 = 1$, $x_1x_2 + x_1x_3 + x_2x_3 = 0$, $x_1x_2x_3 = -4$.
Also $x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2\,(x_1x_2 + x_1x_3 + x_2x_3) = 1$.
From the given equation

$$x_i\,(x_i^2 - x_i) + 4 = 0, \text{ i.e. } x_i - x_i^2 = 4/x_i.$$

$$\text{Thus } y_i - 1 = 4/x_i, \text{ i.e. } x_i = 4/(y_i - 1).$$

Hence the required equation will be obtained from the given equation by substituting $x = 4/(y - 1)$. Thus the required equation is

$$\frac{4^3}{(y-1)^3} - \frac{4^2}{(y-1)^2} + 4 = 0.$$

This simplifies to $y^3 - 3y^2 - y + 19 = 0$.

(7) *If* α, β, γ *are the roots of the equation*

$$2x^3 + x^2 + x + 1 = 0,$$

find the equation whose roots are

$$\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}, \quad \frac{1}{\gamma^3} + \frac{1}{\alpha^3} - \frac{1}{\beta^3}, \quad \frac{1}{\alpha^3} + \frac{1}{\beta^3} - \frac{1}{\gamma^3}.$$

Write $\alpha = 1/a$, $\beta = 1/b$, $\gamma = 1/c$, $a^3 = u$, $b^3 = v$, $c^3 = w$. Then the required equation may be determined by means of two intermediate equations. The three steps are as follows:

(i) The formation of the equation whose roots are a, b, c.

(ii) The determination of the equation whose roots are u, v, w.

(iii) From (ii) is deduced the equation whose roots are

$$v + w - u, \; w + u - v, \; u + v - w.$$

(i) This is found by replacing x by $1/x$ in the given equation. Thus

$$x^3 + x^2 + x + 2 = 0 \text{ has } a, b, c \text{ for its roots.}$$

(ii) The equation whose roots are u, v, w is

$$x^3 - x^2\,\Sigma a^3 + x\,\Sigma a^3b^3 - a^3b^3c^3 = 0,$$

and it is now necessary to calculate Σa^3, Σa^3b^3. Using the formula

$$l^3 + m^3 + n^3 - 3lmn = (l + m + n)\,(l^2 + m^2 + n^2 - lm - mn - nl)$$

it follows that

$$\Sigma a^3 = (\Sigma a)\,\{(\Sigma a)^2 - 3\Sigma ab\} + 3\,abc$$
$$= (-1)\,\{1 - 3\} - 6 = -4$$

Again, using the same formula,

$$\Sigma a^3b^3 = (\Sigma ab)\,\{(\Sigma ab)^2 - 3\,abc\,\Sigma a\} + 3\,a^2b^2c^2$$
$$= \{1 - 3.-2.-1\} + 3.(-2)^2 = 7.$$

Hence the equation whose roots are u, v, w is

$$x^3 + 4x^2 + 7x + 8 = 0.$$

(iii) Now write $y = v + w - u = \Sigma u - 2u = -4 - 2u$.

Hence $u = -\frac{1}{2}(4 + y)$.

Substituting in $u^3 + 4u^2 + 7u + 8 = 0$ we have

$$-\tfrac{1}{8}(4 + y)^3 + (4 + y)^2 - \tfrac{7}{2}(4 + y) + 8 = 0,$$
i.e. $y^3 + 4y^2 + 12y - 16 = 0$.

15·52. Reciprocal Equations

Let $f(x) \equiv x^n + p_1x^{n-1} + p_2{}^{n-2} + \ldots + p_n = 0$, $p_n \neq 0$, be an equation whose roots are $a_1, a_2\ a_3, \ldots, a_n$. Then the equation whose roots are $1/a_1, 1/a_2, 1/a_3, \ldots, 1/a_n$ is $f(1/x) = 0$,

i.e. $p_nx^n + p_{n-1}x^{n-1} + \ldots + p_1x + 1 = 0$.

If the equations $f(x) = 0$, $f(1/x) = 0$ are the same, then $f(x)$ is said to be a *reciprocal equation*, i.e. it is an equation which is unaltered if x be changed into $1/x$. A comparison of the two equations shows that

$$\frac{1}{p_n} = \frac{p_1}{p_{n-1}} = \frac{p_2}{p_{n-2}} = \ldots = \frac{p_{n-1}}{p_1} = \frac{p_n}{1}.$$

From the equality of the first and last ratios it follows that $p_n{}^2 = 1$, i.e. $p_n = \pm 1$.

(*a*) If $p_n = 1$, we have

$$p_1 = p_{n-1},\ p_2 = p_{n-2},\ \ldots,\ p_r = p_{n-r}\ldots$$

i.e. the *coefficients of the corresponding terms taken from the beginning and end are equal.*

(*b*) If $p_n = -1$ then

$$p_1 = -p_{n-1},\ p_2 = -p_{n-2},\ \ldots,\ p_r = -p_{n-r}\ldots$$

i.e. the *coefficients of corresponding terms as counted from the beginning and end are equal in magnitude but opposite in sign.*

In (*b*) if $n = 2m$, m a positive integer, $p_m = -p_m$, i.e. $p_m = 0$, i.e. the coefficient of the middle term is 0.

It is clear that in a reciprocal equation the roots may be grouped in pairs; for if a is a root so is $1/a$.

We now prove that *every reciprocal equation can be reduced to a reciprocal equation of even degree in which the coefficients of the*

first and last terms are equal, in particular we can arrange that both the coefficients are unity.

Consider first an equation of type (*a*). If $n = 2m$ the equation is already in the required form. If $n = 2m + 1$, there must be one root which is its own reciprocal. This value can only be ± 1 and from the form of the equation the value is clearly $x = -1$. On division by the factor $x + 1$ we obtain an equation of the required form.

Next consider type (*b*), which is of the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots - p_2x^2 - p_1x - 1 = 0.$$

This may be written as

$$x^n - 1 + p_1x\,(x^{n-2} - 1) + p_2x^2\,(x^{n-4} - 1) + \ldots = 0.$$

If $n = 2m$, then $x^2 - 1$ is a factor of the polynomial on the right. Division by $x^2 - 1$ gives an equation of the required form.

If $n = 2m + 1$, $x - 1$ is a factor and division by this factor reduces to the equation to one of even degree, with the first and last coefficients both equal to unity.

Example.—*Solve* $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$

[*Madras B.Sc.*]

This is a reciprocal equation of even degree in which the first and last coefficients have opposite signs. Hence $x^2 - 1$ is a factor of the left-hand side.

The equation may be written in the form

$$6\,(x^6 - 1) - 25x\,(x^4 - 1) + 31x^2\,(x^2 - 1) = 0,$$

$$\text{i.e. } (x^2 - 1)\,\{6\,(x^4 + x^2 + 1) - 25x\,(x^2 + 1) + 31x^2\} = 0.$$

The first factor corresponds to the roots $x = \pm 1$. The second factor is

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0,$$

$$\text{i.e. } 6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0.$$

To solve this equation write $y = x + 1/x$. Then

$$6\,(y^2 - 2) - 25y + 37 = 0,$$

$$\text{i.e. } 6y^2 - 25y + 25 = 0, \text{ i.e. } (3y - 5)\,(2y - 5) = 0.$$

Hence $y = 5/3$ or $5/2$.

If $y = 5/3$, $3x^2 - 5x + 3 = 0$, i.e. $x = (5 \pm i\sqrt{11})/6$.

If $y = 5/2$, $2x^2 - 5x + 2 = 0$, i.e. $x = 2$ or $\frac{1}{2}$.

15·53. Equation whose Roots are those of a Given Equation, diminished (or increased) by a Given Quantity

Let $f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n$ be the given equation whose roots are $a_1, a_2, a_3, \ldots a_n$, and suppose we require the equation whose roots are $a_1 - h, a_2 - h, a_3 - h, \ldots a_n - h$.

Now we know that any polynomial of the nth degree may be uniquely expressed in the form

$$a_0(x-h)^n + a_1(x-h)^{n-1} + a_2(x-h)^{n-2} + \dots + a_n.$$

[Chapter IX., § 9·14, Theorem VIII.]

Suppose that $f(x)$ has been expressed in this form so that

$$f(x) = a_0(x-h)^n + a_1(x-h)^{n-1} + a_2(x-h)^{n-2} + \dots + a_n.$$

Changing the notation by writing $x+h$ instead of x we have

$$f(x+h) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n.$$

Now the roots of $f(x+h) = 0$ are those of $f(x)$ diminished by h [§ 15·1]. Hence the roots of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$$

are $\alpha_1 - h,\ \alpha_2 - h,\ \alpha_3 - h,\ \dots,\ \alpha_n - h.$

The method of calculating the values of $a_0, a_1, a_2, \dots, a_n$ has been explained in § 9·14. We give two numerical examples.

Examples.—(1) *Find the equation whose roots are those of*

$$2x^4 - 13x^2 + 10x - 19 = 0$$

decreased by unity.

In this case $h = 1$.

2	0	-13	10	-19
	2	2	-11	-1
	2	-11	-1	-20
	2	4	-7	
	4	-7	-8	
	2	6		
	6	-1		
	2			
	8			

The required equation is $2x^4 + 8x^3 - x^2 - 8x - 20 = 0$.

(2) *Find the equation whose roots are the roots of*

$$x^4 - x^3 - 10x^2 + 4x + 24 = 0$$

increased by 2, and hence solve the equation.

In this case $h = -2$.

1	-1	-10	4	24
	-2	6	8	-24
	-3	-4	12	0
	-2	10	-12	
	-5	6	0	
	-2	14		
	-7	20		
	-2			
	-9			

The required equation is

$$x^4 - 9x^3 + 20x^2 = 0, \text{ i.e. } x^2(x^2 - 9x + 20) = 0.$$

$$\text{Hence } x = 0, 0, 4, 5.$$

It follows that the roots of the original equation are $-2, -2, 2, 3$.

15·54. Removal of a Given Term from an Equation

An example of the method has already been given in § 15·32.

If $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n$, then

$$\begin{aligned} f(x+h) = p_0x^n &+ np_0hx^{n-1} + \frac{n(n-1)}{2!}p_0h^2x^{n-2} + \ldots \\ &+ p_1x^{n-1} + (n-1)p_1hx^{n-2} + \ldots \\ &+ p_2x^{n-2} + \ldots \\ = p_0x^n &+ (np_0h + p_1)x^{n-1} \\ &+ \left\{\frac{n(n-1)}{2}p_0h^2 + (n-1)p_1h + p_2\right\}x^{n-2} + \ldots \end{aligned}$$

If we write $h = -p_1/np_0$ the coefficient of x^{n-1} in $f(x+h)$ will be zero, *i.e.* the equation

$$f(x+h) = 0$$

will have the second term missing. In other words we can obtain from $f(x) = 0$ an equation in which the second term is missing by reducing the roots by h where $np_0h + p_1 = 0$. Similarly if h is one of the roots of

$$\tfrac{1}{2}n(n-1)p_0h^2 + (n-1)p_1h + p_2 = 0.$$

the transformed equation will have the third term missing.

It will be observed that in order to remove the constant term it is necessary to find a root of $f(h) = 0$, i.e. solve the given equation.

Examples.—(1) *Solve the equation* $x^4 - 12x^3 + 48x^2 - 72x + 35 = 0$. [*Camb. Sch.*]

In this example $p_0 = 1$, $p_1 = -12$, $np_0h + p_1 = 4h - 12 = 4(h-3)$. Hence we can remove the second term of the equation by reducing the roots by 3. The numerical working is as follows.

1	−12	48	−72	35
	3	−27	63	−27
	−9	21	−9	8
	3	−18	9	
	−6	3	0	
	3	−9		
	−3	−6		
	3			
	0			

The transformed equation is

$$x^4 - 6x^2 + 8 = 0, \text{ i.e. } x^2 = 2 \text{ or } 4.$$

The roots of the transformed equation are $\pm\sqrt{2}$, ± 2, and the roots of the original equation $3 \pm \sqrt{2}$, 1, 5.

(2) *Transform the equation* $x^4 - 24x^2 - 47x - 22 = 0$ *into one in which the term in* x^3 *is missing.*

Change x into $x + h$. Then the transformed equation is

$$x^4 + 4hx^3 + (6h^2 - 24)x^2 + \ldots = 0.$$

To remove the term in x^2 we take $h = \pm 2$, *i.e.* if we either decrease or increase the roots by 2 the resulting equation will have the required form.

(*a*) Decrease the roots by 2.

$$\begin{array}{ccccc}
1 & 0 & -24 & -47 & -22 \\
 & 2 & 4 & -40 & -174 \\ \hline
 & 2 & -20 & -87 & -196 \\
 & 2 & 8 & -24 & \\ \hline
 & 4 & -12 & -111 & \\
 & 2 & 12 & & \\ \hline
 & 6 & 0 & & \\
 & 2 & & & \\ \hline
 & 8 & & &
\end{array}$$

The transformed equation is $x^4 + 8x^3 - 111x - 196 = 0$.

(*b*) Increase the roots by 2.

$$\begin{array}{ccccc}
1 & 0 & -24 & -47 & -22 \\
 & -2 & 4 & 40 & 14 \\ \hline
 & -2 & -20 & -7 & 8 \\
 & -2 & 8 & 24 & \\ \hline
 & -4 & -12 & 17 & \\
 & -2 & 12 & & \\ \hline
 & -6 & 0 & & \\
 & -2 & & & \\ \hline
 & -8 & & &
\end{array}$$

In this case the transformed equation is

$$x^4 - 8x^3 + 17x + 8 = 0.$$

(3) *Reduce the cubic equation* $ax^3 + 3bx^2 + 3cx + d = 0$ to one in which the term in x^2 is missing.

In this case the roots must be reduced by $-b/a$. Now

$$\begin{aligned} a(x+h)^3 + 3b(x+h)^2 + 3c(x+h) + d &= a(x^3 + 3hx^2 + 3h^2x + h^3) \\ &\quad + 3b(x^2 + 2xh + h^2) + 3c(x+h) + d \\ &= ax^3 + 3(ah+b)x^2 + 3(ah^2 + 2bh + c)x + ah^3 + 3bh^2 + 3ch + d. \end{aligned}$$

Substituting $h = -b/a$ we have

$$a^2x^3 + 3a(ac - b^2)x + a^2d - 3abc + 2b^3 = 0$$

as the required equation. If the roots of the original cubic are α, β, γ, the roots of the transformed equation are $\alpha + b/a$, $\beta + b/a$, $\gamma + b/a$, or since $\alpha + \beta + \gamma = -3b/a$, the roots are

$$\tfrac{1}{3}(2\alpha - \beta - \gamma),\ \tfrac{1}{3}(2\beta - \gamma - \alpha),\ \tfrac{1}{3}(2\gamma - \alpha - \beta).$$

EXERCISES XV

31. If a, b and c are the roots of the equation $x^3 + px^2 + qx + r = 0$ show that the equation whose roots are a^2, b^2, c^2 is

$$x^3 + (2q - p^2)x^2 + (q^2 - 2pr)x - r^2 = 0. \qquad [Camb.\ Sch.]$$

32. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find the equation whose roots are α^3, β^3, γ^3.

33. Find the equation whose roots are the squares of the roots of the equation

$$x^3 - 4x^2 + 13x - 14 = 0.$$

Deduce that the given equation has two imaginary roots.

34. If α, β and γ are the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$, $\dfrac{1}{\beta} + \dfrac{1}{\gamma}$ and $\dfrac{1}{\gamma} + \dfrac{1}{\alpha}$. [*Camb. Sch.*]

35. If α, β, γ are the roots of the equation $x^3 - 3x + 1 = 0$, find the equations whose roots are

(1) $\alpha - 2,\ \beta - 2,\ \gamma - 2$;

(2) $(\alpha - 2)^2,\ (\beta - 2)^2,\ (\gamma - 2)^2$;

(3) $\dfrac{1}{(\alpha - 2)^2},\ \dfrac{1}{(\beta - 2)^2},\ \dfrac{1}{(\gamma - 2)^2}$. [*Madras, B.A.*]

36. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find the equation whose roots are

$$\alpha(\beta + \gamma),\ \beta(\gamma + \alpha),\ \gamma(\alpha + \beta).$$

37. If α, β, γ are the roots of $x^3 - x^2 - 2 = 0$, find the equation whose roots are

$$\alpha/(\beta + \gamma - \alpha),\ \beta/(\gamma + \alpha - \beta),\ \gamma/(\alpha + \beta - \gamma).$$

38. Transform the equation $2x^4 - 2x^3 + x^2 + 2x - 3 = 0$ into one in which the second term is missing.

39. Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 6 = 0$ by transforming it into one in which there is no term in x^3.

40. Explain how to transform the equation

$$f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$$

where $p_0, p_1, p_2, \ldots, p_n$ are fractions, into one in which the coefficients are integers. Apply your method to transform

$$x^4 + \tfrac{5}{6}x^3 + \tfrac{5}{12}x^2 + \tfrac{1}{8}x + 1 = 0$$

into an equation with integral coefficients. What is the relation between the roots of the transformed equation and the original equation?

41. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

find the equation whose roots are 10α, 10β, 10γ, 10δ.

42. Find the cubic equation whose roots are α, β, γ, where

$$\alpha + \beta + \gamma = a, \quad \alpha^2 + \beta^2 + \gamma^2 = b^2 \text{ and } \alpha^3 + \beta^3 + \gamma^3 = c^3.$$

If u_n denotes $\alpha^n + \beta^n + \gamma^n$, express u_n in terms of u_{n-1}, u_{n-2}, u_{n-3}, and a, b, c and prove that u_n is the coefficient of x^n in the expansion in powers of x of

$$\frac{3 - 2ax + \frac{1}{2}(a^2 - b^2)x^2}{1 - ax + \frac{1}{2}(a^2 - b^2)x^2 - \frac{1}{6}(a^3 - 3ab^2 + 2c^3)x^3}. \text{ [Camb. Sch.]}$$

43. Prove that the value of the expression $\dfrac{(x^2 - x + 1)^3}{(x^2 - x)^2}$ is unchanged if either $1 - x$ or $1/x$ is written in place of x. Prove that c, $1 - c$, $1/c$ are three of the roots of the equation

$$\frac{(x^2 - x + 1)^3}{(x^2 - x)^2} = \frac{(c^2 - c + 1)^3}{(c^2 - c)^2}.$$

What are the remaining roots? [*Camb. Sch.*]

15·61. Binomial Equations

An equation of the form $x^n \pm 1 = 0$, where n is a positive integer, is called a *binomial equation.* The roots of such equations have been considered in Chapter VIII. The roots of $x^n - 1 = 0$ are the n nth *roots of unity.*

Consider the case $n = 3$. Then since

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

the cube roots of unity are 1, $\frac{1}{2}(-1 \pm i\sqrt{3})$.

Let ω *denote one of the imaginary roots,* i.e. one of the roots of $x^2 + x + 1 = 0$. Then

$$\omega^2 + \omega + 1 = 0, \quad \omega^3 = 1.$$

Also since the product of the roots of $x^2 + x + 1 = 0$ is unity it follows that if one root is ω the other is ω^2. Thus any product involving real quantities and powers of ω can be represented in one of the forms

$$A + B\omega, \quad A + B\omega^2, \quad A\omega + B\omega^2$$

where A and B are real quantities. In particular

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) \equiv a^3 + b^3 + c^3 - 3abc.$$

For $(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

$$\begin{aligned} &\equiv a^2 + \omega^3 b^2 + \omega^3 c^2 + ab(\omega + \omega^2) \\ &\quad + bc(\omega^4 + \omega^2) + ca(\omega^2 + \omega) \\ &\equiv a^2 + b^2 + c^2 - ab - bc - ca, \end{aligned}$$

since $\omega^3 = 1$, $\omega^4 = \omega^3 . \omega = \omega$, $\omega^2 + \omega = -1$.

Multiplying both sides of the identity by $(a + b + c)$ we obtain the required form.

Examples.—(1) *Prove that if a, b, c are real and rational then*

$$(a + \omega b + \omega^2 c)^3 + (a + \omega^2 b + \omega c)^3$$

is real and rational.

Write $a + \omega b + \omega^2 c = \alpha$, $a + \omega^2 b + \omega c = \beta$. Then

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$$

Now since $\alpha^2 + \alpha\beta + \beta^2 = (\alpha - \omega\beta)(\alpha - \omega^2\beta)$, we obtain on changing the sign of β,

$$\alpha^2 - \alpha\beta + \beta^2 = (\alpha + \omega\beta)(\alpha + \omega^2\beta).$$

$$\begin{aligned} \text{Then } \alpha + \omega\beta &= a + \omega b + \omega^2 c + \omega(a + \omega^2 b + \omega c) \\ &= a(1 + \omega) + b(\omega + \omega^3) + c(\omega^2 + \omega^2) \\ &= a(1 + \omega) + b(1 + \omega) - 2c(1 + \omega). \end{aligned}$$

$$\begin{aligned} \text{Again } \alpha + \omega^2\beta &= a + \omega b + \omega^2 c + \omega^2(a + \omega^2 b + \omega c) \\ &= a(1 + \omega^2) + b(\omega + \omega^4) + c(\omega^2 + \omega^3) \\ &= -a\omega + 2b\omega - c\omega. \end{aligned}$$

$$\text{Also } \alpha + \beta = 2a + b(\omega + \omega^2) + c(\omega + \omega^2) = 2a - b - c.$$

$$\begin{aligned} \text{Hence } \alpha^3 + \beta^3 &= -\omega(1 + \omega)(2a - b - c)(2b - a - c)(2c - a - b) \\ &= (2a - b - c)(2b - a - c)(2c - a - b), \end{aligned}$$

since $-\omega(1 + \omega) = -\omega - \omega^2 = 1$. This is product of three real factors whose terms are all rational.

(2) *Find the values of the determinants:*

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^2 & \omega \\ \omega & \omega^2 & 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega & \omega^2 & 1 \\ \omega^2 & \omega & \omega^2 & 1 & \omega^2 \\ \omega & \omega^2 & 1 & \omega^2 & \omega \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & 1 & 1 \end{vmatrix}$$

where ω is an imaginary cube root of unity. [*Camb. Sch.*]

Let Δ_1, Δ_2 denote the first and second determinants respectively.

Consider Δ_1 and add the second and third columns on to the first. Then using $1 + \omega + \omega^2 = 0$ we have

$$\Delta_1 = \begin{vmatrix} 1 + \omega + \omega^2 & \omega & \omega^2 & \omega^2 & \omega \\ \omega + \omega^2 + 1 & \omega^2 & 1 & \omega & \omega^2 \\ \omega^2 + 1 + \omega & 1 & \omega & \omega^2 & 1 \\ 2\omega^2 + \omega & \omega & \omega^2 & 1 & \omega^2 \\ \omega + \omega^2 + 1 & \omega^2 & 1 & \omega^2 & \omega \end{vmatrix} = \begin{vmatrix} 0 & \omega & \omega^2 & \omega^2 & \omega \\ 0 & \omega^2 & 1 & \omega & \omega^2 \\ 0 & 1 & \omega & \omega^2 & 1 \\ \omega^2 - 1 & \omega & \omega^2 & 1 & \omega^2 \\ 0 & \omega^2 & 1 & \omega^2 & \omega \end{vmatrix}$$

$$= (1 - \omega^2) \begin{vmatrix} \omega & \omega^2 & \omega^2 & \omega \\ \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega^2 & \omega \end{vmatrix}$$

Adding the second and third columns on to the fourth and writing $1 + \omega + \omega^2 = 0$, we find

$$\Delta_1 = (1 - \omega^2)\begin{vmatrix} \omega & \omega^2 & \omega^2 & \omega^2 - 1 \\ \omega^2 & 1 & \omega & 0 \\ 1 & \omega & \omega^2 & 0 \\ \omega^2 & 1 & \omega^2 & 0 \end{vmatrix} = (1 - \omega^2)^2\begin{vmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega^2 \end{vmatrix}.$$

$$= (1 - \omega^2)^2\begin{vmatrix} \omega^2 + 1 + \omega & 1 & \omega \\ 1 + \omega + \omega^2 & \omega & \omega^2 \\ 1 + 2\omega^2 & 1 & \omega^2 \end{vmatrix} = (1 - \omega^2)^2\begin{vmatrix} 0 & 1 & \omega \\ 0 & \omega & \omega^2 \\ 1 + 2\omega^2 & 1 & \omega^2 \end{vmatrix}$$

$$= (1 - \omega^2)^2 (1 + 2\omega^2)\begin{vmatrix} 1 & \omega \\ \omega & \omega^2 \end{vmatrix} = 0.$$

$$\text{Again } \Delta_2 = \begin{vmatrix} 1 + \omega + \omega^2 & \omega & \omega^2 \\ \omega^2 + 1 + \omega & 1 & \omega \\ \omega^2 + 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & 1 & \omega \\ \omega^2 + 2 & 1 & 1 \end{vmatrix}$$

$$= (\omega^2 + 2)\begin{vmatrix} \omega & \omega^2 \\ 1 & \omega \end{vmatrix} = 0.$$

A simpler way of showing that the determinants are zero is by observing that in each of them there are two rows, one of which is a multiple of the other. Thus in Δ_1 consider the second row. If we multiply each term by ω we obtain

$$\omega^2 \quad \omega^3 \quad \omega \quad \omega^2 \quad \omega^3$$
$$\omega^2 \quad 1.$$

This is identical with the third row and so the determinant must vanish. In a similar way we see that the first and second rows of Δ_2 can be made identical.

If in the above discussion we replace unity by any quantity k we obtain the equations

$$x^n \pm k = 0.$$

The n nth roots of these equations are found in a similar way. In particular if k is real and $k^{\frac{1}{3}}$ is the ordinary arithmetical cube root of k then the three roots of $x^3 - k = 0$ are

$$k^{\frac{1}{3}},\ \omega k^{\frac{1}{3}},\ \omega^2 k^{\frac{1}{3}}.$$

Note that the three roots may be obtained from *any one* of them by multiplying it by $1, \omega, \omega^2$.

15·62. The Binomial Equation $x^n - 1 = 0$.

We now prove some special properties of this equation.

(*a*) *If α is an imaginary root of $x^n - 1 = 0$ and m is any positive integer, then α^m is also a root of $x^n - 1 = 0$.*

Now $(\alpha^m)^n = \alpha^{mn} = (\alpha^n)^m = 1^m = 1$. Thus α^m is a root of the equation.

(*b*) *If n is a prime number, a any imaginary root of* $x^n - 1 = 0$ *then the n roots may be expressed in the form*

$$1, \ a, \ a^2, \ \ldots, \ a^{n-1}.$$

The n roots are given by

$$x = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}, \quad r = 0, 1, 2, \ldots (n-1),$$

the value $r = 0$ corresponding to $x = 1$ (Chapter VIII., § 8·41).

Since n is a prime, it follows that for $n > 2$, n is odd. Hence $x = -1$ is not a root of $x^n - 1 = 0$.

Thus we can take a to be

$$\cos\frac{2\rho\pi}{n} + i\sin\frac{2\rho\pi}{n},$$

where ρ is one of the values $1, 2, \ldots, n-1$.

We know from (*a*) that $1, a, a^2, \ldots a^{n-1}$ are roots of $x^n - 1 = 0$.

In order to prove that this series of n values does represent all the roots it is sufficient to show that no two are equal. Let s, t be two positive integers and consider the roots a^s, a^t. Then

$$a^s = \left\{\cos\frac{2\rho\pi}{n} + i\sin\frac{2\rho\pi}{n}\right\}^s = \cos\frac{2s\rho\pi}{n} + i\sin\frac{2s\rho\pi}{n}.$$

$$a^t = \left\{\cos\frac{2\rho\pi}{n} + i\sin\frac{2\rho\pi}{n}\right\}^t = \cos\frac{2t\rho\pi}{n} + i\sin\frac{2t\rho\pi}{n}.$$

If $a^s = a^t$ then $2s\rho\pi/n$ and $2t\rho\pi/n$ must be equal, or differ by an integral multiple of 2π. Hence

$$\frac{2s\rho\pi}{n} - \frac{2t\rho\pi}{n} = 2k\pi,$$

where k denotes 0 or a positive or negative integer.

This equation may be written in the form $\dfrac{\rho(s-t)}{n} = k$.

Since n is prime ρ, $s - t$ have no factor in common with n and hence $\rho(s-t)/n$ cannot be positive or negative integer. Hence $k = 0$. Since $\rho \neq 0$ it follows that $s - t = 0$. Thus the roots a^s, a^t can only be equal if $s = t$.

If follows that the sequence of values $1, a, a^2, \ldots a^{n-1}$ are all different.

(*c*) *If m and n denote positive integers which have no common factor except unity then the equations* $x^m - 1 = 0$; $x^n - 1 = 0$ *have no common root except unity.*

The roots of $x^m - 1 = 0$ are

$$x = \cos\frac{2r\pi}{m} + i\sin\frac{2r\pi}{m}, \quad r = 0, 1, 2, \ldots (m-1).$$

The roots of $x^n - 1 = 0$ are

$$x = \cos\frac{2s\pi}{n} + i\sin\frac{2s\pi}{n}, \quad s = 0, 1, 2, \ldots (n-1).$$

The fractions $2r\pi/m$, $2s\pi/n$ are both positive and less than 2π. It follows that if the equations have a root in common,

$$\frac{2r\pi}{m} = \frac{2s\pi}{n}, \text{ i.e. } \frac{m}{n} = \frac{r}{s}.$$

Thus the fraction m/n is equal to another fraction with a smaller denominator. But since m and n have no common factor, this is impossible. It follows that the two equations $x^m - 1 = 0$, $x^n - 1 = 0$ have no root in common except unity.

(*d*) *If p is the highest common factor of the positive integers m and n, then the roots common to the equations $x^m - 1 = 0$, $x^n - 1 = 0$ are the roots of $x^p - 1 = 0$.*

It will be observed that if in this result we write $p = 1$ we obtain (*c*).

Write $m = \lambda p$, $n = \mu p$, λ and μ being positive integers with no common factor except unity. Using the expressions for the roots given in (*c*), it will be seen that two roots will be equal if, and only, if

$$\frac{2r\pi}{m} = \frac{2s\pi}{n},$$

where $1 \leqslant r \leqslant m-1$, $1 \leqslant s \leqslant n-1$. Substituting $m = \lambda p$, $n = \mu p$ this relation becomes

$$\frac{r}{s} = \frac{\lambda}{\mu}.$$

Then λ/μ is a fraction in its lowest terms since λ, μ have no common factor. Hence $r = \lambda k$, $s = \mu k$ where k is a positive integer; k cannot be a fraction since this would imply that λ and μ had a common factor, r and s being positive integers.

The value of the common root is

$$\cos\frac{2r\pi}{m} + i\sin\frac{2r\pi}{m} = \cos\frac{2k\pi}{p} + i\sin\frac{2k\pi}{p}$$

which is a root of $x^p - 1 = 0$.

A *special nth root of unity* is defined to be a root of $x^n - 1 = 0$ and which satisfies no other equation $x^m - 1 = 0$ where $m < n$. It will then follow that all the roots of $x^n - 1 = 0$ will be giv by the sequence

$$1, \ a, \ a^2, \ . \ . \ . \ a^{n-1}$$

where a is a special nth root of unity.

Example.—*Find the special roots of* $x^{12} - 1 = 0$.

Now 12 may be written as the product of two positive integers (> 1) in only two ways, viz. 2×6, 3×4. Hence the equation $x^{12} - 1 = 0$ has roots in common with $x^2 - 1 = 0$, $x^6 - 1 = 0$, $x^3 - 1 = 0$, $x^4 - 1 = 0$.

Now $(x^2 - 1)$, $(x^3 - 1)$ are factors of $x^6 - 1 = 0$. Thus in determining the special roots of $x^{12} - 1 = 0$ it is sufficient to determine the roots of $x^{12} - 1 = 0$ which are not roots of $x^6 - 1 = 0$, $x^4 - 1 = 0$.

Now $x^{12} - 1 = (x^6 - 1)(x^6 + 1)$. Thus the required roots will be those of $x^6 + 1 = 0$ which are not roots of $x^4 - 1 = 0$.

Hence if we determine the highest common factor of $x^6 + 1$ and $x^4 - 1$, divide $x^6 + 1$ by this factor and equate to zero the quotient obtained by the division we obtain the equation which determines all the special roots of $x^{12} - 1 = 0$.

$$\left.\begin{array}{l} x^4 - 1)x^6 \qquad + 1(x^2 \\ \qquad\quad \underline{x^6 - x^2} \\ \qquad\quad x^2 + 1)x^4 - 1(x^2 - 1 \end{array}\right\} \therefore \ H.C.F. \text{ is } x^2 + 1.$$

Now $(x^6 + 1)/(x^2 + 1) = x^4 - x^2 + 1$. Thus there are four special roots of $x^{12} - 1 = 0$, and these are the roots of $x^4 - x^2 + 1 = 0$. The roots are easily determined from this equation.

Alternatively we may proceed as follows:

$$x^{12} - 1 = (x^6 - 1)(x^6 + 1), \quad x^{12} - 1 = (x^4 - 1)(x^8 + x^4 + 1).$$

The factors $(x^2 - 1)$, $(x^3 - 1)$ need not be considered as they are factors of $x^6 - 1$.

The special roots will be roots of $x^6 + 1 = 0$ and $x^8 + x^4 + 1 = 0$.

The highest common factor equated to zero will give the roots.

$$\begin{array}{l} x^6 + 1)x^8 + x^4 + 1(x^2 \\ \qquad\quad \underline{x^8 + x^2} \\ \qquad\quad\quad x^4 - x^2 + 1)x^6 + 1(x^2 + 1 \\ \qquad\qquad\qquad\qquad\quad \underline{x^6 + 1} \\ \qquad\qquad\qquad\qquad\quad \ \ . \ \ . \end{array}$$

The H.C.F. is $x^4 - x^2 + 1$ and the special roots are the roots of

$$x^4 - x^2 + 1 = 0.$$

EXERCISES XV

44. Find the factors of $x^2 + y^2 + z^2 - xy - yz - zx$.

45. If ω denote one of the imaginary cube roots of unity prove that

(i) $\dfrac{a + \omega b + \omega^2 c}{b + \omega c + \omega^2 a} = \omega$;

(ii) $(1 - \omega^2 + \omega^4)^2 + (1 + \omega^2 + \omega^4)^2 = 4\omega$.

46. Form the cubic equation whose roots are

$$a + b + c,\ a + \omega b + \omega^2 c,\ a + \omega^2 b + \omega c,$$

where ω is one of the imaginary cube roots of unity.

47. Find the special roots of $x^6 - 1 = 0$.

48. Find the special roots of $x^{16} - 1 = 0$.

49. Prove that the special roots of $x^{18} - 1 = 0$ are the roots of the equation $x^6 - x^3 + 1 = 0$.

50. Give an expression for the roots of the equation $x^{2n+1} = 1$. Show that if α is a complex root of the equation $x^7 = 1$, the equation whose roots are $\alpha + \alpha^6$, $\alpha^2 + \alpha^5$, $\alpha^3 + \alpha^4$ is $y^3 + y^2 - 2y - 1 = 0$. [*Lond. B.A.*]

15·71. The Differential Coefficient of a Polynomial

If $f(x)$ is a polynomial of degree n, then the differential coefficient or *derived function* $f'(x)$ is *polynomial of degree* $n - 1$. It follows that $f'(x)$ is a continuous function of x.

If $f(x)$ has a maximum or minimum value when $x = x_0$ then $f'(x_0) = 0$, i.e. the maximum and minimum values of $f(x)$ are obtained by differentiating $f(x)$ and equating to zero the polynomial so obtained. Since $f'(x)$ is of degree $n - 1$ it follows that there are *at most* $n - 1$ maximum and minimum values or expressed graphically there are at most $n - 1$ turning points on the curve $y = f(x)$.

If the $(n - 1)$ roots of $f'(x) = 0$ are *real* and *distinct* there will be precisely $(n - 1)$ such values. For if β be one of the roots then $x - \beta$ is a non-repeated factor of $f'(x)$, i.e.

$$f'(x) = (x - \beta)\,\phi(x)$$

where $\phi(\beta) \neq 0$ and $\phi(x)$ keeps the same sign in the immediate neighbourhood of $x = \beta$. Thus $f'(x)$ must change sign as x passes through the value β.

It will be remembered that in general, $f'(x) = 0$ is a necessary but not a sufficient condition for a maximum or minimum. In addition $f'(x)$ must change sign from $+$ to $-$ for a maximum and from $-$ to $+$ for a minimum as x increases through the value

in question. Another criterion is, if β is a root of $f'(x) = 0$ then $f''(\beta) < 0$ for a maximum, $f''(\beta) > 0$ for minimum, but this fails to give a precise result if $f''(\beta) = 0$.

15·72. Rolle's Theorem

If α and β be two consecutive real roots of $f(x) = 0$ then between α and β there lies at least one real root of $f'(x) = 0$.

Since α and β, $\beta > \alpha$ are consecutive real roots of $f(x) = 0$ it follows that when x lies between α and β, $f(x)$ is always positive or always negative. Suppose the former. Then $f(x) > 0$ when x is slightly greater than α and since $f(\alpha) = 0$ it follows that $f(x)$ is *increasing immediately to the right* of $x = \alpha$. At such points $f'(x) > 0$. Similarly just to the left of $x = \beta$, $f(x)$ is decreasing, for $f(x) > 0$ and $f(\beta) = 0$. Thus *immediately to the left of* $x = \beta$, $f'(x) < 0$.

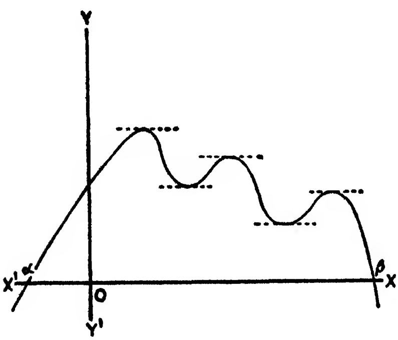


FIG. 40.

Since $f'(x)$ is continuous it can only change from a positive value to a negative value by passing through the value zero. Hence $f'(x) = 0$ for some x inside the interval (α, β).

It is clear from Fig. 40 that a polynomial $f(x)$ may have more than one maximum or minimum value between two consecutive zeros of $f(x) = 0$.

Combined with Rolle's theorem we have the result that between any two consecutive roots of $f(x) = 0$ there is *at least* one root of $f'(x) = 0$; *but there may be more than one.* In other words, between two consecutive roots of $f'(x) = 0$ there need not be a root of $f(x) = 0$.

We can assert further that *between two consecutive roots of $f'(x) = 0$ there is never more than one root of $f(x) = 0$.* For suppose that $f'(a) = 0$, $f'(b) = 0$ and that $f(\alpha) = 0$, $f(\beta) = 0$ where $a < \alpha < \beta < b$. Then from Rolle's theorem it follows that between α and β there must be a root of $f'(x) = 0$. But this is contrary to hypothesis. Hence there cannot be more than one root.

Example.—*Writing* $u_n(x) = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!}$, *show by induction or otherwise, that* $u_n(x) = 0$ *has no real roots when n is even, and one root, which is negative, when n is odd.* [*Lond. B.Sc.*]

Consider first $n = 1$ and $n = 2$. Then

$$u_1 = 1 + x = 0 \text{ when } x = -1.$$

$$u_2 = 1 + x + \tfrac{1}{2}x^2 > 0, \text{ for all values of } x.$$

This follows immediately from the theory of quadratic expressions. Now suppose the result is true for $n = 2p - 1$ and $n = 2p$, i.e. it is assumed that $u_{2p-1}(x)$ has only one real root which is negative, while $u_{2p}(x) > 0$ for all values of x.

Now $u_{2p+1}(0) = 1$, $u_{2p+1}(-\infty) = -\infty$, since the index of the highest power of x is odd. Hence the equation

$$u_{2p+1}(x) = 0$$

has at least one real root which is negative. There cannot be two real roots α, β; for if so there would be a real root of $u'_{2p+1}(x)$ between α and β. But

$$u'_{2p+1}(x) = u_{2p}(x) > 0,$$

so that $u'_{2p+1}(x) = 0$ has no real roots.

Again $u_{2p+2}(x) = u_{2p+1}(x) + \dfrac{x^{2p+2}}{(2p+2)\,!}$

For $x > 0$ it is clear from the definition of $u_n(x)$ that $u_{2p+1}(x) > 0$. To find the least value of $u_{2p+2}(x)$ for $x < 0$ consider

$$u'_{2p+2}(x) = u_{2p+1}(x) = 0.$$

This equation has only one real root α and as x passes through this value $u_{2p+1}(x)$ changes from $-$ to $+$. For $u_{2p+1}(-\infty) = -\infty$, $u_{2p+1}(0) = 1$. Hence $u_{2p+1}(x) = 0$ gives a minimum for $u_{2p+2}(x)$. The least value is $\alpha^{2p+2}/(2p+2)\,!$ which is positive. Since this value is positive it follows that $u_{2p+2}(x) = 0$ has no real roots.

Thus it has been shown that if the result is true for $n = 2p - 1$ and $n = 2p$ it is true for $n = 2p + 1$ and $n = 2p + 2$. Further, it has been proved in the cases $n = 1, 2$, i.e. $p = 1$. Thus the property is true in general.

15·81. Multiple Roots

Let $x = a$ be a r-multiple root of $f(x) = 0$. Then

$$f(x) = (x - a)^r \phi(x), \text{ where } \phi(a) \neq 0.$$

$$f'(x) = (x - a)^r \phi'(x) + r(x - a)^{r-1} \phi(x).$$

$$= (x - a)^{r-1} \{(x - a)\phi'(x) + r\phi(x)\}.$$

The second factor is different from zero when $x = a$. It follows that *a r-multiple root of $f(x) = 0$ is a $(r - 1)$-multiple root of* $f'(x) = 0$.

Continuing the method of argument it follows that if $f^m(x)$ denote the mth derivative of $f(x)$, $m < r$ then *a r-multiple root of $f(x) = 0$ is a $(r - m)$-multiple root of $f^m(x) = 0$.* Further, if n is the degree of $f(x)$ and $n \geqslant m \geqslant r$, then $f^m(a) \neq 0$.

15·82. The Derived Function as a Sum of Partial Fractions

If $\lambda_1, \lambda_2, \ldots \lambda_n$ *are the n roots of the equation* $f(x) = 0$ *then*

$$f'(x) = f(x) \sum_{r=1}^{n} \frac{1}{x - \lambda_r}.$$

We may suppose without loss of generality that the coefficient of x^n in the equation $f(x) = 0$ is unity. Then

$$f(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \ldots (x - \lambda_n).$$

Write $\phi(x) = (x - \lambda_2)(x - \lambda_3) \ldots (x - \lambda_n)$ so that $f(x) = (x - \lambda_1)\phi(x)$. Applying the rule for differentiation of a product,

$$f'(x) = (x - \lambda_1)\phi'(x) + \phi(x) = (x - \lambda_1)\phi'(x) + \frac{f(x)}{x - \lambda_1}.$$

Similarly we may write $\phi(x) = (x - \lambda_2)\psi(x)$ where $\psi(x)$ is $(x - \lambda_3)(x - \lambda_4) \ldots (x - \lambda_n)$. Applying the product rule again,

$$\phi'(x) = (x - \lambda_2)\psi'(x) + \psi(x) = (x - \lambda_2)\psi'(x) + \frac{f(x)}{(x - \lambda_1)(x - \lambda_2)}.$$

Thus $f'(x) = \dfrac{f(x)}{x - \lambda_1} + \dfrac{f(x)}{x - \lambda_2} + (x - \lambda_1)(x - \lambda_2)\psi'(x).$

Continuing this process it follows that

$$f'(x) = \frac{f(x)}{x - \lambda_1} + \frac{f(x)}{x - \lambda_2} + \frac{f(x)}{x - \lambda_3} + \ldots + \frac{f(x)}{x - \lambda_n}.$$

The result may be written in the form $\dfrac{f'(x)}{f(x)} = \displaystyle\sum_{r=1}^{n} \frac{1}{x - \lambda_r}.$

The result may be obtained more simply using logarithmic differentiation. Thus

$$\log f(x) = \sum_{r=1}^{n} \log (x - \lambda_r).$$

$$\text{Differentiating, } \frac{f'(x)}{f(x)} = \Sigma \frac{1}{x - \lambda_r}$$

which is the required result.

If the equation $f(x) = 0$ has a repeated root the equation may be represented in a slightly different way. Suppose that a_1 is a r-multiple root so that $a_1 = a_2 = a_3 = \ldots = a_r$. Then

$$f'(x) = \frac{rf(x)}{x - a_1} + \frac{f(x)}{x - a_{r+1}} + \frac{f(x)}{x - a_{r+2}} + \ldots + \frac{f(x)}{x - a_n}.$$

Similarly for the case in which there is more than one multiple root.

15·83. Sums of Powers of the Roots of an Equation

We now show how the result of the previous section may be used to find the sum of any given integral power of the roots of the equation $f(x) = 0$.

The sum of the mth powers of the roots of the equation $f(x) = 0$ is the coefficient of x^{-m} in the expansion of $xf'(x)/f(x)$ in ascending powers of $1/x$.

Consider the formal expansion of $1/(x - \lambda_r)$ in a series of ascending powers of $1/x$. Then

$$\frac{1}{x - \lambda_r} = \frac{1}{x}\left(1 - \frac{\lambda_r}{x}\right)^{-1}$$

$$= \frac{1}{x}\left\{1 + \frac{\lambda_r}{x} + \frac{\lambda_r^2}{x^2} + \dots\right\}, \quad |\lambda_r| < |x|.$$

$$= \frac{1}{x}\Sigma\frac{\lambda_r^m}{x^m}.$$

$$\text{Thus } \frac{xf'(x)}{f(x)} = \sum_{r=1}^{n}\left(\Sigma\lambda_r^m x^{-m}\right).$$

The coefficient of x^{-m} is $\lambda_1^m + \lambda_2^m + \dots \lambda_r^m + \dots + \lambda_n^m$ which is the sum of the mth powers of the roots.

Examples.—(1) *Find the sum of the cubes of the roots of the equation* $x^3 - 6x^2 + 11x - 6 = 0$. [*Madras, B.A.*]

Write $f(x) = x^3 - 6x^2 + 11x - 6$. Then $f'(x) = 3x^2 - 12x + 11$.

We require the coefficient of x^{-3} in the expansion of

$$\frac{3x^3 - 12x^2 + 11x}{x^3 - 6x^2 + 11x - 6}$$

in ascending powers of $1/x$. Write $y = 1/x$. Then

$$\frac{3x^3 - 12x^2 + 11x}{x^3 - 6x^2 + 11x - 6} = \left(3 - \frac{12}{x} + \frac{11}{x^2}\right)\bigg/\left(1 - \frac{6}{x} + \frac{11}{x^2} - \frac{6}{x^3}\right)$$

$$= (3 - 12y + 11y^2)\{1 - y(6 - 11y + 6y^2)\}^{-1}$$

$$= (3 - 12y + 11y^2)\{1 + y(6 - 11y + 6y^2) + y^2(6 - 11y + 6y^2)^2 + y^3(6 - 11y + 6y^2)^3 + \dots\}$$

$$= (3 - 12y + 11y^2)\{1 + 6y - 11y^2 + 6y^3 + 36y^2 - 132y^3 + \dots + 216y^3 + \dots\}$$

$$= (3 - 12y + 11y^2)(1 + 6y + 25y^2 + 90y^3 + \dots).$$

The coefficient of y^3 is $270 - 300 + 66 = 36$.

Alternatively in a numerical case such as this we may proceed by direct division. Thus we require the coefficient of y^3 in the quotient obtained by dividing $3 - 12y + 11y^2$ by $1 - 6y + 11y^2 - 6y^3$. Thus

$$
\begin{array}{l}
1 - 6y + 11y^2 - 6y^3)\,3 - 12y + 11y^2 \qquad (3 + 6y + 14y^2 + 36y^3 + \\
\qquad\qquad\qquad 3 - 18y + 33y^2 - 18y^3 \\
\qquad\qquad\qquad\qquad 6y + 22y^2 + 18y^3 \\
\qquad\qquad\qquad\qquad 6y - 36y^2 + 66y^3 - 36y^4 \\
\qquad\qquad\qquad\qquad\qquad 14y^2 - 48y^3 + 36y^4 \\
\qquad\qquad\qquad\qquad\qquad 14y^2 - 84y^3 + \dots \\
\qquad\qquad\qquad\qquad\qquad\qquad 36y^3 + \dots
\end{array}
$$

The required coefficient is thus 36. Clearly in the division it is unnecessary to consider y^4 or higher powers of y.

(2) *If S_m denote the sum of the mth powers of the roots of the equation $x^3 + px + q = 0$ find the values of S_3, S_5, S_{-3}.*

Write $f(x) = x^3 + px + q$. Then $f'(x) = 3x^2 + p$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the three roots of the equation. Then

$$\frac{f(x)}{x - \lambda_1} = x^2 + \lambda_1 x + \lambda_1^2 + p.$$

This may be obtained by direct division, observing that $\lambda_1^3 + p\lambda_1 + q = 0$ since $(x - \lambda_1)$ is a factor of $f(x)$. Similarly

$$\frac{f(x)}{x - \lambda_2} = x^2 + \lambda_2 x + \lambda_2^2 + p, \quad \frac{f(x)}{x - \lambda_3} = x^2 + \lambda_3 x + \lambda_3^2 + p.$$

Adding the three equations, and substituting the value of $f'(x)$,

$$3x^2 + p = 3x^2 + (\lambda_1 + \lambda_2 + \lambda_3)\,x + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 3p.$$

Equating corresponding coefficients,

$$S_1 = 0, \quad S_2 + 3p = p, \text{ giving } S_2 = -2p.$$

To find S_m for other values of m we may proceed as follows. Multiply the given equation by x^{m-3}. Then

$$x^m + px^{m-2} + qx^{m-3} = 0.$$

Put $x = \lambda_1, \lambda_2, \lambda_3$ in succession and add. Then

$$S_m + pS_{m-2} + qS_{m-3} = 0.$$

Putting $m = 3$ we have $S_3 + pS_1 + 3q = 0$, giving $S_3 = -3q$.

Putting $m = 4, 5$, we obtain

$$S_4 + pS_2 + qS_1 = 0, \quad S_5 + pS_3 + qS_2 = 0.$$

These equations give

$$S_4 = -pS_2 = 2p^2$$

$$S_5 = -pS_3 - qS_2 = 3pq + 2pq = 5pq.$$

Similarly S_m may be determined for higher values of m.

To find S_{-3} put $m = 2, 1, 0$ in succession. Then

$$S_2 + 3p + qS_{-1} = 0, \quad S_1 + pS_{-1} + qS_{-2} = 0,$$

$$3 + pS_{-2} + qS_{-3} = 0.$$

These equations give $S_{-1} = -p/q$, $S_{-2} = p^2/q^2$, $S_{-3} = -(p^3 + 3q^2)/q^3$.

15·91. Real Roots of the Derived Function

It may be proved that if $f(x) = 0$ *has all its roots real, then the derived equation* $f'(x) = 0$ *has all its roots real.*

If all the roots are distinct the result is an immediate deduction from Rolle's theorem. For $f'(x)$ is a polynomial of degree $n - 1$ and by Rolle's theorem $f'(x) = 0$ has $n - 1$ real roots.

Suppose now that $x = a$ is an r-multiple root and that the other roots are simple. Then $f(x) = (x - a)^r \phi(x)$ where $\phi(x)$ is a polynomial of degree $n - r$. Arranging the roots of $\phi(x) = 0$ in descending order of magnitude and expressing $f'(x)$ in the form $(x - a)^{r-1} F(x)$ it may be shown that $F(x) = 0$ has $n - r$ simple real roots each distinct from a. Hence $f'(x)$ has $(n - r) + (r - 1)$ real roots and since $f'(x)$ is of degree $n - 1$ it follows that all its roots are real.

The result extends directly to the case in which there is more than one multiple root.

15·92. Determination of Repeated (or Multiple) Roots

If the equation $f(x) = 0$ has a repeated root this value of x also satisfies the equation $f'(x) = 0$, i.e. the polynomials $f(x)$ and $f'(x)$ have a common factor. Thus to find a repeated root it is only necessary to find the highest common factor of $f(x)$ and $f'(x)$. This may be determined in numerical cases by the method of Chapter IX., § 9·11.

Examples.—(1) *The equations*

$$6x^3 - 23x^2 + 29x - 12 = 0, \quad 6x^4 - 29x^3 + 38x^2 + 4x - 24 = 0$$

have a common root. Find it, and hence find the remaining roots of the first equation. Find the repeated root of the second equation, and hence all its roots. [*Camb. Sch.*]

To find the common root we first find the H.C.F. of $6x^3 - 23x^2 + 29x - 12$ and $6x^4 - 29x^3 + 38x^2 + 4x - 24$.

$$\begin{array}{r|l} 6x^3 - 23x^2 + 29x - 12 \,) & 6x^4 - 29x^3 + 38x^2 + 4x - 24 \,(\, x - 1 \\ & 6x^4 - 23x^3 + 29x^2 - 12x \\ \hline & - 6x^3 + 9x^2 + 16x - 24 \\ & - 6x^3 + 23x^2 - 29x + 12 \\ \hline & - 14x^2 + 45x - 36 \end{array}$$

The common factor must also be a factor of

$$- 14x^2 + 45x - 36 = - (7x - 12)(2x - 3).$$

It is easily verified that the common factor is $2x - 3$ so that the common root of the two equations is $x = 3/2$.

Now $6x^4 - 29x^3 + 38x^2 + 4x - 24 = (2x - 3)(3x^3 - 10x^2 + 4x + 8)$.

If $f(x) \equiv 3x^3 - 10x^2 + 4x + 8$, $f(\frac{3}{2}) = 3.\frac{27}{8} - 10.\frac{9}{4} + 4.\frac{3}{2} + 8 \neq 0$. Hence $x = 3/2$ is not the repeated root. It follows that $f(x) = 0$ must have a repeated root.

Now $f'(x) = 9x^2 - 20x + 4 = (x - 2)(9x - 2)$. Thus the repeated root must be $x = 2$ or $x = 2/9$.

Now $f(2) = 3.8 - 10.4 + 4.2 + 8 = 0$. Hence the repeated root is $x - 2$ and $f(x) = (x - 2)^2 (3x + 2)$.

It follows that the four roots of the equation

$$6x^4 - 29x^3 + 38x^2 + 4x - 24 = 0$$

are $x = \frac{3}{2}, 2, 2, -\frac{2}{3}$. Again, since

$$\begin{aligned} 6x^3 - 23x^2 + 29x - 12 &= (2x - 3)(3x^2 - 7x + 4) \\ &= (2x - 3)(x - 1)(3x - 4) \end{aligned}$$

the roots of $6x^3 - 23x^2 + 29x - 12 = 0$ are $x = \frac{3}{2}, 1, \frac{4}{3}$.

(2) *If the equation* $x^5 + 5qx^3 + 5rx^2 + t = 0$ *has two equal roots prove that either of them is a root of the quadratic equation*

$$3rx^2 - 6q^2x - 4qr + t = 0. \qquad \text{[Camb. Sch.]}$$

Write $f(x) = x^5 + 5qx^3 + 5rx^2 + t$. Then if $f(x) = 0$ has a repeated root then this value is a root of

$$f'(x) = 5x^4 + 15qx^2 + 10rx = 0.$$

Thus the repeated root is a root of both the equations

$$x^5 + 5qx^3 + 5rx^2 + t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{and } x^3 + 3qx + 2r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

since we may assume that $t \neq 0$ so that $x = 0$ is not the common root. Multiply (ii) by x^2 and subtract from (i). We obtain

$$2qx^3 + 3rx^2 + t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Multiply (ii) by $2q$ and subtract from (iii). Then

$$3rx^2 - 6q^2x - 4qr + t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Since the value of x concerned is a root of both (i) and (ii) it is also a root of (iii) and (iv).

(3) *If the equation* $x^4 - 4ax^3 + 6x^2 + 1 = 0$ *has a repeated root* ξ, *show that* $3a = (\xi^2 + 3)/\xi$. *Hence or otherwise prove that there is only one positive* a *giving a repeated root, and that this value of* a *is* $(\frac{4}{3})^{\frac{3}{4}}$. [M.T.]

Write $f(x) = x^4 - 4ax^3 + 6x^2 + 1$. Then

$$f'(x) = 4x^3 - 12ax^2 + 12x = 4x(x^2 - 3ax + 3).$$

Hence if $f(x) = 0$ has a repeated root, this value must also be a root of $x^2 - 3ax + 3 = 0$, i.e. $3a = (x^2 + 3)/x$

$$
\begin{array}{l}
x^2 - 3ax + 3\,\big)\,x^4 - 4ax^3 + 6x^2 + 1\,\big(x^2 - ax + 3\,(1 - a^2) \\
\qquad\qquad\quad x^4 - 3ax^3 + 3x^2 \\
\qquad\qquad\quad\;\; \overline{\;- ax^3 + 3x^2\qquad\quad} \\
\qquad\qquad\quad\;\; - ax^3 + 3a^2x^2 - 3ax \\
\qquad\qquad\qquad\qquad\qquad 3\,(1 - a^2)\,x^2 + 3ax + 1 \\
\qquad\qquad\qquad\qquad\qquad 3\,(1 - a^2)x^2 - 9a\,(1 - a^2)\,x + 9\,(1 - a^2) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{(12a - 9a^3)\,x + 9a^2 - 8}
\end{array}
$$

Hence the common root is $(9a^2 - 8)/3a\,(3a^2 - 4)$. Substituting in $x^2 - 3ax + 3 = 0$ we obtain

$$\frac{(9a^2 - 8)^2}{9a^2\,(3a^2 - 4)^2} - \frac{3a\,(9a^2 - 8)}{3a\,(3a^2 - 4)} + 3$$

This equation reduces to $27a^4 = 64$. The only positive value of a is

$$\sqrt[4]{\frac{64}{27}} = \sqrt[4]{\frac{4^3}{3^3}} = (\tfrac{4}{3})^{\frac{3}{4}}.$$

EXERCISES XV

51. Given that two roots of the equation $16x^4 - 8x + 3 = 0$ are equal, find these roots and complete the solution of the equation.

[*N.Sc., Prelim.*]

52. Prove that

$$a^2 + b^2 + c^2 - bc - ca - ab = (a + \omega b + \omega^2 c)\,(a + \omega^2 b + \omega c),$$

where ω is a complex cube root of 1.

Prove that, if $(b - c)^n + (c - a)^n + (a - b)^n$ is divisible by $\Sigma a^2 - \Sigma bc$, then n is an integer, not a multiple of 3. Prove that if the same expression is divisible by $(\Sigma a^2 - \Sigma bc)^2$, then n is greater by one than a multiple of 3.

[*M.T.*]

53. Find the condition that the equation $x^4 + 2px^3 + 2rx + pr = 0$ have a pair of equal roots, and if the condition is satisfied, solve the equation completely. [*Camb. Sch.*]

54. Prove that the equation $x^n - k^n = 0$, $k \neq 0$ cannot have a repeated root.

55. If $f(x)$ is a polynomial in x and α is a double root of the equation $f(x) = 0$ prove that α is also a root of $f'(x) = 0$. Hence obtain all the roots of

$$16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0.$$

56. Find the multiple roots of $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$.

[*M.T.*]

57. Show that the equations

$$4x^4 + 4x^3 - 17x^2 - 9x + 18 = 0$$
$$2x^3 + 9x^2 + 13x + 6 = 0$$

have two roots in common and hence solve the equations.

[*Lond. Inter. Econ.*]

58. Factorize $2x^5 - x^4 + 5x^3 - 4x + 3$. [*Camb. Sch.*]

59. If the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$ has a multiple root α, prove that α is a root of the equation

$$p_1x^{n-1} + 2p_2x^{n-2} + 3p_3x^{n-3} + \ldots + np_n = 0.$$

60. (i) If the equation $x^4 - 4c^3x + 1 = 0$ has a pair of equal roots, find c and solve the equation completely.

(ii) If α is a root of the equation $x(x^2 - 9) = k(x^2 - 1)$, express the other two roots as rational functions of α, independent of k. [*Camb. Sch.*]

61. Show that the sum of the eleventh powers of the roots of

$x^7 + 5x^4 + 1 = 0$ is zero. [*Madras, B.A.*]

62. Show that the sum of the mth powers, where $m < n$ of the roots of the equation

$$x^n - 2x^{n-1} - 2x^{n-2} - \ldots - 2x - 2 = 0$$

is $3^m - 1$. [*Madras, B.A.*]

CHAPTER XVI

THEORY OF EQUATIONS (CONTINUED)

IN this chapter we first consider the solution of cubic and biquadratic equations and then pass to other types.

16·1. The Cubic Equation

We take the general cubic equation in the form

$$ax^3 + 3bx^2 + 3cx + d = 0 \quad \ldots\ldots\ldots\ldots\ (\text{i})$$

If we reduce the roots of the equation by $-b/a$, the equation takes the form

$$a^3x^3 + 3a(ac - b^2)x + a^2d - 3abc + 2b^3 = 0 \quad \ldots\ldots\ (\text{ii})$$

If α, β, γ are the roots of the original equation, those of the transformed equation are

$$\alpha + b/a,\ \beta + b/a,\ \gamma + b/a$$

or $\frac{1}{3}(2\alpha - \beta - \gamma)$, $\frac{1}{3}(2\beta - \gamma - \alpha)$, $\frac{1}{3}(2\gamma - \alpha - \beta)$.

[XV., § 15·54, Ex. 3.]

If $H \equiv ac - b^2$, $G \equiv a^2d - 3abc + 2b^3$, then (ii) becomes

$$x^3 + \frac{3H}{a^2}x + \frac{G}{a^3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ (\text{iii})$$

If now we multiply the roots of this equation by a, we obtain

$$x^3 + 3Hx + G = 0 \quad \ldots\ldots\ldots\ldots\ldots\ (\text{iv})$$

The roots of (iv) are $a\alpha + b$, $a\beta + b$, $a\gamma + b$.

16·21. Equation of Squared Differences

Let α, β, γ be the roots of the cubic

$$ax^3 + 3bx^2 + 3cx + d = 0.$$

Then we consider the problem of forming the equation whose roots are $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$, $(\alpha - \beta)^2$. If α', β', γ' are the roots of the transformed equation

$$x^3 + \frac{3H}{a^2}x + \frac{G}{a^3} = 0,$$

then $\beta' - \gamma' = \beta - \gamma$, $\gamma' - \alpha' = \gamma - \alpha$, $\alpha' - \beta' = \alpha - \beta$, that is the differences of the roots are the same for both equations, *i.e.* the equation of squared differences is the same for both.

$$\text{Write } y = (\beta' - \gamma')^2 = \Sigma\alpha'^2 - \alpha'^2 - \frac{2\alpha'\beta'\gamma'}{}$$

$$= -\frac{6H}{a^2} - \alpha'^2 + \frac{2G}{a^3\alpha'},$$

i.e. $a^3 y\alpha' = -6aH\alpha' - a^3\alpha'^3 + 2G.$

Substituting $-a^3\alpha'^3 = 3aH\alpha' + G$ it follows that

$$\alpha' = 3G/(a^3y + 3aH).$$

Hence the required equation will be the cubic in y obtained by substituting $x = 3G/(a^3y + 3aH)$ in

$$x^3 + \frac{3H}{a^2}x + \frac{G}{a^3} = 0.$$

We obtain in simplification

$$y^3 + \frac{18H}{a^2}y^2 + 81\frac{H^2}{a^4}y + \frac{27}{a^6}(G^2 + 4H^3) = 0.$$

This is the equation for the squared differences of the cubic $ax^3 + 3bx^2 + 3cx + d = 0$ where

$$H \equiv ac - b^2, \quad G \equiv a^2d - 3abc + 2b^3.$$

Now $G^2 + 4H^3 \equiv a^2\{a^2d^2 - 6abcd + 4ac^3 - 3b^2c^2 + 4b^3d\}.$

The quantity $(G^2 + 4H^3)/a^2$ is called the *discriminant* of the cubic $ax^3 + 3bx^2 + 3cx + d = 0$ and is usually denoted by Δ.

$$\text{Thus } G^2 + 4H^3 = a^2\,\Delta.$$

Example.—*Find the value of* $\Sigma(\beta - \gamma)^4(\gamma - \alpha)^2$ *where* α, β, γ *are the roots of the cubic* $x^3 + qx + r = 0$. [*Lond. B.Sc.*]

We first form the equation whose roots are

$$\alpha' = (\beta - \gamma)^2, \quad \beta' = (\gamma - \alpha)^2, \quad \gamma' = (\alpha - \beta)^2.$$

Proceeding as above, we have

$$y^3 + 6qy^2 + 9q^2y + 4q^3 + 27r^2 = 0.$$

We now require the value of $\Sigma\alpha'^2\beta'$.

Then $\Sigma\alpha'^2\beta' = (\Sigma\alpha')(\Sigma\alpha'\beta') - 3\alpha'\beta'\gamma'$

$$= (-6q)(9q^2) - 3(4q^3 + 27r^2) = -81r^2 - 66q^3.$$

16·22. Condition for a Repeated Root

If the equation $ax^3 + 3bx^2 + 3cx + d = 0$ has a repeated root then one of the expressions $\beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$ must be zero.

$$\text{Hence } (\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = 0.$$

From the equation of squared differences it follows that

$$(\beta - \gamma)^2 (\gamma - \alpha)^2 (\alpha - \beta)^2 = -\frac{27}{a^6}(G^2 + 4H^3) = -\frac{27}{a^4}\Delta.$$

Hence the condition that the cubic have a repeated root is that

$$G^2 + 4H^3 = 0, \quad \text{or} \quad \Delta = 0.$$

It should be observed that this condition is *both necessary and sufficient*. If in addition $H = 0$, it follows that the three roots are the same. Thus if $H = 0$, $G = 0$ the three roots of the cubic are identical. In other words, the condition that

$$ax^3 + 3bx^2 + 3cx + d$$

be a perfect cube is

$$ac - b^2 = 0, \quad a^2d - 3abc + 2b^3 = 0.$$

The condition for a repeated root may also be deduced from the property that if

$$f(x) \equiv ax^3 + 3bx^2 + 3cx + d$$

then $f(x) = 0$, $f'(x) = 0$ have a root in common.

Now $f'(x) = 3(ax^2 + 2bx + c)$.

$$\begin{array}{lllll}
ax^2 + 2bx + c\,) & ax^3 + 3bx^2 + 3cx & + d\,(x & & \\
 & \underline{ax^3 + 2bx^2 + \;\; cx} & & & \\
 & \qquad bx^2 + 2cx & + d & & \times a \\
 & \qquad abx^2 + 2acx & + ad & (b & \\
 & \qquad abx^2 + 2b^2x & + bc & & \\
 & & & & \\
 & \qquad\qquad 2(ac - b^2)\,x + ad - bc. & & &
\end{array}$$

Hence provided $ac - b^2 \neq 0$, i.e. $H \neq 0$ the common root will be $(bc - ad)/2(ac - b^2)$.

Substituting in $ax^2 + 2bx + c = 0$ we obtain the condition

$$\Delta = a^2d^2 - 6abcd + 4ac^3 - 3b^2c^2 + 4b^3d = 0.$$

If $H = 0$, there cannot be a common root unless

$$ad - bc = 0, \quad \text{i.e.} \quad a/b = c/d.$$

In this case $a^2x^3 + 3a^2bx^2 + 3a^2cx + a^2d = 0$ becomes

$$a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3 = 0, \quad \text{i.e.} \quad (ax + b)^3 = 0.$$

Thus the three roots of the cubic are equal.

If in $G = 0$ we substitute $b^2 = ac$ we obtain $ad - bc = 0$, so that the conditions $G = 0$, $H = 0$ are equivalent to $H = 0$, $ad - bc = 0$, or to

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Note that if a cubic has the form $x^3 + px + q = 0$ the condition for a repeated root is $27q^2 + 4p^3 = 0$.

Example.—*If* $z^3 + 3hz + g = \frac{1}{\mu - \nu}[\mu(z + \nu)^3 - \nu(z + \mu)^3]$ *for all values of* z, *show that*

$$2\mu = \frac{g}{h} + \frac{\sqrt{\Delta}}{h} \text{ and that } 2\nu = \frac{g}{h} - \frac{\sqrt{\Delta}}{h}$$

where $\Delta = g^2 + 4h^3$. *Show also that the cubic* $4z^3 - 27a^2(z + a) = 0$ *has two equal roots.* [*M.T.*]

Note that Δ is the discriminant of the cubic, for in the given case the coefficient of x^3 is unity.

$$\begin{aligned}\mu(z + \nu)^3 - \nu(z + \mu)^3 &= \mu(z^3 + 3\nu z^2 + 3\nu^2 z + \nu^3) \\ &\quad - \nu(z^3 + 3\mu z^2 + 3\mu^2 z + \mu^3) \\ &= (\mu - \nu)z^3 + 3\mu\nu(\nu - \mu)z + \mu\nu(\nu^2 - \mu^2).\end{aligned}$$

Hence $z^3 + 3hz + g \equiv z^3 - 3\mu\nu z - \mu\nu(\mu + \nu)$.

Thus $h = -\mu\nu$, $g = -\mu\nu(\mu + \nu)$, i.e. $\frac{g}{h} = \mu + \nu$ and

$$\frac{\Delta}{h^2} = \frac{g^2}{h^2} + 4h = (\mu + \nu)^2 - 4\mu\nu = (\mu - \nu)^2$$

Taking the positive square root $\mu - \nu = \frac{\sqrt{\Delta}}{h}$.

Combining with $\mu + \nu = \frac{g}{h}$ we obtain

$$2\mu = \frac{g}{h} + \frac{\sqrt{\Delta}}{h}, \quad 2\nu = \frac{g}{h} - \frac{\sqrt{\Delta}}{h}.$$

In $4z^3 - 27a^2(z + a) = 0$ it is easily verified that $\Delta = 0$. Thus $\mu = \nu$. To find the form that $\frac{1}{\mu - \nu}[\mu(z + \nu)^3 - \nu(z + \mu)^3]$ takes when $\mu = \nu$ write $\mu - \nu = t$ and consider the limit when $t \to 0$. The expression becomes

$$\begin{aligned}&\frac{1}{t}[\mu\{(z + \mu) - t\}^3 - (\mu - t)(z + \mu)^3] \\ &= \frac{1}{t}[\mu(z + \mu)^3 - 3\mu t(z + \mu)^2 + 3\mu t^2(z + \mu) - \mu t^3 \\ &\qquad - \mu(z + \mu)^3 + t(z + \mu)^3] \\ &= (z + \mu)^3 - 3\mu(z + \mu)^2 + 3\mu t(z + \mu) - \mu t^2.\end{aligned}$$

Letting $t \to 0$ the expression becomes

$$(z + \mu)^3 - 3\mu(z + \mu)^2 = (z + \mu)^2(z - 2\mu).$$

Thus if $\Delta = 0$ the equation $z^3 + 3hz + g = 0$ has a repeated root equal to $-\mu$.

16·23. Nature of the Roots of a Cubic

Since complex (or imaginary) roots occur in pairs we know that one root of the cubic is real.

We now prove that if $f(x) \equiv ax^3 + 3bx^2 + 3cx + d$, *the cubic* $f(x) = 0$ *has three real roots if* $G^2 + 4H^3 \leqslant 0$ *and two imaginary roots if* $G^2 + 4H^3 > 0$.

We consider the transformed equation whose roots are $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$, $(\alpha - \beta)^2$,

$$\text{i.e.}\quad F(y) \equiv y^3 + \frac{18H}{a^2}y^2 + 81\frac{H^2}{a^4}y + \frac{27}{a^6}(G^2 + 4H^3) = 0.$$

If α, β, γ are all real then the squares of the differences of the roots *must all be positive* so that the equation must have three positive roots. Conversely, if *all the roots of the equation are positive the roots of* $f(x) = 0$ *must all be real.*

For suppose α is real, while β and γ are imaginary. Now β and γ are conjugate complex numbers and may be expressed in the forms

$$\beta = u + iv, \quad \gamma = u - iv \quad \text{where} \quad v \neq 0.$$

$$\text{Then} \quad \beta - \gamma = 2iv, \quad (\beta - \gamma)^2 = -4v^2 < 0.$$

Thus in *the case of an imaginary root of* $f(x) = 0$, *the transformed equation must have a negative root.*

CASE (*a*). $G^2 + 4H^3 < 0$. Since G and H are real this implies that $H < 0$.

$$\text{Now} \quad F(-y) = -y^3 + \frac{18H}{a^2}y^2 - 81\frac{H^2}{a^4}y + \frac{27}{a^6}(G^2 + 4H^3),$$

and since all the coefficients are negative $F(-y)$ has no changes in sign and hence $F(y) = 0$ has no negative root. Hence the three roots of $f(x) = 0$ must be real.

CASE (*b*). $G^2 + 4H^3 = 0$. Either $G = 0$, $H = 0$ or G and H are both different from zero. In the former, the three roots are equal and since one root is real, the three must be real. If the latter conditions are satisfied there is a repeated root which is different from the third root. If two roots are equal all the roots must be real, since complex roots occur in pairs. For if the equal roots were complex, there would have to be two other complex roots, viz. the conjugate numbers. Since the equation has only three roots this is impossible.

CASE (c). $G^2 + 4H^3 > 0$. In this case $F(-\infty) = -\infty$, $F(0) = 27(G^2 + 4H^3)/a^6 > 0$. Hence $F(y) = 0$ has a negative root and $f(x) = 0$ has two complex roots.

It should be observed that if a given cubic has the fo $x^3 + px + q = 0$, then the conditions are as follows.

(a) For three real and distinct roots, $27q^2 + 4p^3 < 0$;

(b) For three real roots not all distinct, $27q^2 + 4p^3 = 0$;

(c) For two complex roots, $27q^2 + 4p^3 > 0$.

Example.—*Find for what values of the constant a the equation*

$$x^3 - 3x + a = 0$$

has three distinct real roots.

Show that if $h > 0$ *the equation* $x^3 - 3x - 2 - 27h = 0$ *has just one real root and that if this is denoted by* $2 + 3\xi$, *then* $0 < \xi < h$; *with the aid of this result obtain the narrower limits*

$$\frac{h}{(1 + h)^2} < \xi < h. \qquad \text{[Camb. Sch.]}$$

The condition that the roots of $x^3 - 3x + a = 0$ are real and distinct, is

$$27a^2 + 4(-3)^3 < 0, \text{ i.e. } a^2 < 4, \text{ or } -2 < a < 2.$$

If $a^2 > 4$ the equation has only one real root.

Since $h > 0$, $-2 - 27h < -2$; hence $x^3 - 3x - 2 - 27h = 0$ has just one real root.

Write $x = 2 + 3\xi$ and the equation becomes on simplification

$$f(\xi) \equiv \xi^3 + 2\xi^2 + \xi - h = 0.$$

Now $f(0) = -h < 0$, $f(h) = h^2(h + 2) > 0$. Hence the real root ξ lies between 0 and h.

Write $\xi = hy$, so that $0 < y < 1$. Then

$$f(hy) = h\{h^2y^3 + 2hy^2 + y - 1\}.$$

The roots of $h^2y^3 + 2hy^2 + y - 1 = 0$ are ξ/h, where ξ is a root of $f(\xi) = 0$.

Now write $y = 1 - z$. The equation reduces to

$$\phi(z) \equiv h^2 + 2h - z(3h^2 + 4h + 1) + z^2(3h^2 + 2h) - h^2z^3 = 0.$$

Thus $\phi(0) = h^2 + 2h > 0$;

$$\begin{aligned}
\phi\{(h^2 + 2h)/(1 + h)^2\} &= -(h^2 + 2h)\left\{\frac{h^2(h^2 + 2h)^2}{(1 + h)^6} - \frac{(h^2 + 2h)(3h^2 + 2h)}{(1 + h)^4} + \frac{3h^2 + 4h + 1}{(1 + h)^2} - 1\right\} \\
&= -(h^2 + 2h)\left[\frac{h(h^2 + 2h)}{(h + 1)^6}\{h(h^2 + 2h) - (3h + 2)(h + 1)^2\} + \frac{2h}{h + 1}\right] \\
&= -\frac{h(h^2 + 2h)}{(h + 1)^6}[(h^2 + 2h)\{h(h^2 + 2h) - (3h + 2)(h + 1)^2\} + 2(h + 1)^5] \\
&= -h(h^2 + 2h)(h^3 + 4h^2 + 6h + 2)/(h + 1)^6.
\end{aligned}$$

Since $h > 0$, this expression is negative. Thus the real root of the cubic in z lies between 0 and $(h^2 + 2h)/(1 + h)^2$.

Now $z = 1 - y = 1 - \xi/h$. Thus

$$0 < 1 - \frac{\xi}{h} < \frac{h^2 + 2h}{(1 + h)^2}, \quad \text{i.e.} \quad \frac{1}{(1 + h)^2} < \frac{\xi}{h} < 1.$$

The condition for real roots may also be determined by considering the maximum and minimum values of the cubic expression

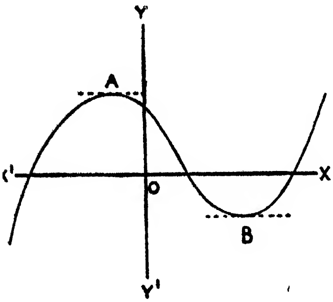


FIG. 41.

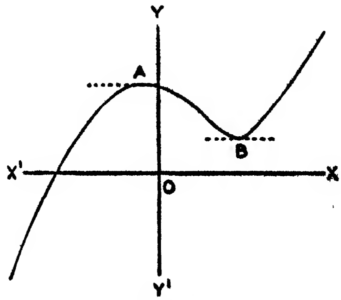


FIG. 43.

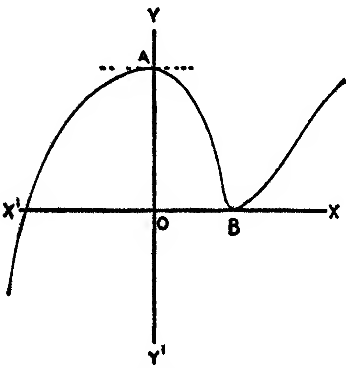


FIG. 42.

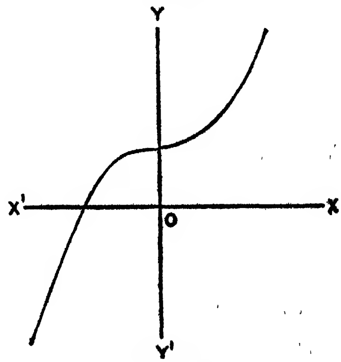


FIG. 44.

involved. Let the cubic equation be $x^3 + px + q = 0$. The general behaviour of the function

$$\psi(x) = x^3 + px + q, \quad p \neq 0, \quad q \neq 0$$

is as follows. When x is large and negative so is $\psi(x)$. As x increases so does $\psi(x)$ until in general a maximum is reached. As x increases beyond this point $\psi(x)$ decreases until a minimum is reached. Beyond this point $\psi(x)$ will increase with x. The general behaviour is indicated in Figs. 41, 42, 43, 44.

In Figs. 41, 42, 43, A corresponds to a maximum, B to a minimum, while Fig. 44 indicates the case in which there is no maximum or minimum. There will be three real roots if A and B lie opposite sides of the X-axis. Corresponding to Fig. 41 the are real and distinct, to Fig. 42 the roots are real but there is a repeated root where the curve touches the X-axis; to Figs. 43, 44 correspond two imaginary roots.

Now $\psi'(x) = 3x^2 + p$. If $p > 0$, $\psi'(x) > 0$ and hence the function $\psi(x)$ has no maximum or minimum value. In this case $\psi(x)$ steadily increases as x increases. Interpreted graphically this asserts that the curve $y = \psi(x)$ steadily rises as x increases and hence can cut the X-axis at only one point (Fig. 44). Thus if $p > 0$ there is only one real root.

Next suppose that $p < 0$ and write $p = -3\lambda^2$ where $\lambda > 0$. For maximum and minimum values,

$$3x^2 - 3\lambda^2 = 0, \text{ i.e. } x = \pm\lambda.$$

Now $\psi''(x) = 6x$ and is positive when x is positive and negative when x is negative. Thus $x = \lambda$ gives a minimum, $x = -\lambda$ a maximum. The corresponding values of $\psi(x)$ are $-2\lambda^3 + q$ and $2\lambda^3 + q$.

It follows that the roots of $\psi(x)$ will all be real if $2\lambda^3 + q > 0$ and $-2\lambda^3 + q < 0$.

If $q > 0$ the first condition is satisfied and the second becomes $2\lambda^3 > q$.

If $q < 0$ the second condition is satisfied and the first reduces to $2\lambda^3 > -q$.

We can reduce these two conditions to one by observing that they are included in $4\lambda^6 > q^2$. Substituting $\lambda^2 = -p/3$ this condition becomes

$$27q^2 + 4p^3 < 0.$$

Arguing in a similar way we can show that the condition for two repeated or coincident roots is $27q^2 + 4p^3 = 0$, while the condition for two imaginary roots is $27q^2 + 4p^3 > 0$.

16·24. Cubic Polynomial as the Sum of two Cubes

We now prove that *the cubic polynomial* $ax^3 + 3bx^2 + 3cx + d$ *can in general be expressed uniquely in the form* $A(x-\lambda)^3 + B(x-\mu)^3$ *where* λ *and* μ *are the roots of the quadratic equation*

$$H \equiv (at + b)(ct + d) - (bt + c)^2 = 0.$$

The function H is of great importance in the theory of the cubic and is called the **Hessian**.

$$ax^3 + 3bx^2 + 3cx + d = A(x-\lambda)^3 + B(x-\mu)^3$$
$$= (A+B)x^3 - 3(A\lambda + B\mu)x^2 + 3(A\lambda^2 + B\mu^2)x - A\lambda^3 - B\mu^3.$$

Equating corresponding coefficients,

$$\left.\begin{array}{l} a = A + B,\ b = -A\lambda - B\mu \\ c = A\lambda^2 + B\mu^2,\ d = -A\lambda^3 - B\mu^3 \end{array}\right\}.$$

The solution of these equations is in general possible, as there are four equations connecting the four unknowns A, B, λ, μ. Eliminating A between the first pair of equations, and also between the second pair, we have

$$\lambda a + b = B(\lambda - \mu) \quad \text{(i)}$$

$$\lambda c + d = B\mu^2(\lambda - \mu) \quad \text{(ii)}$$

$$\therefore\ (\lambda a + b)(\lambda c + d) = \{B\mu(\lambda - \mu)\}^2 \quad \text{(iii)}$$

$$\text{Again,}\ \lambda b + c = B\mu(\lambda - \mu) \quad \text{(iv)}$$

From (iii) and (iv), $(\lambda a + b)(\lambda c + d) - (\lambda b + c)^2 = 0$. Hence λ is a root of $H = 0$. From symmetry it follows that μ is also a root of $H = 0$.

Further, if λ and μ be interchanged it is clear that A and B would also be interchanged. Thus in general the expression of the cubic as the sum of two cubes is unique.

The argument will fail if $\lambda = \mu$, i.e. the roots of $H = 0$ are equal. For in this case the *four* equations obtained by equating coefficients reduce to

$$a = A + B, \qquad b = -\lambda(A + B),$$
$$c = \lambda^2(A + B), \quad d = -\lambda^3(A + B).$$

It will be seen that $\lambda = \mu$ implies that the equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

has a repeated root.

The condition $\lambda = \mu$ is that H should be a perfect square, *i.e*

$$(ad - bc)^2 = 4(ac - b^2)(bd - c^2),$$

i.e. $\Delta \equiv a^2d^2 - 6abcd + 4ac^3 - 3b^2c^2 + 4b^3d = 0$.

This is the condition that the cubic equation should have a repeated root. [§ 16·22.]

16·25. Solution of a Cubic Equation

The method of solution to be adopted depends on the nature of the roots. If two are imaginary, *Cardan's* method is used, while if all the roots are real, a trigonometrical method is employed.

Let the cubic equation be $ax^3 + 3bx^2 + 3cx + d = 0$. The the transformation $y = ax + b$ reduces it to the form

$$y^3 + 3Hy + G = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where $H \equiv ac - b^2$, $G \equiv a^2d - 3abc + 2b^3$. [§ 16·1.]

Equation (i) is taken as the starting point for the solution, in each case.

16·26. Cardan's Solution

In practice this method is used only when $G^2 + 4H^3 > 0$, *i.e.* the equation has two imaginary roots.

Let $y = m^{\frac{1}{3}} + n^{\frac{1}{3}}$ so that $y^3 - 3m^{\frac{1}{3}}n^{\frac{1}{3}}y - (m + n) = 0$.

Comparing the equation with $y^3 + 3Hy + G = 0$ we have

$$H = -m^{\frac{1}{3}}n^{\frac{1}{3}},\ G = -(m + n).$$

Since $-H^3 = mn$ it follows that m, n are the roots of the quadratic equation

$$z^2 + Gz - H^3 = 0.$$

The roots are $\frac{1}{2}\{-G \pm \lambda\}$ where $\lambda^2 = G^2 + 4H^3$.

Taking $m = \frac{1}{2}\{-G + \lambda\}$ the three values of $m^{\frac{1}{3}}$ are

$$\sqrt[3]{\{\tfrac{1}{2}(-G + \lambda)\}},\quad \omega\sqrt[3]{\{\tfrac{1}{2}(-G + \lambda)\}},\quad \omega^2\sqrt[3]{\{\tfrac{1}{2}(-G + \lambda)\}}$$

where ω is an imaginary cube root of unity. The corresponding values of $n^{\frac{1}{3}}$ are given by $n^{\frac{1}{3}} = -Hm^{-\frac{1}{3}}$.

The three roots of the cubic are then given by the three values of $y = m^{\frac{1}{3}} + n^{\frac{1}{3}}$.

If $G^2 + 4H^3 > 0$, λ is real, while if $G^2 + 4H^3 < 0$, λ is imaginary and the three roots are given in imaginary form. But in this latter case the three roots are actually real, so that the method is not convenient for a cubic equation in which all the roots are real.

16·27. Trigonometrical Solutions

In Cardan's method the given cubic is identified with the equation

$$y^3 - 3m^{\frac{1}{3}}n^{\frac{1}{3}}y - (m + n) = 0.$$

In the trigonometrical method we use one or other of the identities

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta,$$
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta.$$

Taking the first of these we write it in the form

$$\cos^3\theta - \tfrac{3}{4}\cos\theta - \tfrac{1}{4}\cos 3\,\theta = 0 \quad \ldots\ldots\ldots\ldots\ (\text{i})$$

Writing $y = \rho\cos\theta,\ \rho > 0$ in $y^3 + 3Hy + G = 0$ we have

$$\cos^3\theta + \frac{3H}{\rho^2}\cos\theta + \frac{G}{\rho^3} = 0 \qquad (\text{ii})$$

Comparing (i) and (ii), $\rho = 2\sqrt{(-H)}$, $\cos 3\theta = -4\,G/\rho^3$.

If $G^2 + 4H^3 < 0$, $H < 0$ so that ρ is real. Further, in order that θ be real

$$4G \leqslant 1, \text{ i.e. } \frac{16G^2}{\rho^6} \leqslant 1.$$

This reduces to $G^2 + 4H^3 \leqslant 0$.

If ϕ is any value of θ satisfying $\cos 3\theta = -4G/\rho^3$, then the general solution is

$$3\theta = 2n\pi \pm 3\phi.$$

Three different values of $\cos\theta$ will be given by $\theta = \phi,\ \frac{2}{3}\pi \pm \phi$. The three solutions of the given cubic will be

$$\rho\cos\phi,\ \rho\cos(\tfrac{2}{3}\pi + \phi),\ \rho\cos(\tfrac{2}{3}\pi - \phi).$$

In a similar way we could use the equation for $\sin 3\theta$ instead of that for $\cos 3\theta$. [See Ex. 2 on p. 668.]

Examples.—(1) *Solve the equation* $x^3 + 6x^2 - 12x + 32 = 0$. [*Camb. Sch.*]

In this case $a = 1$, $b \quad 2$, $c = -4$, $d = 32$. $H = ac - b^2 = -8$, $G = a^2d - 3abc + 2b^3 = 72$. Hence $G^2 + 4H^3 > 0$ and the equation has two imaginary roots.

To remove the term in x^2 reduce the roots by -2.

$$\begin{array}{cccc}
1 & 6 & -12 & 32 \\
 & -2 & -8 & 40 \\
\hline
 & 4 & -20 & 72 \\
 & -2 & -4 & \\
\hline
 & 2 & -24 & \\
 & -2 & & \\
\hline
 & 0 & &
\end{array}$$

The transformed equation is $y^3 - 24y + 72$

Now suppose that $y = m^{\frac{1}{3}} + n^{\frac{1}{3}}$ so that

$$y^3 - 3m^{\frac{1}{3}}n^{\frac{1}{3}}y - (m + n) = 0.$$

Comparing the two equations, $m^{\frac{1}{3}}n^{\frac{1}{3}} = 8$, $m + n = -72$.
Then m and n are the roots of the quadratic equation

$$z^2 + 72z + 512 = 0.$$

$$\text{Thus } z = -8 \text{ or } -64.$$

Taking $m = -8$, the three values of $m^{\frac{1}{3}}$ are -2, -2ω, $-2\omega^2$. The corresponding values of $n^{\frac{1}{3}}$ are given by $m^{\frac{1}{3}}n^{\frac{1}{3}} = 8$ and are -4, $-4\omega^{-1} = -4\omega^2$, $-4\omega^{-2} = -4\omega$. Thus the three roots of the transformed equation are

$$-6, \quad -2(\omega + 2\omega^2), \quad -2(\omega^2 + 2\omega).$$

Hence the roots of the given cubic equation are

$$-8, \quad -2(1 + \omega + 2\omega^2), \quad -2(1 + 2\omega + \omega^2), \quad \textit{i.e.} \ -8, \quad -2\omega^2, \quad -2\omega.$$

(2) *Prove that* $x = 2 \sin 10°$ *is a root of the equation* $x^3 - 3x + 1 = 0$ *and find the other two roots.* [*Camb. Sch.*]

The condition that the three roots of $x^3 + px + q = 0$ are real is that $27q^2 + 4p^3 < 0$. In this case $p = -3$, $q = 1$ and the condition is obviously satisfied.

We compare the given equation with

$$\sin^3 \theta - \tfrac{3}{4} \sin \theta + \tfrac{1}{4} \sin 3\theta = 0.$$

Write $x = \rho \sin \theta$ so that $\sin^3 \theta - \dfrac{3}{\rho^2} \sin \theta + \dfrac{1}{\rho^3} = 0.$

Thus $\rho^2 = 4$, $\rho = 2$ and $\sin 3\theta = 4/\rho^3 = \frac{1}{2}$.

The general solution of this equation is $3\theta = n \times 180° + (-1)^n 30°$.

The values $\theta = -70°, 10°, 50°$ give three distinct values for $\sin \theta$.

Hence the three roots of the equation are

$$-2 \sin 70°, \quad 2 \sin 10°, \quad 2 \sin 50°.$$

(3) *Find the relation between* p *and* q *in order that the equation* $x^3 - px + q = 0$ *may be put in the form*

$$(x^2 + mx + n)^2 = x^4.$$

Hence or otherwise solve the equation $8x^3 - 36x + 27 = 0$. [*Camb. Sch.*]

The equation $(x^2 + mx + n)^2 = x^4$ when expanded, gives

$$2mx^3 + (m^2 + 2n)x^2 + 2mnx + n^2 = 0.$$

Comparing with $x^3 - px + q = 0$,

$$\frac{2m}{1} = \frac{m^2 + 2n}{0} = \frac{2mn}{-p} = \frac{n^2}{q}.$$

Hence assuming $m, n \neq 0$, we have $m^2 + 2n = 0$, $p = -n$, $2m = p^2/q$.
Eliminating m and n, we obtain

$$p^4 - 8pq^2 = 0, \text{ i.e. } p^3 - 8q^2 = 0, \text{ assuming } p \neq 0.$$

This is the required relation between the coefficients.

In the particular case $p = 9/2$, $q = 27/8$, $p^3 - 8q^2 = 0$. Hence the equation may be written in the form

$$(x^2 + mx + n)^2 = x^4$$

where $m = \frac{1}{2} \cdot \frac{81}{4} \cdot \frac{8}{27} = 3$, $n = -\frac{9}{2}$.

Thus the equation becomes $(x^2 + 3x - \frac{9}{2})^2 = x^4$ giving

$$x^2 + 3x - \tfrac{9}{2} = \pm x^2,$$

i.e. $3x - \frac{9}{2} = 0$ or $2x^2 + 3x - \frac{9}{2} = 0$.

Hence $x = \frac{3}{2}$ or $-\frac{3}{4}(1 \pm \sqrt{5})$.

16·3. The Biquadratic Equation

The general equation of the fourth degree is conveniently represented in the form

$$f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0.$$

We consider here a few particular properties of the equation* $f(x) = 0$.

In what follows we suppose that a is *positive*. Now

$$f'(x) = 4(ax^3 + 3bx^2 + 3cx + d).$$

Since $f'(x) = 0$ is a cubic equation its roots may be assumed to be known.

If $f'(x) = 0$ has only one real root, this value must correspond to a minimum of $f(x)$. For $f(x) \to +\infty$ when $x \to -\infty$ or to $+\infty$ and since $f(x)$ is continuous it must have a minimum value.

If $f'(x) = 0$ has three real roots which are distinct, then $f(x)$ possesses two minima and one maximum. If the three roots are real but two are coincident then $f(x)$ has no maximum. In this case $f'(x)$ can be written in the form

$$f'(x) \equiv 4a(x - \alpha)(x - \beta)^2, \quad \alpha \neq \beta.$$

As x passes through the value β, $f'(x)$ does not change sign since $(x - \beta)^2$ is never negative. The value $x = \alpha$ must correspond to a minimum since $f(x)$ must possess at least one minimum value.

If the three roots are coincident and equal to α, then

$$f'(x) \equiv 4a(x - \alpha)^3.$$

As x increases through the value α, $f'(x)$ changes sign from $-$ to $+$ $(a > 0)$ so that $x = \alpha$ gives a minimum.

In any particular biquadratic equation we can determine the number of real roots, and approximate to their values, by calculating the maximum and minimum values of $f(x)$.

* For a more detailed discussion of the biquadratic equation reference should be made to treatises on theory of equations, *e.g.* Burnside and Panton, *Theory of Equations*.

We distinguish two cases:

(i) *One minimum, no maximum.* Denote by λ the minimum value of $f(x)$.

(*a*) If $\lambda > 0$, $f(x) > 0$ for all x, and $f(x) = 0$ has no roots. [See Fig. 45.]

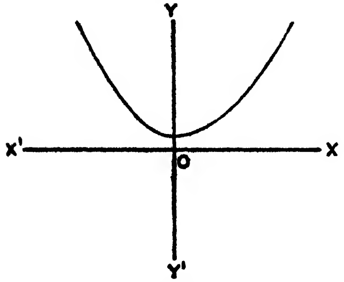


FIG. 45.

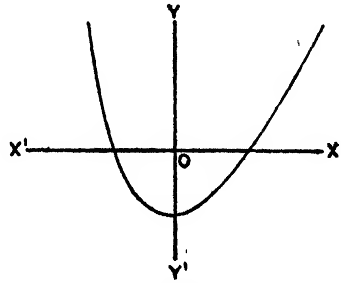


FIG. 46.

(*b*) If $\lambda < 0$, there are two real roots and two imaginary roots. [See Fig. 46.] The limiting case $\lambda = 0$ corresponds to two coincident roots.

(ii) *Two minima, one maximum.* Let $\lambda_1 = f(x_1)$, $\lambda_2 = f(x_2)$ where $x_1 < x_2$ be the two minimum values, ρ the maximum value of $f(x)$.

(*a*) If $\lambda_1 > 0$, $\lambda_2 > 0$ there are no real roots for $\rho > 0$. [See Fig. 47.]

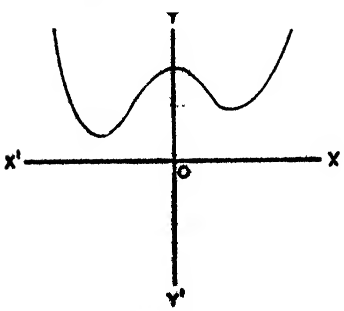


FIG. 47.

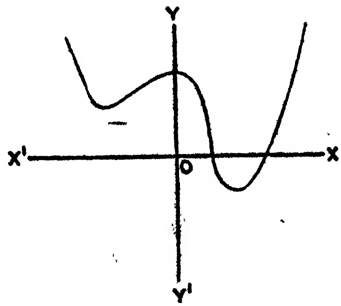


FIG. 48.

(*b*) If $\lambda_1 > 0$, $\lambda_2 < 0$ there are two real roots and two imaginary roots, for $\rho > 0$. [See Fig. 48.] $\lambda_2 = 0$ corresponds to the case in which the two real roots are coincident.

(*c*) If $\lambda_1 < 0$, $\lambda_2 < 0$, $\rho > 0$, the four roots are real. [See Fig. 49.]

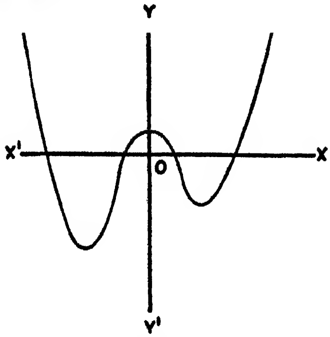


FIG. 49.

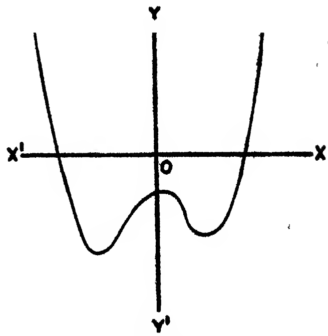


FIG. 50.

(*d*) If $\lambda_1 < 0$, $\lambda_2 < 0$, $\rho < 0$, there are two real and two imaginary roots. [See Fig. 50.]

Examples.—(1) *Find the range of values of k for which all the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ are real.* [*Camb. Sch.*]

$$\text{Write } f(x) = x^4 + 4x^3 - 8x^2 + k.$$
$$f'(x) = 4x^3 + 12x^2 - 16x.$$
$$f''(x) = 12x^2 + 24x - 16.$$

Hence $f'(x) = 0$ when $4x(x^2 + 3x - 4) = 0$, i.e. $x = 0, 1, -4$.

When $x = 0$, $f''(x) < 0$ while when $x = 1$ or -4, $f''(x) > 0$.

Hence $x = 0$ gives a maximum value, $x = 1, -4$ minimum values. The corresponding values of $f(x)$ are k, $k - 3$, $k - 128$.

Hence in order that $f(x) = 0$ have four real roots

$$k > 0,\ k - 3 < 0,\ k - 128 < 0,$$
$$\text{i.e. } 0 < k < 3.$$

(2) *If $f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e$, $H \equiv ac - b^2$ prove that the equation $f(x) = 0$ has at least two imaginary roots if $H > 0$.*

This result may be obtained by transforming the equation into one in which the term in x^3 is missing. Write $y = ax + b$ so that the transformed equation is

$$a\left(\frac{y-b}{a}\right)^4 + 4b\left(\frac{y-b}{a}\right)^3 + 6c\left(\frac{y-b}{a}\right)^2 + 4d\left(\frac{y-b}{a}\right) + e = 0.$$

This equation reduces to

$$y^4 + 6(ac - b^2)y^2 + 4(a^2d - 3abc + 2b^3)y + 6ab^2c - 3b^4 + a^3e - 4a^2bd = 0,$$
$$\text{i.e. } y^4 + 6Hy^2 + 4Gy + a^2I - 3H^2 = 0,$$

where $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$, $I = ae - 4bd + 3c^2$.

Since $H > 0$, it follows from Theorem IX., § 15·32, that the transformed equation has at least two imaginary roots. Since a, b are real it follows that $f(x) = 0$ has at least two imaginary roots.

16·41. The Biquadratic as the Product of Quadratic Factors

Assume $ax^4 + 4bx^3 + 6cx^2 + 4dx + e$

$$\equiv a(x^2 + \lambda x + \mu)(x^2 + \lambda' x + \mu').$$

Expanding the right-hand side, we have

$$ax^4 + a(\lambda + \lambda')x^3 + a(\lambda\lambda' + \mu + \mu')x^2 + a(\lambda\mu' + \lambda'\mu)x + a\mu\mu'.$$

Equating corresponding coefficients

$$\left.\begin{aligned} \lambda + \lambda' &= \frac{4b}{a}, \quad \mu + \mu' = \frac{6c}{a} - \lambda\lambda' \\ \lambda\mu' + \lambda'\mu &= \frac{4d}{a}, \quad \mu\mu' = \frac{e}{a} \end{aligned}\right\}$$

Now the determinant

$$\begin{vmatrix} 2 & \lambda + \lambda' & \mu + \mu' \\ \lambda + \lambda' & 2\lambda\lambda' & \lambda\mu' + \lambda'\mu \\ \mu + \mu' & \lambda\mu' + \lambda'\mu & 2\mu\mu' \end{vmatrix}$$

is the product of the zero determinants

$$\begin{vmatrix} 1 & 1 & 0 \\ \lambda & \lambda' & 0 \\ \mu & \mu' & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 0 \\ \lambda' & \lambda & 0 \\ \mu' & \mu & 0 \end{vmatrix}$$

and so is identically zero. Substitution gives

$$\begin{vmatrix} 2 & \dfrac{4b}{a} & \dfrac{6c}{a} - \lambda\lambda' \\ \dfrac{4b}{a} & 2\lambda\lambda' & \dfrac{4d}{a} \\ \dfrac{6c}{a} - \lambda\lambda' & \dfrac{4d}{a} & \dfrac{2e}{a} \end{vmatrix} = 0$$

i.e.
$$\begin{vmatrix} 2a & 2b & 6c - a\lambda\lambda' \\ 4b & a\lambda\lambda' & 4d \\ 6c - a\lambda\lambda' & 2d & 2e \end{vmatrix} = 0.$$

This determinant when expanded gives a cubic equation in $\lambda\lambda'$. Solving the equation and using the original equations we can obtain values for λ, λ', μ, μ'.

Example.—*Express the biquadratic* $6x^4 - 13x^3 + 24x^2 - 17x + 12$ *as the product of quadratic factors.*

In this case $a = 6,\ b = -\frac{13}{4},\ c = 4,\ d = -\frac{17}{4},\ e = 12.$ Substituting in the determinant and writing $6\lambda\lambda' = t$, we have

$$\begin{vmatrix} 12 & -\frac{13}{2} & 24 - t \\ -13 & t & -17 \\ 24 - t & -\frac{17}{2} & 24 \end{vmatrix} = 0.$$

Expanding by Sarrus' rule [Chap. XI., § 11·42], we have

$$t^3 - 48t^2 + 509t - 1542 = 0.$$

The roots of this cubic are $t = 6,\ 21 \pm 2\sqrt{46}$.

Taking $t = 6$ we have $\lambda\lambda' = 1$. Combining with $\lambda + \lambda' = -13/6$ we find $\lambda = -\frac{2}{3},\ \lambda' = -\frac{3}{2}$.

From the equations $\mu + \mu' = 4 - \lambda\lambda'$, and $\lambda\mu' + \lambda'\mu = -17/6$ it follows that $\mu = 1,\ \mu' = 2$.

The quadratic factors are

$$6\,(x^2 - \tfrac{2}{3}x + 1)\,(x^2 - \tfrac{3}{2}x + 2) = (3x^2 - 2x + 3)\,(2x^2 - 3x + 4).$$

Similarly the values $t = 21 \pm 2\sqrt{46}$ may be considered.

Alternatively the quadratic factors of the biquadratic may be determined by the following method. Assume that

$$\begin{aligned} a\,(ax^4 + 4bx^3 + 6cx^2 + 4dx + e) &\equiv (ax^2 + 2bx + c + 2a\theta)^2 - (2px + q)^2 \\ &\equiv a^2x^4 + 4abx^3 + (2ac + 4a^2\theta + 4b^2 - 4p^2)\,x^2 \\ &\quad + 4\,(bc + 2ab\theta - pq)\,x + c^2 + 4ac\theta + 4a^2\theta^2 - q^2. \end{aligned}$$

Comparing coefficients we have

$$\begin{aligned} 6ac &= 2ac + 4a^2\theta + 4b^2 - 4p^2, \\ 4ad &= 4\,(bc + 2ab\theta - pq), \\ ae &= c^2 + 4ac\theta + 4a^2\theta^2 - q^2. \end{aligned}$$

Hence $p^2 = b^2 - ac + a^2\theta,\quad pq = bc - ad + 2ab\theta,$

$$q^2 = c^2 - ae + 4ac\theta + 4a^2\theta^2.$$

Since $(pq)^2 = p^2q^2$ we obtain on eliminating p and q,

$$(b^2 - ac + a^2\theta)\,(c^2 - ae + 4ac\theta + 4a^2\theta^2) = (bc - ad + 2ab\theta)^2.$$

This equation gives on simplification

$$4a^3\theta^3 - a\theta\,(ae - 4bd + 3c^2) + ace + 2bcd - ad^2 - b^2e - c^3 = 0.$$

Solving this cubic equation for θ we can then determine p and q. The quadratic factors are then

$$\{ax^2 + 2\,(b - p)\,x + 2a\theta - q\}\,\{ax^2 + 2\,(b + p)\,x + 2a\theta + q\}.$$

Thus, *e.g.* in the case of the biquadratic

$$6x^4 - 13x^3 + 24x^2 - 17x + 12$$

the cubic equation gives on simplification

$$2304\theta^3 - 1036\theta + 265 = 0.$$

The roots of this equation are $\theta = \frac{5}{12}$ or $-\frac{5 \pm \sqrt{184}}{24}$.

Taking $\theta = \frac{5}{12}$ we find $p = -\frac{5}{4}$, $q = 3$. Thus

$$\begin{aligned} 36x^4 - 78x^3 + 144x^2 - 102x + 72 &= (6x^2 - \tfrac{13}{2}x + 4 + 5)^2 - (-\tfrac{5}{2}x + 3)^2 \\ &= (6x^2 - \tfrac{13}{2}x + 9 - \tfrac{5}{2}x + 3)(6x^2 - \tfrac{13}{2}x + 9 + \tfrac{5}{2}x - 3) \\ &= (6x^2 - 9x + 12)(6x^2 - 4x + 6). \end{aligned}$$

16·42. The Solution of the Biquadratic Equation

It has been shown in § 16·41 that a biquadratic expression may be represented as the product of two quadratic expressions. By equating to zero each of the quadratic factors the four roots of the biquadratic equation are obtained. In practice it may not be necessary to proceed with the general method in order to determine the quadratic factors as they may sometimes be obtained by inspection. Further, if any relation between the roots is known, the equation can always be reduced to one of lower degree.

Example.—*Solve the biquadratic equation* $x^4 - x^3 + 2x^2 + x + 3 = 0$.

Since the coefficient of x^4 is unity and the constant term is $3 = 1 \times 3$ the factors may be of the form

$$(x^2 + ax \pm 1)(x^2 + bx \pm 3),$$

like signs corresponding.

Taking the $-$ signs the product is

$$x^4 + (a + b)x^3 + (ab - 4)x^2 - (3a + b)x + 3.$$

If this expression equated to zero is identical with the given equation $a + b = -1$, $ab - 4 = 2$, $3a + b = -1$.

The first and last equations give $a = 0$ and this obviously cannot satisfy the second. Thus the three equations are inconsistent.

Taking the $+$ signs we obtain

$$x^4 + (a + b)x^3 + (4 + ab)x^2 + (3a + b)x + 3 = 0.$$

If this is identical with the given equation, $a + b = -1$, $4 + ab = 2$, $3a + b = 1$.

The first and third equations give $a = 1$, $b = -2$. These values satisfy the second equation. Hence the given equation is equivalent to

$$(x^2 + x + 1)(x^2 - 2x + 3) = 0,$$

the roots being $-\frac{1}{2}(1 \pm i\sqrt{3})$, $1 \pm i\sqrt{2}$.

16·43. A Graphical Method of Solution

It has been shown in § 16·3, Ex. 2, that if $H = ac - b^2 > 0$ the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

has at least two imaginary roots. In this case the real solutions may be obtained graphically by considering the intersection of a certain circle with either of the curves $y\sqrt{H} = \pm x^2$. The transformation used in § 16·3, Ex. 2, reduces the equation to the form

$$x^4 + px^2 + 2qx + r = 0,$$

where $p = H > 0$. Multiplying throughout by p the equation may be represented in the form

$$px^4 + (px + q)^2 + pr - q^2 = 0.$$

$$\text{i.e. } \frac{x^4}{p} = \frac{q^2 - pr}{p^2} - \left(x + \frac{q}{p}\right)^2.$$

Taking the square root $\pm \dfrac{x^2}{\sqrt{p}} = \left\{\dfrac{q^2 - pr}{p^2} - \left(x + \dfrac{q}{p}\right)^2\right\}^{\frac{1}{2}}.$

Hence the real roots may be obtained by considering the points of intersection of

$$y^2 = \frac{q^2 - pr}{p^2} - \left(x + \frac{q}{p}\right)^2$$

with either of the curves $y = \pm x^2/\sqrt{p}$.

The former equation represents a circle whose centre is $(-q/p, 0)$ and whose radius is $\sqrt{(q^2 - pr)}/p$. The circle will be real provided $q^2 - pr > 0$.

Example.—*Use the above method to find the approximate values of the real roots of the equation*

$$x^4 + x^2 + 4x - 7 = 0.$$

[*Lond. Inter. Econ.*]

Now $x^4 + x^2 + 4x - 7$
$= x^4 + (x + 2)^2 - 11.$

Thus the real solutions of the equation will be given by the intersections of

$y = \pm x^2, y^2 = 11 - (x + 2)^2.$

The second equation represents a circle whose centre is $(-2, 0)$ and whose radius is $\sqrt{11} = 3{\cdot}32$ approximately.

It is sufficient to draw

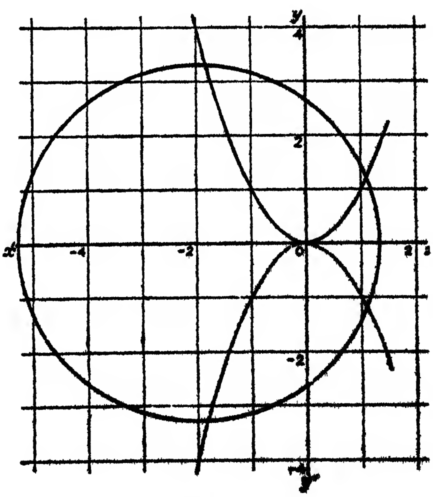


FIG. 51.

the circle and *one* of the curves represented by $y = \pm x^2$.

In Fig. 51, $y = x^2$ is shown by the continuous line, $y = -x^2$ by the dotted line. The graph shows clearly that the values of x obtained are the same in each case. It will be found that $x = 1{\cdot}1$ or $-1{\cdot}82$ approx.

16·44. Reduction to Reciprocal Form

A biquadratic equation can sometimes be conveniently solve by reducing it to the form

$$x^4 + px^3 + qx^2 + px + 1 = 0.$$

The method will be illustrated by two examples.

Examples.—(1) *Find a transformation* $x = \lambda y + \mu$ *which will change the equation*

$$x^4 + 5x^3 + 9x^2 + 5x - 1 = 0$$

into reciprocal form, and hence solve it.

$$(\lambda y + \mu)^4 + 5(\lambda y + \mu)^3 + 9(\lambda y + \mu)^2 + 5(\lambda y + \mu) - 1 = 0,$$

i.e. $\lambda^4 y^4 + (4\mu + 5)\lambda^3 y^3 + (6\mu^2 + 15\mu + 9)\lambda^2 y^2$

$$+ (4\mu^3 + 15\mu^2 + 18\mu + 5)\lambda y + \mu^4 + 5\mu^3 + 9\mu^2 + 5\mu - 1 = 0.$$

If this equation is of reciprocal form

$$\lambda^4 = \mu^4 + 5\mu^3 + 9\mu^2 + 5\mu - 1,$$

$$(4\mu + 5)\lambda^3 = \lambda(4\mu^3 + 15\mu^2 + 18\mu + 5).$$

Eliminating λ, $(4\mu + 5)^2(\mu^4 + 5\mu^3 + 9\mu^2 + 5\mu - 1)$

$$= (4\mu^3 + 15\mu^2 + 18\mu + 5)^2,$$

i.e. $3\mu^3 + 13\mu^2 + 19\mu + 10 = 0.$

This cubic equation is satisfied by $\mu = -2$ giving $\lambda^2 = 1$.

Taking $\lambda = 1$, the transformation $x = y - 2$ will change the given equation into reciprocal form. This transformation is equivalent to increasing the roots of the given equation by 2.

$$\begin{array}{ccccc}
1 & 5 & 9 & 5 & -1 \\
 & -2 & -6 & -6 & 2 \\
\hline
 & 3 & 3 & -1 & 1 \\
 & -2 & -2 & -2 & \\
\hline
 & 1 & 1 & -3 & \\
 & -2 & 2 & & \\
\hline
 & -3 & 3 & & \\
 & -2 & & & \\
\hline
 & -3 & & &
\end{array}$$

The transformed equation is $y^4 - 3y^3 + 3y^2 - 3y + 1 = 0$.

Writing $z = y + 1/y$, this equation becomes

$$z^2 - 3z + 1 = 0.$$

Thus $z = \frac{1}{2}(3 \pm \sqrt{5})$ and the four values of y are determined. If y_1, y_2, y_3, y_4 are these values then the roots of the original equation are

$$y_1 - 2,\ y_2 - 2,\ y_3 - 2,\ y_4 - 2.$$

(2) *By eliminating x between the equations*

$$y = x^2 + px + q,\ x^4 - 2x^2 + 8x - 3 = 0$$

reduce the equations to the form $y^4 + ry^2 + s = 0$ *and hence solve*

$$x^4 - 2x^2 + 8x^3 - 3 = 0.$$

Write $\lambda = q - y$. Then

$$x^2 + px + \lambda = 0 \quad \text{(i)}$$

$$x^4 - 2x^2 + 8x - 3 = 0 \quad \text{(ii)}$$

Multiply (i) by x^2 and subtract (ii) from it:

$$px^3 + (\lambda + 2)x^2 - 8x + 3 = 0 \quad \text{(iii)}$$

Multiply (i) by px and subtract from (iii):

$$(\lambda + 2 - p^2)x^2 + (-8 - \lambda p)x + 3 = 0 \quad \text{(iv)}$$

From (i) and (iv) we have

$$\frac{x^2}{3p - \lambda(-8 - \lambda p)} = \frac{x}{\lambda(\lambda + 2 - p^2) - 3} = \frac{1}{-8 - \lambda p - p(\lambda + 2 - p^2)} \quad (a).$$

Hence

$$(\lambda^2 + 2\lambda - \lambda p^2 - 3)^2 = (-8 - 2p\lambda - 2p + p^3)(3p + 8\lambda + \lambda^2 p),$$

i.e. $\{y^2 + (p^2 - 2q - 2)y + q^2 + 2q - p^2 q - 3\}^2$

$$= \{2py + p^3 - 2pq - 2p - 8\}\{py^2 - y(-8 - 2pq) + pq^2 + 3p + 8q\}.$$

We require the conditions that the coefficients of y^3 and y be zero. Thus

$$2(p^2 - 2q - 2) = 2p^2 \quad \text{(v)}$$

$$2(p^2 - 2q - 2)(q^2 + 2q - p^2 q - 3) = 2p(pq^2 + 3p + 8q)$$
$$+ (-8 - 2pq)(p^3 - 2pq - 2p - 8) \quad \text{(vi)}$$

From (v), $q = -1$. Substituting in (vi) we have

$$2p^2(p^2 - 4) = 2p(4p - 8) + (2p - 8)(p^3 - 8).$$

It is obvious that $p - 2$ is a factor of both sides of the equation and hence $p = 2$ is one solution of the cubic equation in p. Hence the elimination of x between

$$y = x^2 + 2x - 1,\ x^4 - 2x^2 + 8x - 3 = 0$$

gives the form $y^4 + ry^2 + s = 0$.

Substituting $p = 2$, $q = 1$ in the biquadratic for y we obtain

$$(y^2 + 4y)^2 = 4y(2y^2 - 4y), \text{ i.e. } y^2(y^2 + 32) = 0.$$

$$\text{Hence } y = 0,\ 0,\ \pm 4i\sqrt{2}.$$

When $y = 0$, $x^2 + 2x - 1 = 0$, i.e. $x = -1 \pm \sqrt{2}$.

When $y \neq 0$ the corresponding values of x are best obtained from (*a*).

$$\text{Thus } x = \frac{\lambda(\lambda + 2 - p^2) - 3}{-8 - \lambda p - p(\lambda + 2 - p^2)} = \frac{(y+1)(y+3) - 3}{4y}$$
$$= \tfrac{1}{4}(y + 4),\ y \neq 0.$$

Substituting $y = \pm 4i\sqrt{2}$, $x = 1 \pm i\sqrt{2}$.

16·5. Special Series of Equations with Real Roots

The following result is of great importance.

Consider the sequence of symmetrical determinants, $\Delta_1(x)$, $\Delta_2(x)$, $\Delta_3(x)$, $\Delta_4(x)$, . . . where

$$\Delta_1(x) = a - x,\ \ \Delta_2(x) = \begin{vmatrix} a - x & h \\ h & b - x \end{vmatrix},$$

$$\Delta_3(x) = \begin{vmatrix} a - x & h & g \\ h & b - x & f \\ g & f & c - x \end{vmatrix},$$

$$\Delta_4(x) = \begin{vmatrix} a - x & h & g & l \\ h & b - x & f & m \\ g & f & c - x & k \\ l & m & k & d - x \end{vmatrix}, \ldots$$

where $a, b, c, \ldots$ are real constants and x is a variable.

Then the roots of the equations $\Delta_1(x) = 0$, $\Delta_2(x) = 0$, $\Delta_3(x) = 0$, $\Delta_4(x) = 0, \ldots$ *are all real. Further if* n *denote any positive integer* (> 2), *then the roots of* $\Delta_n(x) = 0$ *are separated by those of* $\Delta_{n-1}(x) = 0$.

LEMMA.—We first prove a property of determinants.

If $\Delta_{n-1} = (a_{11}\, a_{22}\, a_{33} \ldots a_{n-1,\, n-1})$, $\Delta_n = (a_{11}\, a_{22}\, a_{33} \ldots a_{nn})$, $\Delta_{n+1} = (a_{11}\, a_{22}\, a_{33} \ldots a_{n+1,\, n+1})$ *are symmetrical determinants such that* Δ_n *is zero, then the product* $\Delta_{n-1}\,\Delta_{n+1}$ *is negative.*

It will be observed that Δ_n is obtained from Δ_{n-1}, and Δ_{n+1} from Δ_n by *bordering* the corresponding determinant symmetrically.

We give the proof for the case $n = 4$, but it will be seen that the method is quite general and will extend to the case in which n has any positive integral value.

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},\ \ \Delta_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

$$\Delta_5 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

where $a_{rs} = a_{sr}$; $r,\ s = 1, 2, 3, 4, 5$ and $\Delta_4 = 0$. Let A_{rs} denote the cofactor of a_{rs} in Δ_5.

We consider the adjoint determinant $\Delta'_5 = (A_{11}A_{22}A_{33}A_{44}A_{55})$ and evaluate the minor $(A_{44}A_{55})$.

Consider the product of the determinants:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \quad \begin{vmatrix} 1 & 0 & 0 & A_{41} & A_{51} \\ 0 & 1 & 0 & A_{42} & A_{52} \\ 0 & 0 & 1 & A_{43} & A_{53} \\ 0 & 0 & 0 & A_{44} & A_{54} \\ 0 & 0 & 0 & A_{45} & A_{55} \end{vmatrix}$$

which is $\Delta_5 (A_{44} A_{55} - A_{45} A_{54})$. Also $A_{55} = \Delta_4 = 0$ and since the determinant Δ_5 is symmetrical, $A_{45} = A_{54}$. Thus the product is $-\Delta_5 A^2_{45}$.

Multiplying the determinants, we have

$$-\Delta_5 A^2_{45} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & \Delta_5 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & \Delta_5 \end{vmatrix}$$

$$= \Delta^2_5 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta^2_5 \Delta_3.$$

Hence $\Delta_5 \Delta_3 = -A^2_{45}$.

Since the elements of A_{45} are real, A^2_{45} is positive. Thus $\Delta_5 \Delta_3$ is negative.

It will be observed that A_{45} is a linear homogeneous function of the bordering elements $a_{15}, a_{25}, a_{35}, a_{45}$.

We return to the theorem enunciated at the beginning of the section.

The root of $\Delta_1(x) = 0$ is $x = a$.

Let α_1, α_2 be the roots of $\Delta_2(x) = 0$. Then α_1, α_2 are the roots of the equation

$$(a - x)(b - x) - h^2 = 0.$$

Also $\Delta_2(-\infty) = +\infty$, $\Delta_2(a) = -h^2$, $\Delta_2(\infty) = +\infty$.

Hence $\Delta_2(x) = 0$ has two real roots, one less than a and the other greater than a. Thus

$$\text{if } \alpha_1 < \alpha_2, \ \alpha_1 < a < \alpha_2.$$

This proves the theorem for the case $n = 2$.

Consider now the case of $n = 3$.

Since $\Delta_2(x) = 0$, $\Delta_3(x)\,\Delta_1(x) < 0$. (Lemma.)

Now $\Delta_3(-\infty) = -\infty$; $\Delta_3(\alpha_1) > 0$ since $\Delta_3(\alpha_1)\,\Delta_1(\alpha_1) < 0$ and $\Delta\,(\alpha_1) < 0$; $\Delta_3(\alpha_2) < 0$ since $\Delta_3(\alpha_2)\,\Delta_1(\alpha_2) < 0$ and $\Delta_1(\alpha_2) > 0$; $\Delta_3(\infty) = +\infty$.

Thus corresponding to

$$-\infty, \quad \alpha_1, \quad \alpha_2, \quad +\infty$$

we have the signs

$$- \quad + \quad - \quad +$$

Hence $\Delta_3(x) = 0$ has three real roots. Further, if the values are $\beta_1, \beta_2, \beta_3$ where $\beta_1 < \beta_2 < \beta_3$, then

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \beta_3.$$

For the case $n = 4$ we argue in the same way as in the previous case.

$\Delta_3(x) = 0$ when $x = \beta_1, \beta_2, \beta_3$ so that $\Delta_4(\beta_1)\,\Delta_2(\beta_1)$, $\Delta_4(\beta_2)\,\Delta_2(\beta_2)$, $\Delta_4(\beta_3)\,\Delta_2(\beta_3)$ are all negative. Also $\Delta_2(\beta_1) > 0$, $\Delta_2(\beta_2) < 0$, $\Delta_2(\beta_3) > 0$. Hence when we substitute $x = -\infty$, $\beta_1, \beta_2, \beta_3, +\infty$ in $\Delta_4(x)$ we obtain the signs

$$+ \quad - \quad + \quad - \quad +$$

Thus there are four real roots which are separated by the roots of $\Delta_3(x) = 0$. Similarly for higher values of n.

The result which has just been proved may be stated as follows. *If the leading elements of a symmetric determinant are all increased or decreased by the same quantity x, the equation in x obtained by equating to zero the determinant so obtained has all its roots real.*

Example.—*Show that there are three real values of λ for which the equations $(a - \lambda)\,x + by + cz = 0$, $bx + (c - \lambda)\,y + az = 0$, $cx + ay + (b - \lambda)\,z = 0$ are simultaneously true, and that the product of these values of λ is D, where*

$$D \equiv \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

[*Lond. B.Sc.*]

Eliminating x, y, z between the three equations

$$E\,(\lambda) \equiv \begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0.$$

The fact that this cubic equation in λ has three real roots follows immediately from the general discussion given above. We give alternative arguments for this particular case.

Method (i). Expanding E (λ) along the first row:

$$E\,(\lambda) = (a-\lambda)\begin{vmatrix} c-\lambda & a \\ a & b-\lambda \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b-\lambda \end{vmatrix} + c\begin{vmatrix} b & c-\lambda \\ c & a \end{vmatrix}$$

$$= (a-\lambda)\,\{c-\lambda)\,(b-\lambda) - a^2\} - \{b^3 + c^3 - 2abc - \lambda\,(b^2 + c^2)\}$$

Let α, β be the roots of the equation $(c - \lambda)(b - \lambda) - a^2 = 0$, where $\alpha < \beta$. It will be observed that the roots must be real. Then

$$2\alpha = b + c - \sqrt{(b + c)^2 - 4bc + 4a^2} = (b + c) - \sqrt{(b - c)^2 + 4a^2}.$$

Also $2\beta = b + c + \sqrt{(b - c)^2 + 4a^2}$.

We may suppose without loss of generality that $b > c$. A similar argument holds if $b < c$.

Then $\sqrt{(b - c)^2 + 4a^2} > b - c$. Thus $2\alpha < b + c - (b - c)$, i.e. $\alpha < c$. Similarly $\beta > b$. Thus $\alpha < c < b < \beta$.

When $\lambda = \alpha$, $b^3 + c^3 - 2abc - \lambda(b^2 + c^2)$

$$= b^3 + c^3 - 2bc\sqrt{(c - \alpha)(b - \alpha)} - \alpha(b^2 + c^2)$$

$$= (b\sqrt{b - \alpha} - c\sqrt{c - \alpha})^2.$$

When $\lambda = \beta$, $b^3 + c^3 - 2abc - \lambda(b^2 + c^2)$

$$= -\{\beta(b^2 + c^2) + 2bc\sqrt{(\beta - c)(\beta - b)} - b^3 - c^3\}$$

$$= -\{b\sqrt{\beta - b} + c\sqrt{\beta - c}\}^2.$$

In $E(\lambda)$ write $\lambda = -\infty$, α, β, $+\infty$. The values of E are $+\infty$, $-(b\sqrt{b - \alpha} - c\sqrt{c - \alpha})^2$, $(b\sqrt{\beta - b} + c\sqrt{\beta - c})^2$, $-\infty$. The signs are $+$, $-$, $+$, $-$. Hence the equation has three real roots, one between $-\infty$ and α, one between α and β, and the other between β and $+\infty$. If $\alpha = \beta$ then $(b - c)^2 + 4a^2 = 0$, i.e. $b = c$, $a = 0$ and the equation becomes $(b - \lambda)(\lambda^2 - \lambda b - 2b^2) = 0$ and there are again three real roots.

If α happens to be a root of the given equation the argument given above fails and it proves only that there are two real roots. Since imaginary roots occur in pairs it follows that the third root must be real. Similarly for the case in which β is a root of $E(\lambda) = 0$.

The product of the roots is $-(a^3 + b^3 + c^3 - 3abc)$ which is D.

Method (ii). Adding the second and third rows to the first and taking out the common factor $a + b + c - \lambda$,

$$E(\lambda) = (a + b + c - \lambda)\begin{vmatrix} 1 & 1 & 1 \\ b & c - \lambda & a \\ c & a & b - \lambda \end{vmatrix}$$

Subtracting the first column from the second and third columns,

$$E(\lambda) = (a + b + c - \lambda)\begin{vmatrix} 1 & 0 & 0 \\ b & c - b - \lambda & a - b \\ c & a - c & b - c - \lambda \end{vmatrix}$$

$$= (a + b + c - \lambda)\begin{vmatrix} c - b - \lambda & a - b \\ a - c & b - c - \lambda \end{vmatrix}$$

$$= (a + b + c - \lambda)(\lambda^2 - a^2 - b^2 - c^2 + ab + bc + ca).$$

Hence $\lambda = a + b + c$ or $\pm(a^2 + b^2 + c^2 - ab - bc - ca)^{\frac{1}{2}}$.

The roots will be real provided

$$a^2 + b^2 + c^2 - ab - bc - ca > 0.$$

Since $a^2 + b^2 > 2ab$, $b^2 + c^2 > 2bc$, $c^2 + a^2 > 2ac$ we obtain on addition the required inequality. The product of the roots is

$$-(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= -(a^3 + b^3 + c^3 - 3abc) = D.$$

EXERCISES XVI

1. Find the equation of squared differences for the cubic
$$x^3 + 9x^2 + 22x + 16 = 0.$$

2. If α, β, γ are the roots of the cubic equation
$$ax^3 + 3bx^2 + 3cx + d = 0$$
find the relation between the coefficients in order that $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$ $(\alpha - \beta)^2$ be in arithmetic progression.

3. If the equations
$$ax^3 + bx^2 + cx + d = 0, \quad a'x^3 + b'x^2 + c'x + d' = 0$$
have two common roots, find the quadratic equation whose roots are these two common roots and prove that
$$\frac{ab' - a'b}{ad' - a'd} = \frac{ac' - a'c}{bd' - b'd} = \frac{ad' - a'd}{cd' - c'd}.$$
[*Madras, Inter.*]

4. Find the value of the determinant
$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$$
and show that its square is
$$\begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & \cdots & \cdots \\ s_2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & \cdots & \cdots & s_{2n-2} \end{vmatrix}$$
where $s_p = a_1^p + \ldots + a_n^p$.

If $n = 3$ and a_1, a_2, a_3 are the roots of $x^3 + px^2 + q = 0$, express the second determinant in terms of p and q and hence find the condition that the two roots of this equation should be equal. [*Camb. Sch.*]

5. Show that the equation $x^3 - 2x^2 - 15x + 36 = 0$ has a multiple root and hence solve it.

6. Prove that if a, b, c are the roots of the equation of the squared differences of $x^3 + 3px + q = 0$ then
$$a^2 + b^2 + c^2 = 2(bc + ca + ab).$$
[*Camb. Sch.*]

7. Prove that the necessary and sufficient condition for the reality of all three roots of the cubic $x^3 + 3px + q = 0$ is that $4p^3 + q^2 < 0$. Discuss $4p^3 + q^2 = 0$. Show that $x^3 + 3x^2 + x + 2 = 0$ has only one real root and locate it between two consecutive integers. [*Camb. Sch.*]

8. If α, β, γ are the roots of $x^3 + bx + c = 0$, find an expression for
$$(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2.$$
Hence show that the roots of $x^3 + px^2 + qx + r = 0$ are real provided
$$p^2q^2 + 18pqr - 4q^3 - 4p^3r - 27r^2 > 0.$$

If this condition is satisfied show that the necessary and sufficient condition for all the roots to be positive is that p and r should be negative and q positive. [*Camb. Sch.*]

9. If $f(x) \equiv x^3 - 3p^2x + q$, p and q being positive, find the maximum and minimum values of $f(x)$. Hence show that if $2p^3 > q$, the equation $x^3 - 3p^2x + q = 0$ has three real roots.

If the roots of the above equation are in the ratio $1 : 2 : -3$, prove that $54p^3 = 7\sqrt{21}.q$. [*Lond. Inter. Econ.*]

10. Prove that the equation $x^3 + 12x + 12 = 0$ has only one real root and find its value.

11. Solve the equation $x^3 + 3x^2 - 27x + 104 = 0$.

12. Solve the equation $x^3 - x^2 - 2x + 1 = 0$.

13. Reduce the equation $x^3 + 3px^2 + 3qx + r = 0$ to the form

$$y^3 \pm 3y + m = 0$$

by assuming $x = \lambda y + \mu$; and solve this equation by assuming $y = z \mp \frac{1}{z}$. Hence prove that the condition for equal roots is

$$4(p^2 - q)^3 = (2p^3 - 3pq + r)^2. \qquad \text{[Camb. Sch.]}$$

14. In the equation $x^3 - 3x - a = 0$ make the substitution $x = y + 1/y$, and show that $y^3 = \frac{1}{2}\{a + (a^2 - 4)^{\frac{1}{2}}\}$. Hence, or otherwise, find a root of the equation $x^3 - 3x - 4 = 0$ to 2 decimal places.

15. If $x + y + z = 0$, $ax^2 + by^2 + cz^2 = 0$, and $\lambda^2 + ab + bc + ca = 0$, show that $(a + \lambda)x$, $(b + \lambda)y$, $(c + \lambda)z$ are the same as $(a - \lambda)x$, $(b - \lambda)y$, $(c - \lambda)z$ in some order or other. [*Madras, B.Sc.*]

16. Form the equation whose roots are $\omega^{-1}p + \omega q$, $\omega p + \omega^{-1}q$, $p + q$ where $\omega^3 = 1, (\omega \neq 1)$. [*Camb. Sch.*]

17. Prove that the condition that the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

may be expressed in the form

$$\lambda(x^2 + 2px + q)^2 + \mu(x^2 + 2px + q) + \nu$$

is $a^2d - 3abc + 2b^3 = 0$.

18. Express $x^4 - 8x^3 + 25x^2 - 36x + 21$ as the product of two quadratic factors in three different ways.

19. Solve the equation $x^4 - 6x^2 - 3x + 4 = 0$ by expressing the left-hand side as the difference of two squares.

20. Find a transformation of the form $x = \lambda y + \mu$, which will change the equation

$$x^4 + 4x^3 - 18x^2 - 44x - 7 = 0$$

into reciprocal form. Hence solve the equation.

21. Show that there is a value of p for which the left-hand side of the equation

$$x^5 - 7x^3 - x^2 + 4x - 1 = 0$$

is divisible by $x^2 + px + 1$. Hence solve the equation. [*Camb. Sch.*]

22. Show that the equation whose roots are the five values of $p\omega + q\omega^{-1}$, where ω is a fifth root of unity, is

$$x^5 - 5pqx^3 + 5p^2q^2x - p^5 - q^5 = 0.$$

Hence obtain all the roots of $16x^5 - 20x^3 + 5x - 1 = 0$.

16·61. Upper and Lower Bounds for the Roots of an Equation

Consider first the positive roots of an equation $f(x) = 0$. Then if M, m, $m < M$ denote positive numbers such that all the positive roots lie in the interval (m, M) then m may be called a *lower bound* for the positive roots, M an *upper bound* for the positive roots. In a similar way we may consider upper and lower bounds for the negative roots of $f(x) = 0$.

It will be observed that m, M are not unique and that the problem in practice is to find the best values for m and M.

Further, if we assume that the coefficient of the highest power of x in $f(x)$ is positive, then $f(x) > 0$ for $x \geqslant M$. Thus one way of finding M would be to arrange the terms of $f(x)$ into groups such that each group is separately positive. Consider, for example, the equation

$$x^5 + x^4 - 100x^3 - 109x^2 + x - 132 = 0.$$

This may be written in the form

$$x^3(x^2 - 100) + x^2(x^2 - 110) + (x - 11)(x + 12) = 0.$$

If $x \geqslant 11$ each one of the groups is positive, except the last, which is zero. Thus we may take $M = 11$.

16·62. General Theorem on Upper Bounds for Positive Roots

Write $f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n$.

In the sequence of numbers $p_1, p_2, p_3, \dots, p_n$ let $p_r, p_s, \dots, p_t$ be negative, where $r < s < \dots < t \leqslant n$. *Then if* $|p_\lambda|$ *is the greatest member of*

$$|p_r|, \ |p_s| \dots \ |p_t|$$

then $f(x) > 0$ *for* $x > \sqrt[r]{|p_\lambda|} + 1$. In other words, $\sqrt[r]{|p_\lambda|} + 1$ *is an upper bound for the positive roots of* $f(x) = 0$.

Consider the group of negative terms

$$p_rx^{n-r} + p_sx^{n-s} + \dots + p_tx^{n-t}.$$

Then assuming $x > 1$

$$\begin{aligned}
| p_r x^{n-r} + p_s x^{n-s} + \dots + p_t x^{n-t} | \\
&\leqslant | p_r | x^{n-r} + | p_s | x^{n-s} + \dots + | p_t | x^{n-t} \\
&\leqslant | p_\lambda | \{x^{n-r} + x^{n-r-1} + x^{n-r-2} + \dots + 1\} \\
&= | p_\lambda | (x^{n-r+1} - 1)/(x - 1). \\
&< | p_\lambda | x^{n-r+1}/(x - 1) \\
&< x^n | p_\lambda |/(x - 1)^r,
\end{aligned}$$

since $x^{r-1} > (x - 1)^{r-1}$, $x > 1$, so that $1/x^{r-1} < 1/(x - 1)^{r-1}$.

Now $f(x) > 0$ if $x^n > x^n | p_\lambda |/(x - 1)^r$, i.e. $(x - 1)^r > | p_\lambda |$, i.e. $x > \sqrt[r]{| p_\lambda |} + 1$.

In the example considered in § 16·61.

$$p_1 = 1, \ p_2 = -100, \ p_3 = -109, \ p_4 = 1, \ p_5 = -132.$$

Hence an upper bound to the positive roots will be

$$\sqrt{132} + 1 = 11\ldots + 1.$$

Taking the integer next above we obtain $M = 13$.

16·63. Bounds for Negative Roots

To discuss the bounds for negative roots of $f(x) = 0$ make the transformation $x = -y$. Then the negative roots of $f(x) = 0$ are the positive roots of $f(-y) = 0$.

Examples.—(1) *Find a lower bound for the negative roots of*

$$x^4 + x^3 - 3x^2 + 5x - 20 = 0.$$

The transformation $x = -y$ gives $y^4 - y^3 - 3y^2 - 5y - 20 = 0$.

We require an upper bound to the positive roots of this equation and the method of § 14·62 gives 21 as an upper bound.

We can obtain a much better result by grouping the terms. Since the only positive term is the highest power of y we distribute this term as evenly as possible among the negative terms. Multiply the terms of the equation by 4 and attach y^4 to each of the four negative terms. Thus

$$4y^4 - 4y^3 - 12y^2 - 20y - 80 = 0,$$

$$\text{i.e.} \ y^3(y - 4) + y^2(y^2 - 12) + y(y^3 - 20) + (y^4 - 80) = 0.$$

If $y > 4$ the expressions in brackets are all positive or zero. Hence 4 is an upper bound to the positive roots.

It follows that -4 is a lower bound to the negative roots of the original equation.

(2) *Find integral values between which lie all the real roots of*

$$x^5 + 6x^4 - 8x^3 - 66x^2 + 7x + 60 = 0.$$

We require an upper bound for the positive roots and a lower bound for the negative roots. The method of § 16·62 gives $\sqrt{66}$ as an upper bound. Taking the integral value next above we obtain 9 for an upper bound.

A better result can be obtained by grouping terms. Thus

$$x^5 + 6x^4 - 8x^3 - 66x^2 + 7x + 60$$
$$\equiv x^3(x^2 - 64) + 4x^3(x^2 - 4) + 2x^3(x - 4) + (14x^2 + 7x + 60).$$

If $x > 4$, the expression on the right is clearly positive. Hence we may take 4 as an upper bound.

Next consider the negative roots and write $x = -y$. We obtain

$$y^5 - 6y^4 - 8y^3 + 66y^2 + 7y - 60 = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The method of § 14·62 gives 60 as an upper bound.

We now proceed to group the terms in order. First multiply the terms of the equation by 2.

$$2y^5 - 12y^4 - 16y^3 + 132y^2 + 14y - 120 = 0.$$

The left-hand side is

$$(y^5 - 6y^4) + (y^5 - 6y^4 - 16y^3 + 131y^2) + (y^2 + 14y - 120)$$
$$= y^4(y - 6) + y^2(y^3 - 6y^2 - 16y + 131) + (y - 6)(y + 20) \quad \ldots \text{(ii)}$$

Now $y^3 - 6y^2 - 16y + 131 = (y - 6)^3 + 12y^2 - 124y + 347$.

The roots of the quadratic equation $12y^2 - 124y + 347 = 0$ are imaginary since $124^2 - 4.12.347 < 0$.

Hence $y^3 - 6y^2 - 16y + 131 > 0$ for $y > 6$.

It follows from (ii) that 6 is an upper bound for the positive roots of (i). Hence -6 is a lower bound for the negative roots of the original equation. Thus the real roots of the given equation lie between -6 and 4.

16·64. Taylor's Expansion for a Polynomial

Let $f(x)$ denote a polynomial of the nth degree, h any number. Then

$$f(x + h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \ldots + \frac{x^n}{n!}f^{(n)}(h)$$
$$= \sum_{r=0}^{n} \frac{x^r}{r!} f^{(r)}(h),$$

where $f^{(r)}(x)$ denotes the rth differential coefficient of $f(x)$ so that $f^{(r)}(h)$ is the value of $f^{(r)}(x)$ when $x = h$. This result may be proved by successive differentiation as follows.

Since $f(x)$ is a polynomial of the nth degree, so also is $f(x + h)$. Hence $f(x + h)$ may be expanded uniquely as a polynomial of the form

$$f(x + h) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \ldots + \frac{a_n}{n!}x^n$$
$$= \sum_{r=0}^{n} \frac{a_r}{r!}x^r.$$

Now the differential coefficient of a polynomial always exists and if all the coefficients are different from zero is itself a polynomial of degree one less than that of the original polynomial. It follows that if we differentiate $f(x+h)$ successively with respect to x we will obtain in order polynomials in x of degrees $n-1$, $n-2$, $n-3$, . . . The nth differential coefficient of $f(x+h)$ will be independent of x and differential coefficients of higher order will be zero. Let the successive derivatives be denoted by

$$f'(x+h), f''(x+h), \ldots, f^{(r)}(x+h), \ldots f^{(n)}(x+h).$$

$$\text{Then } f(x+h) = \sum_{r=0}^{n} \frac{a_r}{r!} x^r,$$

$$f'(x+h) = \sum_{r=1}^{n} \frac{a_r}{(r-1)!} x^{r-1},$$

$$f''(x+h) = \sum_{r=2}^{n} \frac{a_r}{(r-2)!} x^{r-2},$$

$$f^{(n)}(x+h) = a_n.$$

Since the derivatives are polynomials they are continuous, so that the value of the functions obtained by writing $x=0$, will be the value obtained by writing $x=0$ in the corresponding series. Hence

$$f(h) = a_0, \quad f'(h) = a_1, \quad f''(h) = a_2, \ldots, f^{(n)}(h) = a_n.$$

Thus we have the required form for the function $f(x+h)$.

16·65. Newton's Method of Finding an Upper Bound

The advantage of this method is that it gives a good result for the upper bound. It depends on the following theorem. *If h is a number such that $f(h)$, $f'(h)$, $f''(h)$, . . . $f^{(n)}(h)$ are all positive then h is an upper bound to the roots of $f(x)=0$.*

Using the expansion of § 16·64,

$$f(y+h) = f(h) + yf'(h) + \frac{y^2}{2!}f''(h) + \ldots + \frac{y^n}{n!}f^{(n)}(h).$$

Clearly if $y > 0$ and $f(h)$ and all its derivatives are positive, $f(y+h > 0$.

Now write $y + h = x$, so that $y \geqslant 0$ implies $x \geqslant h$. Thus $f(x) > 0$ for $x \geqslant h$, i.e. $f(x) = 0$ has no root greater than or equal to h. Hence h is an upper bound to the roots of $f(x) = 0$.

Method of procedure in a particular case. It should first be observed that $f^{(n)}(x)$, which is a constant, is to be positive. Now consider the smallest integral value of x which makes $f^{(n-1)}(x) > 0$. Let it be h_1.

If $f^{(n-2)}(h_1)$ is not positive find by trial a greater number such that $f^{(n-2)}(h_2) > 0$, and so on. We thus obtain a sequence of numbers $h_1, h_2, h_3, \ldots h_r, \ldots$ corresponding to

$$f^{(n-1)}(x),\ f^{(n-2)}(x),\ f^{(n-3)}(x),\ \ldots\ f^{(n-r)}(x), \ldots$$

such that $f^{(n-r)}(h_r) > 0$ and

$$h_1 \leqslant h_2 \leqslant h_3 \leqslant \ldots \leqslant h_r \leqslant \ldots$$

Clearly $f^{(n-1)}(x),\ f^{(n-2)}(x), \ldots f^{(n-r)}(x)$, will all be positive for $x = h_r$ since h_r does not decrease with r, so that the effect of increasing h will be to leave *unchanged* the signs of the derivatives already considered.

Example.—*Find an upper bound to the roots of*

$$f(x) \equiv x^4 - 6x^3 + 9x^2 - 19x + 12 = 0.$$

$$f^{\mathrm{I}}(x) = 4x^3 - 18x^2 + 18x - 19.$$

$$f^{\mathrm{II}}(x) = 6(2x^2 - 6x + 3).$$

$$f^{\mathrm{III}}(x) = 12(2x - 3).$$

$$f^{\mathrm{IV}}(x) = 24.$$

Here $f^{\mathrm{IV}}(x) > 0$. The value $x = 2$ makes $f^{\mathrm{III}}(x) > 0$.

$$f^{\mathrm{II}}(2) = 6(4 - 12 + 3) < 0.$$

$$f^{\mathrm{II}}(3) = 6(18 - 18 + 3) > 0.$$

$$f^{\mathrm{I}}(3) = 4.27 - 18.9 + 18.3 - 19 < 0.$$

$$f^{\mathrm{I}}(4) = 4.64 - 18.16 + 18.4 - 19 > 0.$$

$$f(4) = 256 - 6.64 + 9.16 - 19.4 + 12 < 0.$$

$$f(5) = 625 - 6.125 + 9.25 - 19.5 + 12 > 0.$$

Hence 5 is an upper bound to the roots of $f(x) = 0$.

EXERCISES XVI

23. By grouping terms, find an upper and lower bound for the roots of the equation $x^4 + 8x^3 + 7x^2 - 15x - 65 = 0$.

24. Find an upper and lower bounds for the roots of the equation

$$x^4 - x^3 + 3x^2 - 7x - 4 = 0.$$

25. Find an upper bound for the roots of the equation

$$5x^5 + 18x^4 + 12x^3 - 45x^2 - 194x - 442 = 0.$$

26. If $f(x) \equiv 2x^4 + 23x^3 + 96x^2 + 171x + 108$, use Newton's method to prove that the equation $f(x) = 0$ has no real roots greater than -1.

27. Apply Newton's method to find an upper bound to the roots of the equation $x^4 + 3x^3 + x^2 - 7x - 30 = 0$.

28. Prove that the real roots of the equation

$$6x^4 - x^3 - 13x^2 + 5x + 4 = 0$$

lie between -2 and 1.

16·71. Solution of Equations with Numerical Coefficients

We now consider the determination of the real roots of an equation *with given numerical coefficients*.

It should be noted that the procedure is quite distinct from the algebraic methods used in the solution of cubic and biquadratic equations. There is in fact no known *general algebraic method* of solving equations of the fifth degree and higher degrees. There is no difficulty in determining the rational roots of a given equation and there is a general method available for finding multiple roots.

The problem then is concerned essentially with the determination of irrational non-repeated roots. The procedure is as follows:

(i) We find an interval which contains *all* the real roots.

(ii) We then proceed to separate the roots, *i.e.* determine intervals which contain only one simple root.

(iii) We then approximate to the root by considering smaller and smaller intervals.

We discuss two particular methods. The first, *Newton's method*, has the advantage that it may be applied to functions other than rational integral functions. Further, the approximation is usually rapid.

For practical purposes the second method, usually called *Horner's method*, is almost invariably used for dealing with rational integral functions. Both rational and irrational roots are determined by the method.

Both methods consist substantially in making successive approximations.

16·72. Mean Value Theorems

In Chap. XIII., § 15·72, it has been proved that if $f(x)$ is any function which possesses a continuous derivative $f'(x)$ in (a, b) where $f(a) = f(b) = 0$ then there exists a number ξ, $a < \xi < b$ such that $f'(\xi) = 0$. [Rolle's theorem.]

The condition that $f'(x)$ be continuous was introduced in order to simplify the argument. The condition is, however, not necessary. *It is sufficient that $f'(x)$ exist at all points of (a, b).** It should be observed that the existence of $f'(x)$ at all points of (a, b) implies that $f(x)$ is continuous in (a, b).

We now deduce from Rolle's theorem two important mean value theorems.

THEOREM (a).†—*If $f(x)$ is a function which possesses a derivative $f'(x)$ at all points of an interval (a, b) then there exists a number ξ, where $a < \xi < b$ such that $f(b) - f(a) = (b-a)f'(\xi)$.*

Consider the function $F(x)$ defined by

$$F(x) = (b-a)\{f(b) - f(x)\} - (b-x)\{f(b) - f(a)\}.$$

Then $F(a) = (b-a)\{f(b) - f(a)\} - (b-a)\{f(b) - f(a)\} = 0$
$F(b) = (b-a)\{f(b) - f(b)\} - (b-b)\{f(b) - f(a)\} = 0.$

From Rolle's theorem it follows that there exists a value ξ, $a < \xi < b$ such that $F'(\xi) = 0$, provided $F'(x)$ exists at all points of (a, b). Now

$$F'(x) = -(b-a)\ f'(x) + f(b) - f(a).$$

$$\text{Thus } f(b) - f(a) = (b-a)f'(\xi).$$

The theorem may be conveniently represented in a slightly different form. Write $b = a + h$. Then since $a < \xi < a + h$ we can write $\xi = a + \theta h$, where $0 < \theta < 1$. The theorem then states that

$$f(a+h) = f(a) + hf'(a + \theta h).$$

THEOREM (b). *If $f(x)$ is a function which possesses a second derivative $f''(x)$ at all points of an interval (a, b) then there exists a number ξ such that*

$$f(b) = f(a) + (b-a)f'(a) + \tfrac{1}{2}(b-a)^2 f''(\xi)$$

where $a < \xi < b$.

Writing $b = a + h$ the result may be expressed in the following form where $0 < \theta < 1$.

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a + \theta h).$$

* For a rigid proof of this result see Hardy, *A Course of Pure Mathematics* (Camb. Univ. Press).

† The reader should consider the geometrical significance of this theorem.

Consider the function

$$\phi(x) = h^2\{f(a+h) - f(x) - (a+h-x)f'(x)\} - (a+h-x)^2\{f(a+h) - f(a) - hf'(a)\}$$

Then $$\phi(a) = h^2\{f(a+h) - f(a) - hf'(a)\} - h^2\{f(a+h) - f(a) - hf'(a)\} = 0.$$

Similarly $\phi(a+h) = 0$.

From Rolle's theorem it follows that there exists a number ξ, $a < \xi < a+h$, such that $\phi'(\xi) = 0$. Now

$$\phi'(x) = h^2\{-f'(x) - (a+h-x)f''(x) + f'(x)\} + 2(a+h-x)\{f(a+h) - f(a) - hf'(a)\}$$

$$= 2(a+h-x)\{f(a+h) - f(a) - hf'(a) - \tfrac{1}{2}h^2 f''(x)\}.$$

Putting $x = \xi$ and observing that $a + h - \xi \neq 0$ since $a < \xi < a+h$ it follows that

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(\xi).$$

16·73. Newton's Method of Approximation

As stated in § 16·71, the method is of use in dealing with equations which involve transcendental functions as well as those which involve algebraic functions only. Suppose that the given equation is $f(x) = 0$, that the equation has a *simple* root near $x = a$ and that there are no other roots in the immediate neighbourhood of $x = a$. Let $a + h$ be the required root, h being small. It should be observed that h may be positive or negative. Then provided $f(x)$ possesses a second order derivative near $x = a$,

$$0 = f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a + \theta h), \quad 0 < \theta < 1.$$

$$\text{Thus } h = -\frac{f(a)}{f'(a)} - \tfrac{1}{2}h^2\frac{f''(a+\theta h)}{f'(a)}.$$

Since $a + h$ is a simple root of $f(x) = 0$, $f'(a+h) \neq 0$. Hence by taking h small enough we can find a positive number A such that $|f'(x)| > A$ for all values of x under consideration. As h is small, $f(a)$ will be of the first degree of smallness and the term $h^2 f''(a + \theta h)$ will be of the second degree of smallness. Neglecting this second order term, $h = -f(a)/f'(a)$. Hence the new approximation to the root is $a - f(a)/f'(a)$.

If we represent this value by β, then we may repeat the argument and obtain a further approximation $\beta - f(\beta)/f'(\beta)$. Proceeding in this way we may approximate as closely as we please to the required root.

We may consider the method from a graphical point of view. AB (Fig. 52) represents the graph of the function $y = f(x)$ in the neighbourhood of the point C where the graph cuts the X-axis. Then there is a real root of $f(x) = 0$ corresponding to the point C.

Let P be a point on the X-axis near C, $OP = a$, PQ the ordinate at P so that $PQ = f(a)$. The tangent at Q cuts the X-axis in T. If $\angle PTQ = \psi$, then $f'(a) = \tan\psi$. Hence

$$TP = PQ \cot\psi = f(a)/f'(a)$$

$$\text{and } OT = OP - TP = a - f(a)/f'(a).$$

It is clear from the figure that T is nearer to C than P, so that the value of x corresponding to T gives a closer approximation to the root than the value corresponding to P.

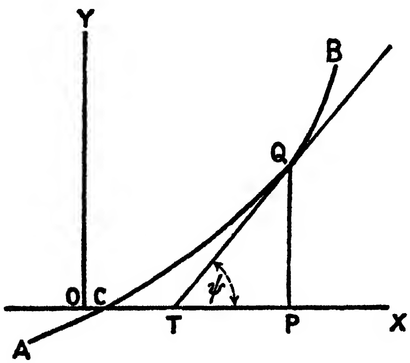


FIG. 52.

Example.—*Show that the equation $x^3 + x - 3 = 0$ has only one real root, and find its approximate value, correct to two decimal places.* [*Lond. Inter. Econ.*]

Write $f(x) \equiv x^3 + x - 3$. Since there is only one change of sign from $+$ to $-$ the equation has at most one positive root.

Again,

$$f(-x) = -x^3 - x - 3.$$

Since there are no changes in sign there are no negative roots.

Now $f(1) = -1$, $f(2) = 6$. Thus the root lies between 1 and 2. Before applying Newton's method we approximate more closely. $f(1{\cdot}2) = -{\cdot}072$, $f(1{\cdot}3) = {\cdot}497$. Thus $x = 1{\cdot}2$ is a closer approximation to the root.

Write $a = 1{\cdot}2$. Now $f'(x) = 3x^2 + 1$. Thus $f'(a) = 5{\cdot}32$ and

$$a - f(a)/f'(a) = 1{\cdot}2 + {\cdot}072/5{\cdot}32 = 1{\cdot}2 + {\cdot}014 \text{ approx.} = 1{\cdot}21$$

correct to two decimal places.

Let us consider the magnitude of the term which has been neglected, viz. $\frac{1}{2}h^2 f''(a + \theta h)/f'(a)$. Now $f''(x) = 6x + 1$. Since $x < 1{\cdot}3$, $f''(x) < 7{\cdot}8$. Hence

$$\tfrac{1}{2}h^2 f''(a + \theta h)/f'(a) < \tfrac{1}{2} \times ({\cdot}014)^2 \times 7{\cdot}8/5{\cdot}32 < ({\cdot}014)^2 = {\cdot}000196.$$

Clearly this term will not affect the second decimal place, so that the required root correct to two decimal places is $1{\cdot}21$.

In applying Newton's method it is necessary to obtain a fairly close approximation to the root if a good result is to be obtained by one application of the method. For this purpose it is frequently

best to first obtain an approximation by graphical methods as illustrated in the following example.

Example.—*Solve graphically the equation* $\sin \frac{1}{4}(2x - 3) = 2x - x^2$, *showing that it has two real roots, one positive and one negative. Find the approximate values of these roots from the graph and determine more accurately the value of the positive root by Newton's method.* [*Lond. B.Sc., Eng.*]

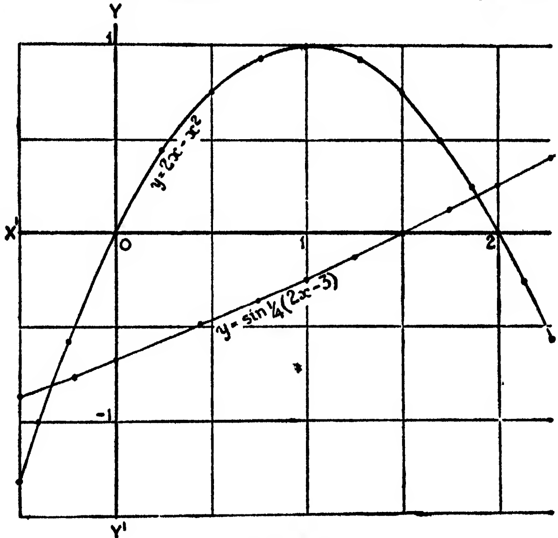


FIG. 53.

The solutions are found graphically by determining the points of intersection of the curves $y = \sin \frac{1}{4}(2x - 3)$, $y = 2x - x^2$. Since

$$-1 < \sin \tfrac{1}{4}(2x - 3) < 1$$

it is only necessary to consider values of x for which $-1 < 2x - x^2 < 1$.

Now $2x - x^2 = 1 - (1 - x)^2$. Hence $2x - x^2$ has a maximum value 1 at $x = 1$ and steadily decreases as x recedes from $x = 1$, in both directions. Further, the curve is symmetrical about the line $x = 1$.

Again, $2x - x^2 = -1$ when $x = 1 \pm \sqrt{2}$ so that it is only necessary to consider values of x between $1 - \sqrt{2}$ and $1 + \sqrt{2}$.

The curves (Fig. 53) are plotted from the following tables, x being measured in radians.

x	-0.5	-0.25	0	0.25	0.5	0.75
$\frac{1}{4}(2x - 3)$	-1.0	-0.875	-0.75	-0.625	-0.5	-0.375
$\sin \frac{1}{4}(2x - 3)$	-0.842	-0.768	-0.678	-0.585	-0.480	-0.366
$2x - x^2$	-1.250	-0.563	0	0.438	0.750	0.938

x	1·0	1·25	1·5	1·75	2·0	2·25
$\frac{1}{4}(2x-3)$	−0·25	−0·125	0	0·125	0·25	0·375
$\sin\frac{1}{4}(2x-3)$	−0·248	−0·125	0	0·125	0·248	0·366
$2x-x^2$	1·0	0·938	0·750	0·438	0	−0·563

Clearly the curves intersect at two and only two points. Thus there are two real roots: one is positive and the other negative, the corresponding values of x being 1·9 and − 0·33 approximately.

To determine the positive root more accurately by Newton's method, write

$$f(x) = 2x - x^2 - \sin\tfrac{1}{4}(2x-3).$$
$$f'(x) = 2 - 2x - \tfrac{1}{2}\cos\tfrac{1}{4}(2x-3).$$
$$f(1{\cdot}9) = 3{\cdot}8 - 3{\cdot}61 - \sin(0{\cdot}2) = -{\cdot}0088,$$
$$f'(1{\cdot}9) = 2 - 3{\cdot}8 - \tfrac{1}{2}\cos(0{\cdot}2) = -2{\cdot}29.$$

Hence $f(1{\cdot}9)/f'(1{\cdot}9) = 0{\cdot}00384\ldots$ This gives the new approximation to the root as $1{\cdot}89616\ldots$

To determine the accuracy of the result consider an upper bound to the error involved by neglecting the term in h^2.

The error term is

$$\tfrac{1}{2}h^2f''(x)/f'(1{\cdot}9), \text{ i.e. } \tfrac{1}{2}h^2\{-2+\tfrac{1}{8}\sin\tfrac{1}{4}(2x-3)\}/(-2{\cdot}29).$$

Taking the least favourable value for $\sin\frac{1}{4}(2x-3)$, viz. -1 and replacing 2·29 by 2, we obtain $-9h^2/16$ as an upper bound for the error.

Now $|9h^2/16| < 9\times(0{\cdot}004)^2/16 = {\cdot}000009$. Clearly this will not, e.g. affect the third decimal place in the determination of the root. Thus we we can say that the positive root is 1·896 correct to 3 decimal places.

16·8. Roots of Equations, some of whose Coefficients are Small

Let ϵ be a small quantity, positive or negative. We consider the determination of the roots of equations, some of whose coefficients have ϵ as a factor. It is assumed that the solutions may be expanded in powers of ϵ, *i.e.* if x is a root of the given equation,

$$x = a_0 + a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \ldots$$

where the coefficients $a_0, a_1, a_2, a_3 \ldots$ are independent of ϵ.

Clearly a_0 is a root of the equation obtained by writing $\epsilon = 0$ in the given equation. In this way the values of a_0 may be determined. The number of these values will depend on the degree of the given equation, and it will be necessary to consider each value separately.

In order to find a_1 neglect ϵ^2 and higher powers, *i.e.* write $x = a_0 + a_1\epsilon$ in the given equation, and expand the terms neglecting powers of ϵ higher than the first. In this way we obtain an equation which determines a_1. Thus suppose the given equation is

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where some of the coefficients $p_1, p_2, \ldots p_n$, have ϵ as a factor. The substitution gives

$$(a_0 + a_1\epsilon)^n + p_1 (a_0 + a_1\epsilon)^{n-1} + p_2 (a_0 + a_1\epsilon)^{n-2} + \ldots + p_n = 0 \ldots . \text{(ii)}$$

To make the argument explicit, suppose that p_1 is of the form ϵq_1, where q_1 is independent of ϵ, and that the other coefficients $p_2, p_3, \ldots, p_n$ are independent of ϵ. Thus neglecting ϵ^2 and higher powers we obtain from (ii), on expansion by the binomial theorem

$$\{a_0^n + n\epsilon a_0^{n-1} a_1\} + \epsilon q_1 a_0^{n-1} + p_2\{a_0^{n-2} + \epsilon (n-2) a_0^{n-3} a_1\} + \ldots + p_n = 0,$$

i.e. $$\{a_0^n + p_2 a_0^{n-2} + p_3 a_0^{n-3} + \ldots + p_n\} + \epsilon a_1\{n a_0^{n-1} + p_2 (n-2) a_0^{n-3} + p_3 (n-3) a_0^{n-4} + \ldots + p_{n-1}\} + \epsilon q_1 a_0^{n-1} = 0 \ldots\ldots\ldots \text{(iii)}$$

Now the expression in the first bracket is zero since a_0 is a root of (i) obtained by writing $\epsilon = 0$, in which case (i) becomes

$$x^n + p_2 x^{n-2} + p_3 x^{n-3} + \ldots + p_n = 0.$$

Since $\epsilon \neq 0$ it follows from (iii) that

$$a_1 = q_1 a_0^{n-1} / \{n a_0^{n-1} + p_2 (n-2) a_0^{n-3} + p_3 (n-3) a_0^{n-4} + \ldots + p_{n-1}\}$$

Hence a_1 is *uniquely* determined when a_0 is given.

To find a_2 write $x = a_0 + a_1\epsilon + a_2\epsilon^2$ and neglect ϵ^3 and higher powers of ϵ. Substituting in (i) and proceeding as before it will be seen that a_2 is uniquely determined when a_0 and a_1 are given. Proceeding in this way we may obtain the roots to any required degree of approximation.

Example.—*In the equation* $x^2 - x - 2 = \epsilon x^3$ *the number* ϵ *is small. Show that* $-1 + \frac{1}{3}\epsilon - \frac{8}{27}\epsilon^2$ *is an approximation to a root and determine the corresponding approximation to the root which is near* 2. *Write down an approximation to the third root.* [*Camb. Sch.*]

When $\epsilon = 0$ the equation becomes $x^2 - x - 2 = 0$, i.e. $x = -1$ or 2. It follows that the given equation has one root near $x = -1$ and another near $x = 2$. Substitute $x = -1 + h\epsilon + k\epsilon^2$ in the given equation and neglect ϵ^3 and higher powers. Then

$$(-1 + h\epsilon + k\epsilon^2)^2 - (-1 + h\epsilon + k\epsilon^2) - 2 = \epsilon(-1 + h\epsilon + k\epsilon^2)^3,$$

$$\text{i.e. } \epsilon(1 - 3h) + \epsilon^2 (h^2 - 3k - 3h) = 0.$$

Equating to zero the coefficients of ϵ and ϵ^2 we obtain $h = \frac{1}{3}$, $k = -\frac{8}{27}$ as required.

For the root near $x = 2$ write $x = 2 + \lambda\epsilon + \mu\epsilon^2$. Substituting in the given equation and neglecting ϵ^3 and higher powers we have

$$(3\lambda - 8)\epsilon + (\lambda^2 + 3\mu - 12\lambda)\epsilon^2 = 0.$$

Equating to zero the coefficients of ϵ and ϵ^2,

$$\lambda = \frac{8}{3}, \quad \mu = \frac{224}{27}.$$

Hence the approximation for the second root is

$$2 + \frac{8}{3}\epsilon + \frac{224}{27}\epsilon^2.$$

Now the product of the three roots of the equation is $-2/\epsilon$. Hence $1/\epsilon$ is a first approximation to the third root.

16·81. Application of Taylor's Expansion

We now give an example to show how the Taylor expansion* may be used to approximate to roots of equations of the type considered in the previous section.

Example.—*Explain how to find successive approximations to the values of x, when*

$$x = a_0 + \epsilon f(x) + \epsilon^2 g(x) + \ldots$$

where ϵ is small and $f(x)$, $g(x)$, . . . are given functions of x. Show that if ϵ is small, the equation

$$x^2 - 1 = \epsilon(x+1)^2 + \epsilon^2 x^3$$

has a root approximately equal to $1 + 2\epsilon + \frac{5}{2}\epsilon^2$. Find an approximation correct as far as ϵ^2 to the root which is nearly equal to -1 and also find an approximate form for the third root. [*Camb. Sch.*]

When $\epsilon = 0$, $x = a_0$ so that $x = a_0$ is a first approximation.

Let the next approximation be $x = a_0 + a_1\epsilon$. Using Taylor's expansion

$$f(x) = f(a_0 + a_1\epsilon) = f(a_0) + a_1\epsilon f'(a_0) + \frac{a_1^2\epsilon^2}{2!}f''(a_0) + \ldots$$

$$g(x) = g(a_0 + a_1\epsilon) = g(a_0) + a_1\epsilon g'(a_0) + \frac{a_1^2\epsilon^2}{2!}g''(a_0) + \ldots$$

. . . .

Neglecting ϵ^2 and higher powers $a_1\epsilon = \epsilon f(a_0)$, $a_1 = f(a_0)$.

For the third approximation we take $x = a_0 + a_1\epsilon + a_2\epsilon^2$. Substituting in the given equation, neglecting ϵ^3 and higher powers,

$$a_1\epsilon + a_2\epsilon^2 = \epsilon f(a_0) + a_1\epsilon^2 f'(a_0) + \epsilon^2 g(a_0),$$

$$\text{i.e. } a_2 = a_1 f'(a_0) + g(a_0).$$

Proceeding in this way we may obtain the successive coefficients a_3 a_4, . . .

For the particular case, consider first the root near $x = 1$. The equation may be written in the form

$$x - 1 = \epsilon(x+1) + \epsilon^2 x^3/(1+x).$$

Then $f(x) = x + 1$, $g(x) = x^3/(1+x)$, $f'(x) = 1$ so that $f(1) = 2$, $f'(1) = 1$.

Also $g(1) = 1/2$. Thus correct to the term in ϵ^2 the approximation is

$$x = 1 + 2\epsilon + (2.1 + \tfrac{1}{2})\epsilon^2 = 1 + 2\epsilon + \tfrac{5}{2}\epsilon^2.$$

* We assume in this section that Taylor's expansion is valid for functions other than polynomials. A general proof of the theorem is given in Hardy, *A Course of Pure Mathematics* (Camb. Univ. Press).

In order to determine the root near $x = -1$ we write the equation in the form

$$x + 1 = \frac{\epsilon(x+1)^2}{x-1} + \frac{\epsilon^2 x^3}{x-1}.$$

In this case $f(x) = (x+1)^2/(x-1)$, $f'(x) = (x+1)(x-3)/(x-1)^2$, $g(x) = x^3/(x-1)$. Thus

$$f(-1) = 0, \ f'(-1) = 0, \ g(-1) = \tfrac{1}{2}.$$

Hence the approximation is $x = -1 + \frac{1}{2}\epsilon^2$.

Again, the given equation may be written in the form

$$\epsilon^2 x^3 + (\epsilon - 1)x^2 + 2x\epsilon + 1 + \epsilon = 0.$$

Thus the sum of the three roots is $(1-\epsilon)/\epsilon^2$. Hence the third root is

$$\frac{1-\epsilon}{\epsilon^2} - 1 - 2\epsilon - \tfrac{5}{2}\epsilon^2 + 1 - \tfrac{1}{2}\epsilon^2 = \frac{1-\epsilon}{\epsilon^2} \text{ approximately.}$$

16·9. Horner's Method of Approximation

The method applies whether the roots can be represented by a terminating decimal or not and is concerned in the first instance with positive roots. The procedure is to determine the root, figure by figure. Thus the integral part is first determined; then the first decimal place is determined, then the second, and so on, until the root terminates or the root has been obtained to the required degree of approximation. The methods involved have already been considered. The two transformations required are (*a*) diminution of the roots of an equation, Chapter XV., § 15·53, and (*b*) multiplication of the roots of an equation by 10.

The first step is to determine between what two integers the real root, in which we are interested, lies. The procedure for the case in which there is *more than one* real root between two integers will be indicated in § 16·92. Having determined the integral part of the root we now transform the given equation by reducing the roots by this integral value. In the new equation the root must lie between 0 and 1. In order to avoid decimals in the working, the transformed equation now has its roots multiplied by 10 so that the figure required lies between 0 and 10. This integral value is determined, and thus the first decimal figure of the required root is determined. Proceeding in this way we may obtain any number of decimal places or else the root terminates.

The method and presentation will be clear from the examples given below.

Examples.—(i) *Calculate to two places of decimals the root lying between 2 and 3 of the equation*

$$x^4 - 12x^3 + 12x - 3 = 0. \qquad \text{[*Madras, B.Sc.*]}$$

If $f(x) = x^4 - 12x^2 + 12x - 3$, $f(2) = -11$, $f(3) = 6$ so that there is a root between 2 and 3. Thus the integral part of the root is 2 and the first step is to reduce the roots of the equation by 2. The work is set out as indicated earlier. Note that since there is no term in x^3 we must insert 0 as the coefficient of x^3.

1	0	− 12	12	− 3
	2	4	− 16	− 8
	2	− 8	− 4	− 11
	2	8	0	
	4	0	− 4	
	2	12		
	6	12		
	2			
	8			

The transformed equation is

$$x^4 + 8x^3 + 12x^2 - 4x - 11 = 0.$$

Multiplying the roots of this equation by 10 we obtain

$$F(x) \equiv x^4 + 80^3 + 1200x^2 - 4000x - 110000 = 0.$$

$$\text{Now } F(8) = 4096 + 40960 + 76800 - 32000 - 110000 < 0,$$

$$F(9) = 6561 + 58320 + 97200 - 36000 - 110000 > 0.$$

Thus $F(x) = 0$ has a root between 8 and 9 and the first decimal place for the required root is 8.

We now reduce the roots of $F(x) = 0$ by 8.

1	80	1200	− 4000	− 11000
	8	704	15232	89856
	88	1904	11232	− 20144
	8	768	21376	
	96	2672	32608	
	8	832		
	104	3504		
	8			
	112			

The transformed equation is

$$x^4 + 112x^3 + 3504x^2 + 32608x - 20144 = 0.$$

Multiplying the roots of this equation by 10 we obtain

$$G(x) \equiv x^4 + 1120x^3 + 350400x^2 + 32608000x - 201440000 = 0.$$

A study of the magnitudes of the coefficients of the terms in the equations $f(x) = 0$, $F(x) = 0$, $G(x) = 0$ shows that as the process continues, the dominating terms which will determine the required value of x, will be the last two terms, *i.e.* the term in x and the constant term. Thus, e.g. in $G(x) = 0$, since x must lie between 0 and 10 the term in x^4 cannot exceed 10^4, *i.e.* 10,000, which is small in comparison with $-$ 201,440,000.

Now $201{,}440{,}000/32{,}608{,}000 = 6\cdot1\ldots$ This suggests that $G(x) = 0$ has a root between 6 and 7 and that this is the case is easily verified. Further, this figure 6 is correct to the *nearest integer* as may be seen by continuing the working. Thus correct to two decimal places the root is $2\cdot86$.

16·91. Simplifications in the case of Horner's Method

(*a*) In the actual presentation of the numerical working we can omit completely the powers of x, writing down the coefficients only of the various transformed equations. The setting out is indicated in the example given below.

(*b*) The example of § 16·9 shows that if it is necessary in each case to determine the required figure by actual substitution, the numerical work involved may be considerable. This is, however, unnecessary and one of the practical advantages of Horner's method is that after the second or third figure of the root has been determined the *transformed equation itself suggests by inspection the next figure for the root.* This is called the principle of the *trial-divisor.* The reason for it has been indicated at the end of the example in § 14·9. The *second last coefficient* of each transformed equation (after the roots have been multiplied by 10) is called the *trial divisor.* This divided into the constant term give the suggested value.

(*c*) It is clear that beyond a certain stage in the working the second last coefficient and the constant term must have opposite signs; for as the process of evolution of the root proceeds the influence of the other terms becomes smaller and smaller. After the first transformation, *i.e. after the equation has had its roots reduced by the integral part of the root* under consideration, the absolute term *must not change sign.* The sign may change at the first transformation. Thus, e.g. the equation

$$2x^3 - 7x^2 - 4x + 14 = 0$$

has a root between 3 and 4. Reducing the roots by 3 we obtain

$$2x^3 + 11x^2 + 8x - 7 = 0.$$

The reason for the change in sign is that we have passed over another root of the equation, viz. that between 1 and 2. But assuming that the original equation has *only one* root between 3 and 4, the transformed equation

$$2x^3 + 110x^2 + 800x - 7000 = 0$$

has one and *only one* root between 0 and 10. Thus further transformations of the type considered will not affect the sign of the

constant term. We have thus the following rule. The figure to be taken is the *highest number* which in the process of transformation will leave *unchanged the sign of the constant term.* If the number taken were too small the next figure suggested would be greater than 9, as in ordinary division.

Example.—*Find by Horner's method the real root of the equation*

$$x^3 + 2x - 10 = 0$$

correct to four significant figures. [*Lond. Inter. Econ.*]

Descartes' rule of signs shows that the equation has no negative roots and at most, one positive root. It is easily seen that this root lies between 1 and 2.

We first set out the working and then comment on it.

1	0	2	− 10 (1·847
	1	1	3
	1	3	− 7000
	1	2	6432
	2	500	568000
	1	304	477504
	30	804	− 90496000
	8	368	85368423
	38	117200	− 5127577000
	8	2176	
	46	119376	
	8	2192	
	540	12156800	
	4	38689	
	544	12195489	
	4	38738	
	548	1223422700	
	4		
	5520		
	7		
	5527		
	7		
	5534		
	7		
	55410		

We first diminish the roots by 1. The transformed equation when the roots have been multiplied by 10 is

$$x^3 + 30x^2 + 500x - 7000 = 0.$$

The first trial divisor 500 gives no real indication of the first decimal place as is to be expected. It will be seen that if 9 is taken instead of 8 the sign of the constant term would have been changed from − to +.

After the roots have been reduced by 8 and the roots of the resulting equation multiplied by 10, the transformed equation is

$$x^3 + 540x^2 + 117200x - 568000 = 0.$$

The trial divisor is 117200 and $568000/117200 = 4{\cdot}8\ldots$ and the third decimal place is given correctly. The close approximation to the true result, 4·7, is to be expected because of the relative smallness of the first two terms.

The trial divisor for the next transformed equation gives 7·4, suggesting that the third and fourth decimal places are 74. That the 4 is correct is verified by the next trial divisor.

Hence correct to four significant figures the root is 1·847.

16·92. Roots nearly Equal

The method of procedure in such cases is to separate the roots and then proceed as before.

Consider the following example. The equation

$$x^3 - 3x^2 - 4x + 13 = 0$$

has two roots between 2 and 3. Find each of them correct to two decimal places.

Diminishing the roots of the equation by 2 and multiplying the roots of the new equation by 10 we obtain

$$f(x) \equiv x^3 + 30x^2 - 400x + 1000 = 0.$$

This equation must have *two* roots between 0 and 10.

$$\text{Now } f(3) = 27 + 270 - 1200 + 1000 > 0.$$
$$f(4) = 64 + 480 - 1600 + 1000 < 0.$$
$$f(6) = 216 + 1080 - 2400 + 100 < 0.$$
$$f(7) = 343 + 1470 - 2800 + 1000 > 0.$$

It follows that the transformed equation has one root between 3 and 4 and another between 6 and 7. Thus the two roots are $2{\cdot}3\ldots,\ 2{\cdot}6\ldots$

Proceeding as before it will be seen that correct to two decimal places the roots are 2·36, 2·69.

EXERCISES XVI

29. If a root of the equation $f(x) = 0$ is known to lie between two values of x, a, and $a + h$, where h is small, and if $f(a) = \xi$ and $f(a + h) = -\eta$, where ξ and η are small, show that a good approximation to the root is $a + \dfrac{h\xi}{\xi + \eta}$.

Prove that the equation $x^3 - 4x + 1 = 0$ has a root lying between 1 and 2, and find it correct to two places of decimals. [*Lond., B.Sc. Eng.*]

30. Plot in the same diagram the curves

$$y = x^2 \text{ and } y = 3 + \frac{2}{x - 1}$$

for positive values of x. Deduce approximate values of the positive roots of the equation

$$x^3 - x^2 - 3x + 1 = 0.$$

Find by any method, a better approximation to the smaller of these roots. [*Lond. Inter. Econ.*]

31. Find the real root of $x^5 + 3x - 5 = 0$ correct to two decimal pla [*Madras, B.A.*]

32. Find correct to three decimal places, the root of $x^4 - 8x - 60 = 0$ which is nearly equal to 3. [*Camb. Sch.*]

33. Find to four significant figures the real root of the equation

$$x^3 - 4x - 7 = 0.$$

34. If x is measured in radians the equation $\tan x = x$ has a root near $x = 3\pi/2$. Find by Newton's method the value of the root correct to two decimal places.

35. If a is small, the equation $\cos x = ax$ has a root near $x = \frac{1}{2}\pi$. Prove that $\frac{\pi}{2}(1 - a)$ is a better approximation.

36. In the equation $x(x^2 - 1) - \epsilon(x + 2)$, ϵ is a small number. Prove that an approximation to one of the roots is $1 + \frac{3}{2}\epsilon - \frac{21}{8}\epsilon^2$, the third power of ϵ being neglected. Find the corresponding approximations to the other roots. [*Camb. Sch.*]

37. Show that, if ϵ is small, one root of the equation

$$x^3 - 6x^2 + 11x - 6 = \epsilon$$

is $1 + \frac{1}{2}\epsilon$ approximately; and find approximations to the other roots to the same order in ϵ. [*Lond., B.A.*]

38. If $x^{3 + \epsilon} = 8$ show that x differs from 2 by less than $\frac{1}{2}\epsilon$ if ϵ is so small that its square may be neglected. [*Camb. Sch.*]

39. Extract the fifth root of 7 correct to 4 significant figures by using Horner's method. [*Lond., Inter. Econ.*]

40. Show that the equation $x^3 = 3x + 4$ has a root between 2 and 3, and use Horner's method to calculate the root to *three* places of decimals. [*Lond., Inter. Econ.*]

41. On the same diagram draw the graphs of $y = x^3$ and $y = x + 9$. Show that the abscissa of the point of intersection satisfies the equation $x^3 - x - 9 = 0$, and hence obtain an approximate value of the real root of this equation. By Horner's method, or otherwise, find the value of the real root of $x^3 - x - 9 = 0$ correct to two decimal places. [*Lond., Inter. Econ.*]

42. Find between what consecutive positive or negative integers the roots of the equation $x^3 - x^2 - 7x - 3 = 0$ must lie, and use Horner's method to find the positive root correct to the second decimal place.

[*Lond., Inter. Econ.*]

43. By drawing the graphs of $y = x^3$ and $y = x^2 - 2x + 3$, prove that the equation $x^3 - x^2 + 2x - 3 = 0$ has only one real root. By Horner's method, or otherwise, find the value of this root correct to the third decimal place. [*Lond., Inter. Econ.*]

44. Draw on the same diagram the graphs of

$$y = x - 1 \text{ and } y = x(x - 2)(x - 3).$$

Hence obtain approximate values of the roots of the equation

$$x^3 - 5x^2 + 5x + 1 = 0.$$

By Horner's method obtain the value of the root between 0 and 2, correct to three decimal places. [*Lond., Inter. Econ.*]

45. Obtain, correct to four significant figures, the root of the equation $x^3 - 2x - 3 = 0$ lying between 1 and 2 *either* (i) by a purely algebraic method of approximation, *or* (ii) by a graphical method followed by a closer algebraic approximation. [*Lond., Inter. Econ.*]

46. Show that the equation $x^4 - 3x + 1 = 0$ has only two real roots, and that they lie, one between 0 and 1, and the other between 1 and 2. Find the root between 0 and 1 correct to three places of decimals.

[*Lond., B.Sc.*]

47. Given an algebraic equation

$$x^n + p_1x^{n-1} + \ldots + p_n = 0,$$

write down equations whose roots are (1) the roots of this equation diminished by a, (2) the roots of the original equation multiplied by b. Find by Horner's method, to three significant figures, the root between 2 and 3 of $x^4 - 30x + 18 = 0$. [*M.T.*]

48. Show that the equation $x^3 - 7x - 4 = 0$ has three real roots, and find the positive one correct to three significant figures.

49. Calculate to three places of decimals the real positive roots of the equation $x^3 - 8x - 40 = 0$. [*Madras, B.Sc.*]

50. If $f(x) = x^3 + x^2 - 2x - 1$ and $f(a) = 0$, then $f(a^2 - 2) = 0$. Find the greatest real root of $f(x) = 0$ correct to three decimal places.

[*Madras, B.A.*]

51. Show that the equation $x^3 - 7x + 1 = 0$ has three real roots, and find the largest root correct to one per cent. [*Camb. Sch.*]

ANSWERS

EXERCISES I

Infinite Sequences and Series

3. a_n tends to values which oscillate between $\pm \frac{1}{2}$.
   b_n tends to oscillate infinitely.

4. If $\{a_n\}$ is monotonic decreasing so is $\{b_n\}$.

12. Compare with series $\Sigma \frac{1}{n}$.

26. $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \frac{9563}{3044}$.

27. (i) All limits zero; (ii) double limit zero, repeated limits do not exist; (iii) double limit does not exist, repeated limits 0, 1; (iv) double limit does not exist, repeated limits both 0.

28. Sum by columns $= 1$.

30. (i) $|x| < 1$; (ii) All finite values of x.

EXERCISES II

Theorem on Limits and Continuous Functions

1. 0.

2. a/l.

5. $3x^2$, $1/x^2$.

6. (i) $\lim\limits_{x \to 1+0} f(x) = -\infty$, $\lim\limits_{x \to 1-0} f(x) = +\infty$; (ii) $\lim\limits_{x \to \infty} f(x) = \infty$.

7. $1/\sqrt{a}$.

14. (i) Limit exists and $= 1$; (ii) function oscillates between 0 and 1; (iii) function oscillates between $+1$ and -1; (iv) limit exists and $= 0$.

16. $x = 1$, $x = 3$.

17. Continuous for all values of x in the interval (a, b) except at $x = b$.

19. $\frac{2}{3}a^{-\frac{1}{3}}$; 10.

20. $\frac{5}{3}$.

21. 1; 1.

EXERCISES III

The Binomial Theorem for a Rational Index

3. $1 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{128}x^3$.

4. $1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3$.

5. $a^{-2} + x^2a^{-4} + x^4a^{-6} + x^6a^{-8}$.

6. $a^{-\frac{3}{5}}\left\{1 + \frac{3}{10}\frac{x}{a} + \frac{39}{200}\frac{x^2}{a^2} + \frac{299}{2000}\frac{x^3}{a^2}\right\}$.

7. $a^{-3} + 3a^{-5}x^{-1} + 6a^{-7}x^{-2} + 10a^{-9}x^{-3}$.

8. $c^{\frac{3}{2}} - \frac{3}{4}x^2c^{-\frac{1}{2}} - \frac{3}{32}x^4c^{-\frac{5}{2}} - \frac{5}{128}x^6c^{-\frac{9}{2}}$.

9. $a^{-2} + 6a^{-\frac{7}{3}}x^{\frac{1}{3}} + 21a^{-\frac{8}{3}}x^{\frac{2}{3}} + 56a^{-3}x$.

10. $a^{-1} + \frac{1}{5}x^5a^{-6} + \frac{3}{25}x^{10}a^{-11} + \frac{11}{125}x^{15}a^{-16}$.

11. $1 - 4x + \frac{17}{2}x^2 - 14x^3$.
13. $(r + 1)(2r + 1)$.

16. 2nd term.
17. 20th term.
18. 9th term.

19. 2nd term.
20. $(3n - 27)/20$; $n = 43/3$; 17.

21. $1/x$.
23. 15th term; 27th term; 1st term.

24. 2nd term $= 1\frac{1}{4}$.
25. 3rd term $= 7\frac{87}{225}$.

26. 4th $=$ 5th term $= 1\frac{2369}{6561}$.
28. (i) 6; (ii) 0·242.
29. $n = 3$; $n = r$.

30. 121.
31. $2^{r-3}(r^2 + 7r + 8)$.

32. $3n^2 + n + 1$.
34. $n(n + 1) \ldots (n + r - 1)\, a^r/r!$

37. $1.3.5 \ldots (2p - 1)/2^p(p!)$, where $p = \frac{1}{2}r$ or $\frac{1}{2}(r - 1)$ according as r is even or odd.
45. $1 + \frac{3}{4}x$.
46. (a) 15; 18.

47. $\frac{1}{2}(n + 1)(n + 2)$.
52. $(1 + x)^2$; $x > -\frac{1}{2}$.
53. $1 + x + x^2 + x^3$.

54. 0·02000.
55. 9·99333.
56. 1·4422.
57. 1·2599.

58. 3·0723.
74. (a) 0·974; (b) 0·986; (c) 0·997.

EXERCISES IV

Infinite Series whose Terms are Functions of a Variable

1. The sum is 0.
8. $S(x) = x/(1 - x^2)$, $0 < x < 1$; $S(0) = S(1) = 0$.

9. Not uniformly convergent at $x = 0$.

EXERCISES V

The Exponential and Logarithmic Series

1. $a = \log p$, $b = q/p$, $c = (2rp - q^2)/2p^2$; $a = \log 2$, $b = 3/2$, $c = 7/8$.

6. 0·521.
7. 1; 0·3679; 0·0183; 0·0001.
10. 1.
12. $3e$.

13. $\frac{1}{2}(e - e^{-1})$.
14. 25/24.
16. $2\frac{17}{24}$.

17. (b) (i) $\frac{1}{2}(\cosh x + \cos x)$; (ii) $\frac{1}{2}(\sinh x - \sin x)$.
19. $(-1)^{r-1}(a^r + b^r + c^r)$

20. 2·303.
24. $\log \frac{7}{5} = 0·33647$; $\log_{10} 7 = 0·84510$.

25. $n < -1$ or $n > 0$.
26. $2x - \frac{4}{3}x^3$.
27. 0·2469.

28. $|x| < \frac{1}{3}$; $\Sigma(-1)^{n-1}(3^n + 1)x^n/n$; 0·153.
29. 1·6094.
30. 1·105171.

33. 4·6151205.
37. $2\{x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{2}{9}x^9 + \frac{1}{11}x^{11} + \ldots\}$.

38. $\log 3 = 1·0986$; $\log_{10}e = 0·4343$.
40. 11 terms.

41. $1 + x + x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4$.
44. $1 - \log 2$.
45. $\frac{3}{4} - \log 2$.

46. 20.
54. $\log \frac{5}{3}$.
56. $\frac{1}{18}$.

EXERCISES VI

Determination of Functions from Empirical Data

The results obtained by graphical methods are only approximate.

2. $a = b = \frac{1}{8}$. 3. $a = 2{\cdot}8$, $b = 3{\cdot}4$. 4. $a = 7{\cdot}5$, $b = 5{\cdot}3$.

5. $a = 0{\cdot}95$, $b = 0{\cdot}69$. 6. $a = 39{\cdot}8$, $b = 11{\cdot}5$.

7. $a = 2{\cdot}5$, $b = 10{\cdot}8$. 8. $a = 0{\cdot}304$, $b = 0{\cdot}0004$.

9. $a = 0{\cdot}0075$, $n = 0{\cdot}87$. 10. $a = 0{\cdot}4$, $n = 1{\cdot}5$.

11. $a = 28120$, $n = -2{\cdot}04$. 12. $a = 0{\cdot}0425$, $n = 1{\cdot}163$.

13. $n = 1{\cdot}3$.; $C = 200$. 14. $a = 2{\cdot}82$, $n = 1{\cdot}75$.

15. $a = 50$, $b = 1{\cdot}149$. 16. $a = 5$, $b = 12$, $n = -\frac{1}{2}$.

17. $b = 5$, $a = 10$, $n = 2$. 18. $c = 10$, $a = 3{\cdot}1$, $b = -0{\cdot}1$.

EXERCISES VII

Further Theorems on Convergence

5. (i) D.; (ii) C. if $a < -1$, D. if $a > -1$; (iii) C.; (iv) C. if $x < e^{-1}$, D. if $x > e^{-1}$.

6. (i) C.; (ii) D.; (iii) C. if $p > 1$, D. if $p < 1$. 7. (i) D.; (ii) D.; (iii) D.

8. D. 9. C. if $-1 < x < 1$, otherwise D. 10. C. for all finite x.

12. D. 13. C. if $b > a > 0$; D. if $a > b > 0$.

14. (i) C. if $a < -1$, D. if $a > -1$, except when a is a negative integer. (ii) C. if $a < -2$, D. if $a > -2$, except when a is a negative integer.

16. C. for all finite p. 17. C. 18. $-\log x$; $0 < x < 2$. 23. Sum is unity.

EXERCISES VIII

Complex Numbers

1. $2(i + 1)$. 2. $a^2 - b^2 + 2abi$; $(a^3 - 3ab^2) + i(3a^2b - b^3)$; $(a^4 - 6a^2b^2 - b^4) + i(4a^3b - 4ab^3)$. 3. 56. 4. -36.

5. $(x - y)(x + y)(x + iy)(x - iy)$. 7. $4abi/(a^2 + b^2)$. 8. $-1 + i$.

9. (i) $\pm(1 + i\sqrt{2})$; (ii) $\pm(2 + i\sqrt{5})$.

12. (i) 2; (ii) 1025; (iii) $\frac{5}{13}$; (iv) $1\frac{47}{85}$.

14. $x = (a_1 + b_1)\cos nt - (a_2 - b_2)\sin nt$; $y = (a_1 - b_1)\sin nt + (a_2 + b_2)\cos nt$.

17. (i) $\frac{1}{5}\sqrt{26}$, $-\tan^{-1} 5$; (ii) $\tan\frac{1}{2}\theta$, $\frac{1}{2}\pi$.

18. $|z_1| = |z_2| = 2$; amp. $z_1 = \frac{1}{3}\pi$, amp. $z_2 = \frac{1}{4}\pi$; if L_1, L_2, L_3 represent z_1, z_2, z_3 then OL_2 bisects L_1OL_3. 21. $AB.AC = AD.AE$.

23. $2^{\frac{1}{12}}\left\{\cos\dfrac{(8r+3)\pi}{24} + i\sin\dfrac{(8r+3)\pi}{24}\right\}$, $r = 0, 1, \ldots, 5$.

24. $2{\cdot}185 + 0{\cdot}476i$.

25. (i) $\dfrac{-16+18i}{29}$; (ii) $-128 + 128i\sqrt{3}$; (iii) $\frac{1}{2}\cosh(\log 2) - i\dfrac{\sqrt{3}}{2}\sinh(\log 2)$.

26. $u = \lambda\left(\dfrac{1}{R} + R\right)$, $v = \mu\left(\dfrac{1}{R} - R\right)$, where $\lambda = (x-1)\cos\alpha + y\sin\alpha$, $\mu = (x-1)\sin\alpha - y\cos\alpha$, $R^2 = (x-1)^2 + y^2$.

28. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

29. $x = -\frac{1}{2}(1 - i\sqrt{7}), y = -\frac{1}{2}(1 + i\sqrt{7})$.

30. $x = \frac{1}{2}\left\{\cos\dfrac{2r\pi}{n} - 1 - i\sin\dfrac{2r\pi}{n}\right\}\Big/\left(1 - \cos\dfrac{2r\pi}{n}\right)$, $r = 0, 1, \ldots, n-1$.

34. Series converges under the following conditions:—(i) $b < 0$; all a, c; (ii) $b = 0$; $a < 0$, $c \neq 0$ or $2n\pi$, $a < -1$ if $c = 0$ or $2n\pi$.

41. $\log 2 + i\pi$; $\frac{1}{2}\pi + i\cosh^{-1} 2$.

42. 1, $\tan^{-1} 3/4$; locus is a straight line through O inclined at an angle $\tan^{-1} b/a$ to the real axis.

43. $e^{\frac{1}{n}}\left\{\cos\dfrac{\phi+2r\pi}{n} + i\sin\dfrac{\phi+2r\pi}{n}\right\}$, $r = \sqrt{(a^2+b^2)}$, $\phi = \cos^{-1} a/r$ $= \sin^{-1} b/r$; $e^{\pm} 2^{\frac{1}{4}}(\cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi)$.

44. $i\cosh^{-1} 2$; $e^{2\log 10}\{\cos(3\log 10) + i\sin(3\log 10)\}$; $\log 4 + \pi i$.

EXERCISES IX

PARTIAL FRACTIONS, RECURRING SERIES, DIFFERENCE EQUATIONS

1. $(a-\alpha)(a-\beta)(a-\gamma)(a-\delta)/(a-b)(a-c)(a-d)$, with three similar expressions for the other constants.

2. $a = 1, b = -10, c = 25$.

3. $1 - \dfrac{3}{x-2} + \dfrac{8}{x-3}$; $\dfrac{x+\sqrt{2}}{2\sqrt{2}(x^2+\sqrt{2}x+1)} + \dfrac{-x+\sqrt{2}}{2\sqrt{2}(x^2-\sqrt{2}x+1)}$; $\dfrac{1}{x} - \dfrac{1}{x-1} + \dfrac{1}{(x-1)^2}$.

5. $\dfrac{x}{x-1} - \dfrac{7}{x-} + \dfrac{\frac{13}{2}}{x-3}$; $\dfrac{-2}{(x-1)^2} + \dfrac{-3}{x-1} + \dfrac{3}{x-2}$.

6. $\dfrac{\frac{1}{2}}{x^3} + \dfrac{-\frac{1}{4}}{x^2} + \dfrac{\frac{1}{8}}{x} + \dfrac{-\frac{1}{8}}{x+2}$; $\frac{1}{6} - \dfrac{\frac{1}{2}}{1+2x} + \dfrac{\frac{1}{3}}{1+3x}$; $\dfrac{\frac{1}{12}}{x-2} + \dfrac{\frac{11}{12}x+\frac{2}{3}}{x^2+2x+4}$.

7. $\dfrac{1}{x-1} - \dfrac{1}{x+2} - \dfrac{3}{(x+2)^2}$.

8. Given function is

$$\frac{2}{x+2}+\frac{x^3}{(x^2-1)^2}.$$

Coefficient of $x^{2r} = 2^{-2r}$;
coefficient of $x^{2r+1} = -2^{-2r-1} - \frac{1}{2}r(r+1)$.

10. $\dfrac{2}{1+2x}+\dfrac{6}{(1+2x)^2}+\dfrac{1}{3-x}$; $8\frac{1}{3} - 27\frac{8}{9}x + 80\frac{1}{27}x^2$.

11. $\sum\limits_{n=0}^{\infty} x^n \{\frac{1}{9}.2^n + (-1)^n \frac{1}{3}(n+1) + (-1)^{n+1}\frac{4}{9}\}$.

12. $\dfrac{3}{2-x}-\dfrac{x-1}{x^2+1}$; $2{\cdot}486092$.

13. $\dfrac{10}{(x-1)^2}+\dfrac{\frac{35}{3}}{2x-1}+\dfrac{-\frac{2}{3}}{5x-1}$; $-1+30x^3+280x^4$.

14. $a_n = -(n+2^{-n-1})$.

15. $\dfrac{\frac{1}{3}}{x-2}+\dfrac{\frac{1}{3}}{x+2}-\dfrac{1}{x+1}$; $1-4^{-n}$; $|x|<1$.

17. $\dfrac{-\frac{1}{5}}{(1+2x)^2}+\dfrac{\frac{2}{25}}{1+2x}+\dfrac{\frac{3}{25}}{1-3x}$; $\sum\limits_{n=1}^{\infty}\frac{1}{25}x^n\{(-1)^{n-1}2^n(5n+3)+3^{n+1}\}$.

18. $-\left\{(-1)^n\frac{1}{27}\cdot\dfrac{1}{2^{n+1}}+\frac{1}{27}+\frac{1}{9}(n+1)+\frac{2}{3}(n+1)(n+2)\right\}$.

19. $\dfrac{1}{1-3x}+\dfrac{1}{1+2x}+\dfrac{1}{1+x}$; $\sum\limits_{n=0}^{\infty}\{3^n+(-1)^n(2^n+1)\}x^n$; $|x|<\frac{1}{3}$.

20. $a=\frac{1}{8}$, $b=-\frac{1}{2}$, $c=\frac{4}{3}$.

21. $\dfrac{1}{(a-b)^3}\left\{\dfrac{-2}{x-a}+\dfrac{2}{x-b}+\dfrac{a-b}{(x-a)^2}+\dfrac{a-b}{(x-b)^2}\right\}$.

22. $1-4x+9x^2-17x^3$.

23. $\dfrac{4}{2x+1}-\dfrac{2}{x-2}-\dfrac{10}{(x-2)^2}$; $\frac{5}{2}-10x+14\frac{5}{8}x^2-33\frac{1}{8}x^3$.

24. $\frac{1}{13}\left\{\dfrac{6}{3-2x}+\dfrac{3x-2}{1+x^2}\right\}$.

25. $\dfrac{\frac{5}{2}}{1-x}-\dfrac{\frac{1}{2}}{1+x}-\dfrac{1+x}{1+x^2}$; $1+2x+3x^2+4x^3+x^4$.

26. $\frac{1}{4}\left\{\dfrac{1}{x-1}-\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}\right\}$; $\frac{1}{4}\{1+(-1)^n+2(n-1)(-1)^{n-1}\}$;
$|x|>1$.

27. $p_n = 2^{n+1}-1$.

28. $(1+2^n)x^n$; $(2-3x)/(1-3x+2x^2)$.

30. $\dfrac{1-3^nx^n}{1-3x}+\dfrac{1-4^nx^n}{1-4x}$.

31. $u_n = n-1+2^n$.

32. $\dfrac{1 - x\cos\theta - x^n\cos n\theta + x^{n+1}\cos(n-1)\theta}{1 - 2x\cos\theta + x^2}$; $\dfrac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2}$.

33. $\dfrac{x\sin\theta}{1 - 2x\cos\theta + x^2}$; $\dfrac{x\sin\theta - x^{n+1}\sin(n+1)\theta + x^{n+2}\sin n\theta}{1 - 2x\cos\theta + x^2}$.

34. $(\frac{4}{5})^{n-1} + (\frac{3}{5})^{n-1}$; $15/2$.

35. $(3 + 17x)/(3 - x)(1 - x)^2$.

36. $1 + (p + r)x + (q + s + pr)x^2 + (ps + qr)x^3 + qsx^4$
$= (1 + px + qx^2)(1 + rx + sx^2)$.

39. $u_n = A + 2^nB + 2^nC\cos\dfrac{2n\pi}{3} + 2^nD\sin\dfrac{2^n\pi}{3}$;

$u_n = 1 + 2^{n+1} - 2^n\cos\dfrac{2n\pi}{3} + 2^n\sqrt{3}\sin\dfrac{2n\pi}{3}$.

EXERCISES X

FINITE DIFFERENCES

5. 1^n; $2^n - 2$; $3^n - 3.2^n + 3$.

6. $e^x(x + 3x^2 + x^3)$.

8. $u_n = 12 - 12n^{(1)} - 2n^{(2)} + 2n^{(3)}$; $\frac{1}{6}n(n-1)(3n^2 - 7n - 46)$.

9. $55 - 62x + 21x^2 - 2x^3$.

10. $f(12{\cdot}4)$ should be $0{\cdot}00947$; $f(12{\cdot}72) = 0{\cdot}0183$.

11. $f(x) = 2 + 3(x)_{1,2} + (x)_{2,2} + (x)_{3,2}$.

12. $u_n = c_1 2^n + c_2 k^n$, $k \neq 2$; $u_n = (c_1 + c_2 n)2^n$, $k = 2$; $u_n = (n - 1)2^n$ is particular solution.

13. $y = c_1(-4)^x + (c_2 + c_3 x)5^x$; $f(x) = \frac{5}{9}(-4)^x + (x - \frac{5}{9})5^x$.

14. $u_x = c4^x - \frac{1}{3}x + \frac{14}{9}$.

15. $c_1(-1)^x + c_2 4^x + \dfrac{a^x}{(a-4)(a+1)}$, $a \neq 4, -1$;

$c_1(-1)^x + c_2 4^x + \dfrac{x}{20}4^x$, $a = 4$; $c_1(-1)^x + c_2 4^x + \dfrac{x}{5}(-1)^x$, $a = -1$.

EXERCISES XI

SUMMATION OF SERIES

2. 31940.

3. $n(n+1)x^2 - 2(n+1)x + 2$.

4. $\frac{1}{6}n(2n^3 + 4n^2 - n - 1)$.

5. $\frac{1}{6}n(n+1)(3n^2 + 5n + 1)$.

6. $\frac{1}{4}n(n+1)^2(n+2)$.

12. $\frac{4}{3}\sqrt{3} - 2\frac{1}{4}$.

13. (i) $\dfrac{r}{r!} - \dfrac{r^{n+1}}{(r+n)!}$; (ii) 12.

14. (i) $\frac{1}{4}n(n+1)(n^2 + 9n + 20)$; (ii) $(\frac{5}{3})^{\frac{3}{2}}$.

16. $\sqrt{3}$.

17. $(1 + x)e^x$.

18. (i) $1 + e^{-1}$; (ii) $1/(1 - x)(1 - xy)$, $|x| < 1$, $|xy| < 1$.

19. $4e - 3$.

20. $\frac{3}{2}e$.

22. $15e - 1$.

23. $\frac{2}{3}\log 2 - \frac{5}{18}$.

24. (i) $5e$; (ii) $\frac{1}{42} + \frac{1}{18}\cdot\dfrac{1}{4n+3} - \frac{5}{18}\cdot\dfrac{1}{4n+7}$.

25. $4e - 5$.

26. $\dfrac{\ }{4x-3} - \dfrac{\ }{4x-2} + \dfrac{\ }{4x-1}$,

29. $\log \frac{4}{3}$.

31. (i) $(\frac{3}{2})^{\frac{3}{2}}$; (ii) $1 - \left(1 - \dfrac{1}{x}\right) \log (1 - x)$, $|x| < 1$.

32. $x/(1 - x)^2$, $|x| < 1$; $\frac{1}{2}\dfrac{xe^{\theta}}{(1 - xe^{\theta})^2} + \frac{1}{2}\dfrac{xe^{-\theta}}{(1 - xe^{-\theta})^2}$, $|x| < e^{-\theta}$, if $\theta > 0$ and $|x| < e^{\theta}$ if $\theta < 0$.

34. $4e - 2$.

35. $\dfrac{1}{n} - \dfrac{1}{2} \cdot \dfrac{1}{n - 1} + \dfrac{1}{4} \cdot \dfrac{1}{(n - 1)^2} - \dfrac{1}{2} \cdot \dfrac{1}{n + 1} - \dfrac{1}{4} \cdot \dfrac{1}{(n + 1)^2}$.

36. $\frac{1}{140} - \frac{1}{48}\left(\dfrac{1}{2n + 3} + \dfrac{1}{2n + 5} + \dfrac{1}{2n + 7}\right) + \frac{1}{16} \cdot \dfrac{1}{2n + 9}$; $\frac{1}{140}$.

37. $5e$.

38. (i) $\frac{1}{18} - \dfrac{1}{6(n + 1)} + \dfrac{1}{3(n + 2)} - \dfrac{1}{6(n + 3)}$; (ii) $\frac{7}{12}$.

39. If $s(x)$ denote the sum of the series,

$$s(x) = \left(\frac{1}{6x} - \frac{3}{2x^3} + \frac{4}{3x^4}\right) \log (1 - x) - \frac{11}{36x} - \frac{5}{6x^2} + \frac{4}{3x^3},$$

for $-1 < x < 0$ and $0 < x < 1$; $s(0) = 0$; $s(1) = \frac{7}{36}$.

40. $4e - 5$.

41. $|x| < 1$; if $s(x)$ denote the sum,

$$s(x) = \tfrac{1}{3} + \tfrac{1}{2}\frac{x^2 - 1}{x^3}\{\tfrac{1}{2} \log (1 + x) - \tfrac{1}{2} \log (1 - x) - x\},$$

$x \neq 0$; $s(0) = \frac{1}{3}$; $s(\pm 1) = \frac{1}{3}$.

42. (i) $\frac{1}{8} - \dfrac{1}{16(2n + 1)} - \dfrac{3}{16(2n + 3)}$.

43. $s(x) = e^x\{3x^{-2} - 3x^{-1} + 1\} - 3x^{-2} + \frac{1}{2}$, $x \neq 0$; $s(0) = 0$.

44. $s(x) = \left(\dfrac{1}{2x} - \dfrac{1}{2x^3}\right) \log (1 - x) - \frac{1}{2} + \dfrac{1}{4x} + \dfrac{1}{2x^2}$, $x \neq 0$; $s(0) = 0$.

45. (i) $\frac{1}{2} \log 3$; (ii) $15e$.

46. 1.

47. $A = 0$, $B = \frac{1}{6}$, $C = -\frac{1}{2}$, $D = \frac{1}{3}$; $\frac{1}{6}n(n + 1)(2n + 1)$.

48. (i) $\frac{1}{4} - \dfrac{1}{2(n + 1)(n + 2)}$; (ii) $n(2n^3 + 8n^2 + 7n - 2)$.

49. $\frac{1}{18} - \dfrac{1}{3(n + 1)(n + 2)(n + 3)}$.

50. $A = 16/5$, $B = -8/3$, $C = 7/15$; $\frac{16}{5}n^5 - \frac{8}{3}n^3 + \frac{7}{15}n$.

51. $1 - \dfrac{1}{3^n(n + 1)}$; 1.

53. $\frac{1}{2}\left\{n(n + 1)\cos\theta + \dfrac{\cos\theta - \cos 3\theta - \cos(2n + 1)\theta + n\cos(2n + 3)\theta}{1 - \cos 2\theta}\right\}$.

54. $u_n = n(n^2 - 2n + 3)$; $s_n = \frac{1}{12}n(3n^3 - 2n^2 + 9n + 14$.

55. $n^3 - 3n^2 - 2n + 1$. 56. $\frac{1}{12}n(n+1)^2(n+2)$.

58. $\frac{1}{2} n \cos a + \frac{1}{2}\dfrac{\cos(n+2)a \sin na}{\sin a}$.

60. $(1 - x)\cos\theta/(1 - 2x\cos 2\theta + x^2)$.

61. $\{x\cos\theta - x^2 - x^{n+1}\cos(n+1)\theta + x^{n+2}\cos n\theta\}/(1 - 2x\cos\theta + x^2)$.

62. (i) $\cos na \sec^n a - 1 + \sin na \sec^n a \cot a$; (ii) $e^{\cos a}\sin(\sin a)$.

63. $\cos\frac{1}{2}\theta/\sqrt{2\cos\theta}$. 64. $\sin\frac{1}{2}n(\pi - \theta)/(2\sin\frac{1}{2}\theta)^n$

69. $\sin\theta/(1 - 2x\cos\theta + x^2)$. 70. $x(1 - x^2)\sin\theta/(1 - 2x\cos\theta + x^2)^2$.

71. $\{(2a_0 - a_1)\cos a - a_0\cos(a - \beta)$
$+ (2a_{n-1} - a_{n-2})\cos\{a + (n-1)\beta\} - a_{n-1}\cos(a + n\beta)\}/4\sin^2\frac{1}{2}\theta$.

73. $[1 + x - (2n+1)x^n + (2n-1)x^{n+1}]/(1-x)^2$.

EXERCISES XII

DETERMINANTS

2. (i) $(a-b)(b-c)(c-a)$; (ii) $(a-b)(b-c)(c-a)$.

3. $(a-b)(b-c)(c-a)(a+b+c)$. 4. -575.

6. $(x - a^2 + 2a - 1)(x - \alpha)(x - \beta)$, where α, β are the roots of
$x^2 + (a^2 + 4a + 1)x + 2(a+1)^2(a-1) = 0$.

7. $(x+1)(x + 1 + i\sqrt{7})(x + 1 - i\sqrt{7})$.

8. $n\pi$, $2n\pi \pm \frac{1}{2}\pi$, $n\pi \pm \frac{1}{3}\pi$, $n\pi \pm \frac{1}{4}\pi$.

9. $a^3(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)/\Pi a_i\ \Pi(a - a_i)^2$. 10. 0.

11. $\{m^2 + \frac{1}{2}(1 - s)\}\{k^2 + \frac{1}{2}(1 - s)m^2\} - \{\frac{1}{2}(1 + s)mk\}^2$
$+ k^2(1 - s^2) + \frac{1}{2}(1 - s)m^2$.

14. $3\lambda\mu(\lambda - \mu)(a-b)(b-c)(c-a)$. 16. 160.

17. The second determinant is $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)\lambda$ where λ is the sum of the homogeneous products of n dimensions in a, b, c, d.

18. $1 + x_1 + x_2 + x_3 + x_4$. 19. (i) -3 22. $0, -5 \pm 3i\sqrt{3}$.

23. $0, -11$. 24. (i) $0, \frac{1}{2}\{-3 \pm \sqrt{205}\}$. 25. $\frac{2}{3}$.

26. 3, 18. 27. $0, 0, 3a$.

28. $2abc/(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca)$. 30. $xy + yz + zx + xyz$.

31. $\frac{1}{16}\{-51 \pm i\sqrt{215}\}$. 38. $\begin{matrix} bc - f^2 & fg - hc & hf - \\ fg - hc & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{matrix}$

40. $x = 0, y = -3, z = -1$. 41. $x = 1, y = 2, z = 0$.

42. $x = 3, y = 5, z = 2$. 45. $x = \pm 1$.

EXERCISES XIII

MATRICES

6. $$AB = \begin{bmatrix} -3 & 3 & 17 \\ 0 & 3 & 11 \\ 0 & 3 & 13 \end{bmatrix}, \qquad BA = \begin{bmatrix} 8 & 13 & 5 \\ -3 & -11 & 12 \\ 1 & -5 & 16 \end{bmatrix},$$

$$A^{-1} = \tfrac{1}{8}\begin{bmatrix} -8 & 7 & 5 \\ 6 & -3 & -3 \\ 2 & -1 & 1 \end{bmatrix}, \quad B^{-1} = \tfrac{1}{3}\begin{bmatrix} -5 & -11 & 10 \\ 13 & 25 & -23 \\ -3 & -6 & 6 \end{bmatrix},$$

$$A^{-1}B^{-1} = \tfrac{1}{18}\begin{bmatrix} 116 & 233 & -211 \\ -60 & -123 & 111 \\ -26 & -53 & 49 \end{bmatrix}, \quad B^{-1}A^{-1} = \tfrac{1}{8}\begin{bmatrix} -2 & -4 & 6 \\ 0 & 13 & -11 \\ 0 & -3 & 3 \end{bmatrix}.$$

7. $x = -1,\ y = 3,\ z = 2.$

8. $$\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a\gamma - \beta & -\gamma & 1 \end{bmatrix}.$$

9. $$\text{adj.}\, A = \begin{bmatrix} 4 & -42 & 20 \\ 20 & 50 & -30 \\ -9 & -3 & 20 \end{bmatrix}, \qquad |A| = 130.$$

14. $$AB = \begin{bmatrix} 1+i & 1+i \\ 1-i & 1-i \end{bmatrix}, \qquad BA = \begin{bmatrix} 1-i & 1+i \\ 1-i & 1+i \end{bmatrix}.$$

15. (i) 2; (ii) 2.

16. $17/4x^{(1)} + 9/4x^{(2)} - 5/2x^{(3)}.$

17. $x = 3z - 2,\ y = 1 - z.$

18. $x = 0,\ y = z.$

19. $b \neq 0,\ a \neq 1$ or -2, one solution; $a = b = 1$ or $a = b = -2$, an infinity of solutions; in all other cases, no solution.

20. Solutions exist if any one of the following sets of conditions is satisfied:— $a = b = c$; $a = b \neq 1,\ c = 1$; $b = c \neq 1,\ a = 1$; $c = a \neq 1,\ b = 1$; $a = b = 1$; $b = c = 1$; $c = a = 1$.

23. $$X = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}, \qquad Y = \tfrac{1}{3}\begin{bmatrix} 43 & -7 \\ 118 & -19 \end{bmatrix}.$$

24. Hyperboloid of one sheet, semi-axes $\sqrt{7}$, $\sqrt{7}$, $\sqrt{2}$, principal planes $x - 2y = 0,\ 2x + y = 0,\ z = 0.$

25. $a^2 \neq b^2,\ b^2 \neq c^2,\ c^2 \neq a^2$, one solution. $a = b \neq -c$; or $b = c \neq -a$; or $c = a \neq -b$, an infinity of solutions. In all other cases, no solution.

26. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$,

$\begin{bmatrix} a & b(a+3) \\ -(a+1)/b & -(a+4) \end{bmatrix}$,

where a and b are arbitrary.

28. $U = \begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{bmatrix}$, where ω is a complex cube root of unity.

29. $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $A^r = \begin{bmatrix} 3^r & 0 \\ 0 & 3^r \end{bmatrix}$ (r even),

$A^r = \begin{bmatrix} -3^{r-1} & 2.3^{r-1} \\ 4.3^{r-1} & 3^{r-1} \end{bmatrix}$ (r odd).

30. $x_1 = \alpha$, $x_2 = \beta$, $x_3 = 9/7\alpha - 5/7\beta$, $x_4 = -5/7\alpha + 16/7\beta$, where α and β are arbitrary.

(*a*) $\{1, -1, 2, -3\}$. (*b*) None.

31. $\lambda > 2$.

32. $\alpha = 1$, $\beta = -1$, $\gamma = 2$. $B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

35. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & \sqrt{3}/6 & \sqrt{3}/6 & \sqrt{3}/3 \\ 1/2 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & \sqrt{3}/3 & \sqrt{3}/3 & -\sqrt{3}/3 \\ -1/2 & 1/2 & 1/2 & -1/2 & 0 \\ -1/2 & -1/2 & \sqrt{3}/6 & \sqrt{3}/6 & \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

36. $x_1 = (a + \lambda_1)(b + \lambda_1)(\lambda_2 - \lambda_3)$,
$x_2 = (a + \lambda_2)(b + \lambda_2)(\lambda_3 - \lambda_1)$,
$x_3 = (a + \lambda_3)(b + \lambda_3)(\lambda_1 - \lambda_2)$.

38. $1, \ -\cos^2 \dfrac{\theta - \phi}{2} \pm i \sin \dfrac{\theta - \phi}{2} \sqrt{1 + \cos^2 \dfrac{\theta - \phi}{2}}$.

EXERCISES XIV

ELIMINATION

1. The equations are not independent since any one of them may be deduced from the other two. Thus the equations are not sufficient for elimination.

2. $t = 0, 3, 3$; when $t = 3$ the equations become identical.

5. $(a' - a)(a'b - ab') - (b' - b)^2$.

7. The determinant is $\begin{vmatrix} 0 & 1 & p & q & r \\ -1 & p & q & r & 0 \\ 0 & 0 & 1 & b & c \\ 0 & 1 & b & c & 0 \\ -1 & b & c & 0 & 0 \end{vmatrix}$

8. $\begin{array}{cccccc} 0 & 0 & & b_1 & c_1 & d_1 \\ 0 & a_1 & & c_1 & d_1 & 0 \\ a_1 & b_1 & & d_1 & 0 & 0 \\ & & & b_2 & c_2 & d_2 \\ & & & c_2 & d_2 & 0 \\ a_2 & \bar{b}_2 & \bar{c}_2 & d_2 & 0 & 0 \end{array}$

9. $(a^2 + b^2 - 4a)^2 + 16b^2 = 9(a^2 + b^2)^2$.

10. $b^2 \{(d + a)^2 + c^2\} = (d^2 - a^2 + c^2)^2$.

11. $2b(1 + c) = (a^2 + b^2)(1 - c)$.

12. y^2 13. $y^2 = a(2x + a)$.

14. $x^2(h'a - a'h) - xy(ba' - ab') + y^2(hb' - bh') = 0$.

19. $a^2 + b^2 - 6ab = 0$. 20. $3(a^2 + b^2 + c^2) - 4d^2 = 2(ab + bc + ca)$.

21. $a^2 + b^2 + c^2 = abc + 4$. 22. $ab = 1 + c$.

EXERCISES XV

The Theory of Equations

3. Two negative roots, two imaginary roots.

7. One positive, one negative, two imaginary roots.

10. $2\cos\frac{7}{15}\pi$, $2\cos\frac{11}{15}\pi$, $2\cos\frac{13}{15}\pi$. 12. $\pm\frac{2}{3}$, $\frac{1}{3}(3 \pm i\sqrt{6})$.

13. ± 1, $\pm\sqrt{3}$, $\pm\sqrt{3}$; $1 - 7y^2 + 15y^4 - 9y^6 = 0$.

14. $\pm\frac{3}{2}$, 2, 2, -4. 15. $\frac{1}{2}(1 \pm i\sqrt{7})$, $-1 \pm i$.

16. 3, $\frac{1}{2}(7 \pm \sqrt{13})$. 17. $-\frac{5}{3}$, $-\frac{2}{3}$, $\frac{1}{3}$, $\frac{4}{3}$.

18. $a = 1$ or $-\frac{1}{3}$; $b = 6$ or $\frac{74}{9}$. 20. $3q$.

21. $\Sigma\alpha^2 = -2p$, $\Sigma\alpha^3 = 3q$. 23. $\sin 10°$, $\sin 50°$, $\sin 250°$.

25. $-15b^6 + 16ab^4c - 2a^2b^4c^2 + a^4c^4$. 28. $3x^3 + 9x^2 + 6x + 2 = 0$.

32. $x^3 + (p^3 - 3pq + 3r)x^2 + (3r^2 + q^3 - 3pqr)x + r^3 = 0$.

33. $x^3 + 10x^2 + 57x - 196 = 0$. 34. $r^2x^3 + 2qrx^2 + q^2x - r = 0$.

35. (i) $x^3 + 6x^2 + 9x + 3 = 0$; (ii) $x^3 - 18x^2 + 45x - 9 = 0$;
(iii) $9x^3 - 45x^2 + 18x - 1 = 0$.

36. $x^3 - 2qx^2 + (pr + q^2)x - pqr + r^2 = 0$. 37. $17x^3 + 25x^2 + 12x + 2 = 0$.

38. $2x^4 + 0{\cdot}25x^2 + 2{\cdot}25x - 2{\cdot}4609375 = 0$.

39. $-1 \pm \sqrt{3}$, $-1 \pm i\sqrt{2}$. 40. $x^4 + 4x^3 + 54x^2 + 36x + 1296 = 0$.

41. $ax^4 + 10bx^3 + 100cx^2 + 1000dx + 10000e = 0$.

42. $x^3 - ax^2 + \frac{1}{2}(a^2 - b^2)x + \frac{1}{6}(-a^3 + 3ab^2 - 2c^3) = 0$;
$u_n = au_{n-1} - \frac{1}{2}(a^2 - b^2)u_{n-2} - \frac{1}{6}(-a^3 + 3ab^2 - 2c^3)u_{n-3}$.

43. $1/(1 - c)$, $(c - 1)/c$, $c/(c - 1)$.

44. $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$, where ω is an imaginary cube root of unity.

46. $x^3 - 3ax^2 + 3(a^2 - bc)x - (a^3 + b^3 + c^3 - 3abc)$.

47. $\frac{1}{2}(1 \pm i\sqrt{3})$. 48. $\cos\frac{(2r+1)\pi}{p} + i\sin\frac{(2r+1)\pi}{p}$, $r = 0, 1, 2, \ldots, p$.

50. $x = 1, \cos\frac{2r\pi}{2n+1} \pm i\sin\frac{2r\pi}{2n+1}$, $r = 1, 2, 3, \ldots n$.

51. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}(-1 \pm i\sqrt{2})$. 53. Assuming $p, r \neq 0$ $p^2 + r = 0$; $-p, -p, -p, p$.

55. $-\frac{1}{4} \pm \frac{1}{4}\sqrt{5}$, $-\frac{1}{4} \pm \frac{1}{4}\sqrt{5}$. 56. $-2, -2, 3, 3$.

57. $-2, -\frac{3}{2}, 1, \frac{3}{2}; -1, -2, -\frac{3}{2}$. 58. $(2x + 3)(x^2 - x + 1)^2$.

60. (i) $c = 3^{-\frac{1}{4}}$; $3^{-\frac{1}{4}}, 3^{-\frac{1}{4}}, -3^{-\frac{1}{4}} \pm i\,2^{\frac{1}{2}}3^{-\frac{1}{4}}$;

(ii) $\{4a \pm (a^2 + 3)\}/(1 - a^2)$, $\{4a \mp (a^2 + 3)/(1 - a^2)$.

EXERCISES XVI

THEORY OF EQUATIONS (*continued*)

1. $y^3 - 30y^2 + 225y - 68 = 0$. 2. $G^2 + 2H^3 = 0$.

3. $x^2(a'b - ab')^2 + x(a'c - ac')(a'b - ab') + (a'c - ac')^2 + (bc' - b'c)(a'b - ab')$.

4. $\pm\Pi(a_r - a_s)$, $r = 1, 2, \ldots n$; $s = 1, 2, \ldots n$, $r \neq s$; $4p^3 + 27q = 0$.

5. The roots are $x = 3, 3, -4$. 7. The root lies between -2 and -3.

8. $\Pi(\alpha - \beta)^2 = -b^3 - 27c^2$.

9. Max. value $= q + 2p^3$. Min. value $= q - 2p^3$. 10 $x = -2\sqrt[3]{3} + \sqrt[3]{4}$.

11. $x = -8$ or $-1 - 2\omega - 5\omega^2$ or $-1 - 2\omega^2 - 5\omega$, where ω is an imaginary cube root of unity.

12. $x = \frac{1}{3}(1 + 2\sqrt{7}\cos\frac{1}{3}\alpha)$ or $\frac{1}{3}\{1 + 2\sqrt{7}\cos(\frac{2}{3}\pi \pm \frac{1}{3}\alpha)\}$ where $\cos\alpha = (-1/2\sqrt{7})$.

14. $x = (2 + \sqrt{3})^{\frac{1}{3}} - (2 - \sqrt{3})^{\frac{1}{3}} = 2{\cdot}20$ to 2 dec. places.

16. $x^3 - 3pqx - (p^3 + q^3) = 0$.

18. $(x^2 - 3x + 3)(x^2 - 5x + 7)$; $\left(x^2 - 4x + \frac{5}{2} + \frac{i\sqrt{3}}{2}\right)\left(x^2 - 4x + \frac{5}{2} - \frac{i\sqrt{3}}{2}\right)$;

$\{x^2 + x(i\sqrt{3} - 4) + 3 - 2i\sqrt{3}\}\{x^2 - x(i\sqrt{3} + 4) + 3 + 2i\sqrt{3}\}$.

19. $x = \frac{1}{2}(1 \pm \sqrt{17})$, $\frac{1}{2}(-1 \pm \sqrt{5})$.

20. One transformation is $x = 2y + 1$; $x = 3 \pm 2\sqrt{2}, -1 \pm 2\sqrt{2}$.

21. $p = -3$; $x = -1, -1 \pm \sqrt{2}, \frac{1}{2}(3 \pm \sqrt{5})$.

22. $x = \frac{1}{2}(\omega + \omega^{-1})$, $\omega^5 = 1$. 23. $-8, 2$. 24. $-\frac{1}{2}, 4$. 25. 4.

27. 2 (one root equal to 2). 29. $1{\cdot}85$. 30. $2{\cdot}15, 0{\cdot}311$. 31. $1{\cdot}12$.

32. $3{\cdot}036$. 33. $2{\cdot}591$. 34. $4{\cdot}49$.

36. $-2\epsilon + 2\epsilon^2$; $-1 + \frac{1}{2}\epsilon + \frac{5}{8}\epsilon^2$. 37. $2 - \epsilon$; $3 + \frac{1}{3}\epsilon$.

39. 1·476. 40. 2·195. 41. 2·24.

42. Roots lie between (*a*) − 1, − 2; (*b*) − 1, 0; (*c*) 3, 4; 3·35.

43. 1·270. 44. − 0·17, 1·69, 3·95; 1·689. 45. 1·893. 46. 0·338.

47. $\Sigma p_r (y + a)^{n-r}$; $\Sigma p_r b^r y^{n-r}$; 2·87. 48. 2·90.

49. 4·189. 50. 1·246. 51. 2·57.

INDEX TO VOLUMES I. AND II.

The numerals I. or II. refer to the volume, the other numbers being page references.